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Adaptive discretization of fractional order viscoelasticity using sparse time history

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Abstract

An efficient numerical method to integrate the constitutive response of fractional order viscoelasticity is developed. The method can handle variable time steps. To overcome the problem of the growing amount of data that has to be stored and used in each time step we introduce sparse quadrature. We use an internal variable formulation of the viscoelastic equations where the internal variable is of stress type. The rate equation that governs the evolution of the internal variable involves a fractional integral and can be identified as a Volterra integral equation of the second kind with a weakly singular kernel. For the numerical integration of the rate equation we adopt the finite element method in time, in particular the discontinuous Galerkin method with piecewise constant basis functions is used. A priori and a posteriori error estimates are proved. An adaptive strategy based on the a posteriori error estimate is developed. Finally, the precision and effectiveness of the method are demonstrated by comparing the numerical solutions with analytical solutions.

Keywords: Fractional order viscoelasticity; Volterra equation; Sparse quadrature; Error estimates; Adaptivity

1. Introduction

Fractional order operators (integrals and derivatives) have proved to be very suitable for modeling memory effects of various materials and systems of technical interest. In particular, they are very useful for modeling viscoelastic materials, see, e.g., [1,2]. Polymers in general show a weak frequency dependence of their viscoelastic characteristics. This frequency dependence is difficult to describe with classical viscoelasticity based on integer order operators in the rate laws for the internal variables. A large number of derivative operators (or internal variables), resulting in many parameters, is required to obtain a reasonably accurate description of the observed viscoelastic characteristics. By introducing fractional order operators in the constitutive relations, the number of parameters can be significantly reduced. The fractional order

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viscoelastic model has successfully been fitted to experimental data over a broad frequency range for several polymers using only four parameters (two "elastic" constants, one relaxation constant, and the non-dimensional fractional order of differentiation) in the uniaxial case, see, e.g., [1]. The fractional order viscoelastic model has also been successfully fitted to time domain rubber data at small strains in, e.g., [3].

A motivation for using fractional order operators in viscoelasticity is that a whole spectrum of viscoelastic mechanisms can be included in a single internal variable. The stress relaxation spectrum for the fractional order model is continuous with the relaxation constant as the most probable relaxation time, while the order of the operator plays the role of a distribution parameter. Note that the spectrum is discrete for the classical model that is based on integer order derivatives. By a suitable choice of material parameters for the classical viscoelastic model it is observed both numerically and analytically that the classical model with a large number of internal variables (each representing a specific viscoelastic mechanism) converges to the fractional model with a single internal variable [4].

The drawback of the fractional order model is that when numerically integrating the constitutive response the whole history of the internal variables must be saved and included in each time step. The most commonly used algorithms for this integration are based on the Lubich [5,6] convolution quadrature for fractional order operators (see, e.g., [7–9]). Discretization of fractional order differential equations using the Lubich approach are also studied in [10–12]. The Lubich convolution quadrature requires uniformly distributed time steps or alternatively logarithmically distributed time steps as outlined in [12]. These are cumbersome restrictions because it is not possible to use adaptivity and goal oriented error estimation. From an engineering viewpoint it is important to assess the quality of the numerical algorithm with respect to its capability to predict responses with high accuracy. This means that goal oriented error estimations should be used.

The starting point for the present study is the numerical integration method for an integro-differential equation with a weakly singular convolution kernel developed in a previous study [13] which is based on the work in [14]. Here we develop an adaptive algorithm with a priori and a posteriori error estimates for the integration of fractional order viscoelastic constitutive equations. The rate equation for the internal variable is written in a form that involves a fractional integral operator (rather than a fractional derivative operator) and may be identified as a Volterra integral equation with a weakly singular kernel. Fractional integral operators are advantageous as they are less singular than fractional derivative operators and therefore easier to handle numerically. For the numerical integration of the rate equation we adopt the discontinuous Galerkin method with piecewise constant basis functions. This method is in particular well suited for singular kernels in that the convolution integrals enter in the form of averages instead of point values. The a posteriori error estimate forms the basis for the adaptive strategy. Following [13,14] we use sparse quadrature in order to overcome the problem with the growing amount of data that has to be stored and used in each time step. Sparsely distributed time steps are used in the distant part of the history while small steps are used in the most recent part. The idea is to break up the convolution structure by using piecewise linear interpolants of the kernel between the large steps in the distant part of the history. This kind of sparse quadrature was first studied in [15,16]. A posteriori error analysis similar to the present work was done in [17], but without sparse quadrature and without emphasis on the singular kernel.

In a few numerical examples, we consider the constitutive response. In particular, the stress relaxation function (i.e., the stress response upon a unit strain) is calculated and compared to the analytical solution.

2. Fractional order viscoelasticity

We will now formulate a time domain linear viscoelastic model based on a single internal variable with rate equation involving a fractional integral. The model is equivalent to the fractional derivative model of viscoelasticity (see, e.g., [18,19]) in the sense that the models give the same constitutive stress response on a given strain history. Isothermal conditions and isotropy are assumed throughout the present study. For constitutive modeling of anisotropic fractional order viscoelastic response we refer to [8]. First we define fractional integration and differentiation. The Riemann–Liouville definition of fractional integration of order α is (see, e.g., [20])

$$D^{-\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{y(s)}{(t-s)^{1-\alpha}}\,\mathrm{d}s.$$

The same definition can be used for fractional differentiation of order α by a formal replacement of $-\alpha$ by α $(\alpha \neq 1, 2, 3, \ldots)$,

$$D^{\alpha}y(t) = \frac{1}{\Gamma(-\alpha)}\int_0^t \frac{y(s)}{(t-s)^{1+\alpha}}\,\mathrm{d}s.$$

The convolution integral above is in general divergent and needs to be interpreted in the sense of its regularization. A convergent expression for the fractional derivative operator is obtained by splitting the derivative operator into an integer order derivative and a fractional integral operator

$$D^{\alpha} = D^{N-\rho} = D^N D^{-\rho},$$

where N is the integer that satisfies $\alpha < N < \alpha + 1$ and $0 < \rho < 1$. Specializing to $0 < \alpha < 1$, which is the interesting interval in viscoelasticity, the definition of the fractional derivative can be written as

$$D^{\alpha}y(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t \frac{y(s)}{(t-s)^{\alpha}}\,\mathrm{d}s\right].$$

We note that the fractional differential operator is not a local operator, i.e., the derivative depends on the whole history of the function.

Now consider the simplest uniaxial fractional derivative viscoelastic model that can reproduce instantaneous and long time elastic responses. By using the concept of internal, or non-observable, variables the constitutive equation is formulated as two coupled equations (see [8])

$$\sigma(t) = E_1(\varepsilon(t) - \varepsilon^{\mathrm{v}}(t)) + E_2\varepsilon(t), \tag{2.1}$$

$$D^{\alpha}\varepsilon^{\mathrm{v}}(t) + \frac{1}{\tau^{\alpha}}\varepsilon^{\mathrm{v}}(t) = \frac{1}{\tau^{\alpha}}\varepsilon(t), \quad \varepsilon^{\mathrm{v}}(0) = 0, \tag{2.2}$$

where σ is the stress and ε is the (macroscopic) strain, while ε^{v} is an internal variable of strain type representing a distribution of irreversible microstructural processes in the material, $E_1 > 0$ and $E_2 > 0$ are elastic stiffnesses, $\tau > 0$ is the relaxation constant, and $0 < \alpha \leqslant 1$. When $\alpha < 1$, the formal initial condition to (2.2) would be a fractional integral $D^{\alpha-1}\varepsilon^{\mathrm{v}}(0) = 0$, rather than $\varepsilon^{\mathrm{v}}(0) = 0$. However, (having in mind that the strain must be bounded)

$$\varepsilon^{\mathrm{v}}(0) = 0 \text{ implies } D^{\alpha-1}\varepsilon^{\mathrm{v}}(0) = 0.$$

The initial condition above means that the model predicts an initial response following Hooke's elastic law,

$$\sigma(0) = (E_1 + E_2)\varepsilon(0) = E_{(0)}\varepsilon(0), \tag{2.3}$$

where $E_{(0)} = E_1 + E_2$ is the instantaneous stiffness of the model.

The present method for evaluating the convolution integral requires that the kernel is integrable; we therefore reformulate the rate equation to contain a fractional integral rather than a fractional derivative. By applying the fractional integral operator $D^{-\alpha}$ to the rate equation (2.2) with $\varepsilon^{\mathrm{v}}(0) = 0$, while using the composition rule for fractional order operators (see [20]) and making the change of variable $E_1(\varepsilon - \varepsilon^{\mathrm{v}}) \mapsto \sigma^{\mathrm{v}}$, the constitutive equations (2.1) and (2.2) can be written as

$$\sigma(t) = \sigma^{\mathrm{v}}(t) + E_2\varepsilon(t), \tag{2.4}$$

$$\sigma^{\mathrm{v}}(t) + \frac{1}{\tau^{\alpha}} D^{-\alpha}\sigma^{\mathrm{v}}(t) = E_1\varepsilon(t), \tag{2.5}$$

where σ^{v} can be regarded as an internal variable of stress type. Note that these equations need not be accompanied by initial conditions because no derivatives are involved. However, with $t = 0$ in (2.5) we get $\sigma^{\mathrm{v}}(0) = E_1\varepsilon(0)$ which together with (2.4) leads to Hooke's law (2.3). The rate equation (2.5) can be identified as a Volterra integral equation of the second kind

$$\begin{aligned} &\sigma^{\mathrm{v}}(t) + \frac{1}{\tau^{\alpha}} \int_0^t \beta(t-s)\sigma^{\mathrm{v}}(s)\,\mathrm{d}s = E_1\varepsilon(t), \quad t \in (0,T), \\ &\text{with } \beta(t) = \frac{1}{\Gamma(\alpha)} t^{-1+\alpha}, \quad 0 < \alpha \leqslant 1. \end{aligned} \tag{2.6}$$

The kernel function β in the fractional integral formulation (2.6) is weakly singular but integrable and belongs to $L_1(0,T)$, while the kernel function in the fractional derivative formulation (2.2) ($\sim t^{-1-\alpha} \notin L_1(0,T)$). This makes the fractional integral formulation easier to handle numerically and suitable for the present method based on the discontinuous Galerkin method.

It is convenient to work with non-dimensional quantities. We therefore introduce the replacements: $\sigma/E_{(0)} \mapsto \sigma$, $\sigma^{\mathrm{v}}/E_{(0)} \mapsto \sigma^{\mathrm{v}}$, $E_1/E_{(0)} \mapsto E_1$, $E_2/E_{(0)} \mapsto E_2$, $t/\tau \mapsto t$, and $T/\tau \mapsto T$. The change of variables does not affect the expression for the total stress (2.4). The rate equation (2.6), however, is now given by

$$\begin{aligned} &\sigma^{\mathrm{v}}(t) + \int_0^t \beta(t-s)\sigma^{\mathrm{v}}(s)\,\mathrm{d}s = E_1\varepsilon(t), \quad t \in (0,T), \\ &\text{with } \beta(t) = \frac{1}{\Gamma(\alpha)} t^{-1+\alpha}, \quad 0 < \alpha \leqslant 1. \end{aligned} \tag{2.7}$$

For the sake of greater generality, and for the convenience of notation, we study the equation (2.7) in a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We also use the space $L_2(0,T;H)$ of square integrable functions with values in H, equipped with the norm

$$\|v\|_{[L_2(0,T)]} = \left(\int_0^T \|v(t)\|^2\,\mathrm{d}t \right)^{1/2}.$$

In structural mechanics, the stress and strain are defined in a spatial domain Ω and the natural Hilbert space is $H = L_2(\Omega)$. However, it is customary to consider the constitutive response pointwise, i.e., the response is regarded to be only a function of time and not a function of spatial coordinates. As a consequence, in this case, $H = \mathbf{R}$ (the space of real numbers) and $(u,v) = uv$ and $\|u\| = |u|$. This is the situation in our numerical experiments in Section 7.

We now derive a stability estimate for the solution of the continuous problem. It is well known that the kernel function is positive definite for $\alpha \in (0,1]$ in the sense that for any $T \geqslant 0$ (a proof can be found in [4]),

$$\int_0^T \int_0^t \beta(t-s)\varphi(s)\varphi(t)\,\mathrm{d}s\,\mathrm{d}t \geqslant 0, \quad \forall \varphi \in L_2(0,T;\mathbf{R}). \tag{2.8}$$

Take the inner product of (2.7) with σ^{v} and integrate with respect to t,

$$\int_0^T (\sigma^{\mathrm{v}}(t), \sigma^{\mathrm{v}}(t))\,\mathrm{d}t + \int_0^T \int_0^t \beta(t-s)(\sigma^{\mathrm{v}}(s), \sigma^{\mathrm{v}}(t))\,\mathrm{d}s\,\mathrm{d}t = \int_0^T E_1(\varepsilon(t), \sigma^{\mathrm{v}}(t))\,\mathrm{d}t. \tag{2.9}$$

By expanding σ^{v} in an ON-basis, $\sigma^{\mathrm{v}}(t) = \sum_{i=1}^{\infty} \sigma_i^{\mathrm{v}}(t) e_i$, while using that the kernel function β satisfies (2.8), we get

$$\int_0^T \int_0^t \beta(t-s)(\sigma^{\mathrm{v}}(s), \sigma^{\mathrm{v}}(t))\,\mathrm{d}s\,\mathrm{d}t = \sum_{i=1}^{\infty} \int_0^T \int_0^t \beta(t-s)\sigma_i^{\mathrm{v}}(s)\sigma_i^{\mathrm{v}}(t)\,\mathrm{d}s\,\mathrm{d}t \geqslant 0. \tag{2.10}$$

Upon using (2.9) and (2.10) together with the Cauchy–Schwartz inequality, we obtain

$$\int_0^T \|\sigma^{\mathrm{v}}(t)\|^2\,\mathrm{d}t \leqslant \int_0^T E_1|(\varepsilon(t), \sigma^{\mathrm{v}}(t))|\,\mathrm{d}t \leqslant \int_0^T E_1\|\varepsilon(t)\|\|\sigma^{\mathrm{v}}(t)\|\,\mathrm{d}t,$$

and we arrive at

$$\|\sigma^{\mathrm{v}}\|^2_{L_2(0,T)} \leqslant E_1\|\varepsilon\|_{L_2(0,T)}\|\sigma^{\mathrm{v}}\|_{L_2(0,T)}.$$

Finally, we have

$$\|\sigma^{\mathrm{v}}\|_{L_2(0,T)} \leqslant E_1\|\varepsilon\|_{L_2(0,T)}. \tag{2.11}$$

This implies that the solution is unique. Since the integral operator in (2.7) is compact in $L_2(0,T;H)$, this also implies existence of a solution, by the Fredholm theory. Thus, for any $\varepsilon \in L_2(0,T;H)$ there is a unique solution $\sigma^{\mathrm{v}} \in L_2(0,T;H)$ to (2.7).

2.1. Three-dimensional formulation

In isotropic linear viscoelasticity it is customary to consider viscoelastic behavior under conditions of pure shear and pure dilatation separately. This is done by decomposing the stress tensor σ_{ij} and the strain tensor ε_{ij} in their deviatoric and volumetric parts,

$$\sigma_{ij} = s_{ij} + \tfrac{1}{3}\delta_{ij}\sigma_{kk} \quad \text{and} \quad \varepsilon_{ij} = e_{ij} + \tfrac{1}{3}\delta_{ij}\varepsilon_{kk},$$

where δ_{ij} is the unity tensor and the summation convention is used.

The three-dimensional generalization of the viscoelastic model in (2.4) and (2.5) then reads, in dimensional units,

$$s_{ij} = s_{ij}^{\mathrm{v}} + 2G_2 e_{ij}, \quad s_{ij}^{\mathrm{v}} + \frac{1}{\tau_{\mathrm{G}}^{\alpha_{\mathrm{G}}}} D^{-\alpha_{\mathrm{G}}} s_{ij}^{\mathrm{v}} = 2G_1 e_{ij}$$

and

$$\sigma_{kk} = \sigma_{kk}^{\mathrm{v}} + 3K_2\varepsilon_{kk}, \quad \sigma_{kk}^{\mathrm{v}} + \frac{1}{\tau_{\mathrm{K}}^{\alpha_{\mathrm{K}}}} D^{-\alpha_{\mathrm{K}}} \sigma_{kk}^{\mathrm{v}} = 3K_1\varepsilon_{kk},$$

where K_1, K_2, G_1, and G_2 are elastic stiffnesses, and τ_{G} and τ_{K} are relaxation constants in shear and volumetric responses, respectively, while α_{G} and α_{K} are the fractional integral exponents in shear and volumetric responses, respectively. The initial response follows Hooke's elastic law with

$$s_{ij}(0) = 2(G_1 + G_2)e_{ij}(0) = 2G_{(0)}e_{ij}(0)$$

and

$$\sigma_{kk} = 3(G_1 + G_2)\varepsilon_{kk}(0) = 3K_{(0)}\varepsilon_{ij}(0),$$

where $G_{(0)} = G_1 + G_2$, and $K_{(0)} = K_1 + K_2$ are identified as the instantaneous shear and bulk moduli, respectively.

In the following, we develop methods for computing the constitutive response and error estimates for the uniaxial case (2.4) and (2.7). However, essentially the same methods will apply for the three-dimensional case by letting $H = \mathbf{R}^{3\times 3}$ with inner product and norm $(u, v) = u_{ij}v_{ij}$ and $\|u\| = (u_{ij}u_{ij})^{1/2}$, respectively.

3. The discontinuous Galerkin method

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$ be a temporal mesh with time intervals, $I_n = (t_{n-1}, t_n)$, and time steps, $k_n = t_n - t_{n-1}$. (Recall that we use non-dimensional time steps.) We define the finite element space as

$$\mathscr{W}_D = \{w : w(t) = w_n \text{ for } t \in I_n, w_n \in H, n = 1, \dots, N\}.$$

Note that $w \in \mathscr{W}_D$ may be discontinuous at $t = t_n$; we write $w_n = w_n^- = w_{n-1}^+$.

The finite element approximation $\Sigma^{\mathrm{v}} \in \mathscr{W}_D$ of the exact solution σ^{v} of (2.7) is given by,

$$\begin{aligned} &\Sigma^{\mathrm{v}} \in \mathscr{W}_D, \quad \text{for } n = 1, \dots, N, \\ &\int_{I_n} \left(\Sigma^{\mathrm{v}}(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Sigma^{\mathrm{v}}(s)}{(t-s)^{1-\alpha}} \,\mathrm{d}s - E_1 \varepsilon(t), v(t) \right) \mathrm{d}t = 0 \quad \forall v \in \mathscr{W}_D. \end{aligned} \tag{3.1}$$

Since the functions in $\mathscr{W}_D$ are piecewise constant, we get

$$\Sigma_n^{\mathrm{v}} + q_n(\Sigma^{\mathrm{v}}) - E_1 \bar{\varepsilon}_n = 0, \tag{3.2}$$

where

$$\begin{aligned} \bar{\varepsilon}_n &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \varepsilon(t) \,\mathrm{d}t, \\ q_n(\Sigma^{\mathrm{v}}) &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Sigma^{\mathrm{v}}(s)}{(t-s)^{1-\alpha}} \,\mathrm{d}s \,\mathrm{d}t \\ &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \sum_{j=1}^{n} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j \wedge t} (t-s)^{\alpha-1} \Sigma_j^{\mathrm{v}} \,\mathrm{d}s \,\mathrm{d}t = \sum_{j=1}^{n} k_j \omega_{nj} \Sigma_j^{\mathrm{v}}, \\ \omega_{nj} &= \frac{1}{k_n k_j} \int_{t_{n-1}}^{t_n} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j \wedge t} (t-s)^{\alpha-1} \,\mathrm{d}s \,\mathrm{d}t, \quad t_j \wedge t = \min(t_j, t). \end{aligned} \tag{3.3}$$

Thus, in each time step we have to solve the following equation for Σ_n^{v}:

$$(1 + k_n \omega_{nn}) \Sigma_n^{\mathrm{v}} = E_1 \bar{\varepsilon}_n - \sum_{j=1}^{n-1} k_j \omega_{nj} \Sigma_j^{\mathrm{v}}.$$

As we can see the discontinuous Galerkin method results in discrete equations where the terms enter in form of averages instead of point values. This approach is in particular suitable for singular kernels. The integrals in the expression for the weights ω_{nj} can be integrated analytically. Note that in the case of a uniform mesh the set of weights from the previous time step can be reused in the subsequent time steps. This is not possible with a non-uniform mesh. However, the computational cost to compute the weights is comparatively low. The integration of the source term ε must in general be carried out by a suitable quadrature rule.

In addition to the finite element space $\mathscr{W}_D$ we introduce the space $\mathscr{W}$ of functions that are piecewise smooth with respect to the temporal mesh. We may note that $\mathscr{W}_D \subset \mathscr{W}$. Further, we define the bilinear form $B : \mathscr{W} \times \mathscr{W} \to \mathbf{R}$ by

$$\begin{aligned} B(w,v) &= \sum_{n=1}^{N} \int_{I_n} \left(w(t) + \int_0^t \beta(t-s)w(s)\,\mathrm{d}s, v(t) \right) \mathrm{d}t \\ &= \sum_{n=1}^{N} \int_{I_n} \left(w(t), v(t) + \int_t^T \beta(s-t)v(s)\,\mathrm{d}s \right) \mathrm{d}t. \end{aligned} \tag{3.4}$$

Using the first variant of (3.4), the finite element problem (3.1) becomes

$$\Sigma^{\mathrm{v}} \in \mathscr{W}_D : \quad B(\Sigma^{\mathrm{v}}, v) - \int_0^T (E_1\varepsilon, v)\,\mathrm{d}t = 0 \quad \forall v \in \mathscr{W}_D. \tag{3.5}$$

For later reference we note the weak form of (2.7),

$$\sigma^{\mathrm{v}} \in \mathscr{W} : \quad B(\sigma^{\mathrm{v}}, v) - \int_0^T (E_1\varepsilon, v)\,\mathrm{d}t = 0 \quad \forall v \in \mathscr{W}. \tag{3.6}$$

4. Sparse quadrature

We are now in the position to introduce sparse quadrature for the convolution term q_n in (3.3), and thereby reduce the computational cost. The idea is to create an additional time mesh with steps larger than the original steps, and break up the convolution character by using linear interpolation of the kernel β in (2.7) between the large steps. We introduce time levels $0 = M_0 < M_1 < M_2 < \cdots$, and replace the kernel $\beta(t-s)$ by a piecewise linear interpolant

$$\tilde{\beta}(t,s) = \begin{cases} \beta(t - t_{M_{l-1}})\phi_{1,l}(s) + \beta(t - t_{M_l})\phi_{2,l}(s), & s \in [t_{M_{l-1}}, t_{M_l}], \quad l = 1,\ldots,L, \\ \beta(t-s), & s \in [t_{M_L}, t], \end{cases}$$

where

$$\phi_{1,l}(s) = \frac{t_{M_l} - s}{K_l}, \quad \phi_{2,l}(s) = \frac{s - t_{M_{l-1}}}{K_l}, \quad K_l = t_{M_l} - t_{M_{l-1}}.$$

In order to have a margin from the singularity at $s = t$ in β we take L to be the largest integer such that $t - t_{M_L} \geqslant 1$, see Fig. 1.

We now define the quadrature formula, $\tilde{q}_n(\varphi) \approx q_n(\varphi)$ for $\varphi \in \mathscr{W}_D$, as

$$\begin{aligned} \tilde{q}_n(\varphi) &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \tilde{\beta}(t,s)\varphi(s)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\left(\int_0^{t_{M_L}} \tilde{\beta}(t,s)\varphi(s)\,\mathrm{d}s + \int_{t_{M_L}}^t \beta(t-s)\varphi(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \sum_{j=1}^{M_L}\left(\frac{1}{k_nk_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j}\tilde{\beta}(t,s)\,\mathrm{d}s\,\mathrm{d}t\,k_j\varphi_j\right) + \sum_{j=M_L+1}^{n}\left(\frac{1}{k_nk_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j\wedge t}\beta(t-s)\,\mathrm{d}s\,\mathrm{d}t\,k_j\varphi_j\right) \\ &= \sum_{j=1}^{M_L}\tilde{\omega}_{nj}k_j\varphi_j + \sum_{j=M_L+1}^{n}\omega_{nj}k_j\varphi_j, \end{aligned} \tag{4.1}$$

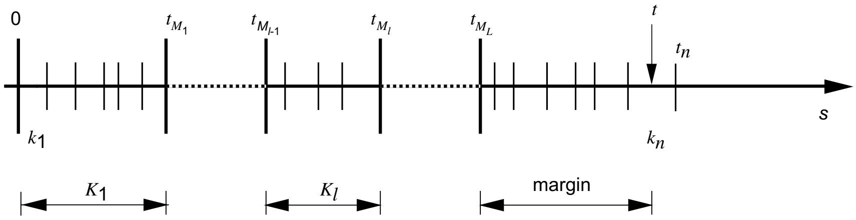

Fig. 1. Time mesh including original time steps k_n and sparse time steps K_l. Note the margin.

where the weights ω_{nj} are defined in (3.3) and

$$\tilde{\omega}_{nj} = \frac{1}{k_n k_j} \int_{t_{n-1}}^{t_n} \int_{t_{j-1}}^{t_j} \tilde{\beta}(t,s)\,\mathrm{d}s\,\mathrm{d}t. \tag{4.2}$$

The first part (where sparse quadrature is used) can be computed as

$$\sum_{j=1}^{M_L} \tilde{\omega}_{nj} k_j \varphi_j = \sum_{l=1}^{L} \left(\tilde{\beta}_{nl,1} \tilde{\varphi}_{l,1} + \tilde{\beta}_{nl,2} \tilde{\varphi}_{l,2} \right),$$

where

$$\tilde{\beta}_{nl,1} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \beta(t - t_{M_{l-1}})\,\mathrm{d}t, \quad \tilde{\beta}_{nl,2} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \beta(t - t_{M_l})\,\mathrm{d}t$$

and

$$\tilde{\varphi}_{l,i} = \sum_{j=M_{l-1}+1}^{M_l} \int_{t_{j-1}}^{t_j} \phi_{i,l}(s)\,\mathrm{d}s\varphi_j, \quad i = 1,2.$$

By using $\tilde{\varphi}_{l,i}$ the number of data that needs to be stored and used in the first sum of the last row in (4.1) is reduced from $\mathrm{O}(n)$ to $\mathrm{O}(L)$. The finite element problem including sparse quadrature takes the form

$$\Sigma^{\mathrm{v}} \in \mathscr{W}_D: \quad \widetilde{B}(\Sigma^{\mathrm{v}}, v) - \int_0^T (E_1\varepsilon, v)\,\mathrm{d}t = 0 \quad \forall v \in \mathscr{W}_D,$$

where

$$\widetilde{B}(w,v) = \sum_{n=1}^{N} \int_{I_n} \left(w(t) + \int_0^t \tilde{\beta}(t,s) w(s)\,\mathrm{d}s, v(t) \right) \mathrm{d}t.$$

So instead of (3.2) we solve

$$\Sigma_n^{\mathrm{v}} + \tilde{q}_n(\Sigma^{\mathrm{v}}) - E_1\bar{\varepsilon}_n = 0.$$

We now estimate the quadrature error.

Theorem 4.1. *The quadrature error of the fractional integral is bounded by*

$$\left\| \int_0^t (\tilde{\beta}(t,s) - \beta(t-s))\varphi(s)\,\mathrm{d}s \right\| \leqslant \sum_{j=1}^{M_L} \epsilon_{nj} k_j \|\varphi_j\|, \quad \forall \varphi \in \mathscr{W}_D, \tag{4.3}$$

where

$$\begin{aligned}&\epsilon_{nj} = \frac{1}{8}\max_{s\in I_{nl}}|\beta''(s)|K_l^2, \quad \textit{if } j = M_{l-1}+1,\ldots,M_l, \quad \textit{and}\\ &I_{nl} = [t_{n-1} - t_{M_l}, t_n - t_{M_{l-1}}].\end{aligned} \tag{4.4}$$

The global quadrature error is bounded by

$$|\widetilde{B}(w,v) - B(w,v)| \leqslant \left(\sum_{n=1}^{N} k_n\left(\sum_{j=1}^{M_L}\epsilon_{nj}k_j\|w_j\|\right)^2\right)^{1/2}\|v\|_{L_2(0,t_N)}, \quad \forall w\in\mathscr{W}_D, \quad \forall v\in\mathscr{W}.$$

Proof. By using the standard interpolation error formula, with $\tilde{s}\in[t_{M_{l-1}}, t_{M_l}]$, we get

$$\begin{aligned}\left\|\int_0^t(\tilde{\beta}(t,s)-\beta(t-s))\varphi(s)\,\mathrm{d}s\right\| &\leqslant \sum_{l=1}^{L}\sum_{j=M_{l-1}+1}^{M_l}\frac{1}{k_j}\int_{I_j}|\tilde{\beta}(t,s)-\beta(t-s)|\,\mathrm{d}s k_j\|\varphi_j\|\\ &\leqslant \frac{1}{2}\sum_{l=1}^{L}\sum_{j=M_{l-1}+1}^{M_l}\frac{1}{k_j}\int_{I_j}|\beta''(t-\tilde{s})|(s-t_{M_{l-1}})(t_{M_l}-s)\,\mathrm{d}s k_j\|\varphi_j\|\\ &\leqslant \frac{1}{8}\sum_{l=1}^{L}\sum_{j=M_{l-1}+1}^{M_l}\max_{s\in I_{nl}}|\beta''(s)|K_l^2k_j\|\varphi_j\|.\end{aligned}$$

The bound for the global quadrature error is obtained by using the definitions of B and $\widetilde{B}$ together with (4.3)

$$\begin{aligned}|\widetilde{B}(w,v)-B(w,v)| &= \left|\int_0^{t_N}\left(\int_0^t(\tilde{\beta}(t,s)-\beta(t-s))w(s)\,\mathrm{d}s, v(t)\right)\mathrm{d}t\right|\\ &\leqslant \int_0^{t_N}\left\|\int_0^t(\tilde{\beta}(t,s)-\beta(t-s))w(s)\,\mathrm{d}s\right\|\|v(t)\|\,\mathrm{d}t\\ &\leqslant \left(\int_0^{t_N}\left\|\int_0^t(\tilde{\beta}(t,s)-\beta(t-s))w(s)\,\mathrm{d}s\right\|^2\mathrm{d}t\right)^{1/2}\left(\int_0^{t_N}\|v(t)\|^2\,\mathrm{d}t\right)^{1/2}\\ &\leqslant \left(\sum_{n=1}^{N}k_n\left(\sum_{j=1}^{M_L}\epsilon_{nj}k_j\|w_j\|\right)^2\right)^{1/2}\|v\|_{L_2(0,t_N)},\end{aligned}$$

where in the last step we used that ϵ_{nj} is piecewise constant with respect to the original mesh. □

The factor (4.4) that enters in the quadrature error can explicitly be written as

$$\max_{s\in I_{nl}}|\beta''(s)| = |\beta''(t_{n-1}-t_{M_l})| = \frac{(1-\alpha)(2-\alpha)}{\Gamma(\alpha)}(t_{n-1}-t_{M_l})^{-3+\alpha}. \tag{4.5}$$

The factor in (4.5) will be large if sparse quadrature is used close to the singularity point at $s = t$, which motivates the previous choice of margin $t - t_{M_L} \geqslant 1$. Note that in the special case of $\alpha = 1$ (ordinary integration) this factor becomes zero. This means that the quadrature error is zero, as expected.

The modified kernel $\tilde{\beta}(t,s)$ is not necessarily positive definite in the sense of (2.8). The positive definiteness is needed in order to prove a priori error estimates (stability) of the present numerical method. As in [13,14] we therefore add a positive term $\delta(t) = \delta_n$ for $t\in I_n$ to the quadrature formula:

$$\hat{q}_n(\varphi) = \tilde{q}_n(\varphi) + \delta_n \varphi_n.$$

The following lemma is identical to [14, Lemma 5.2]. The proof is a combination of (2.10) and (4.3).

Lemma 4.2. *Assume that the numbers δ_j are positive and increasing with $\delta_j \geqslant \epsilon_{Nj} t_N/2$, where ϵ_{Nj} is defined in* (4.4). *Then we have the following analog of* (2.8):

$$\int_0^{t_N} \left(\int_0^t \tilde{\beta}(t,s)\varphi(s)\,\mathrm{d}s + \delta(t)\varphi(t), \varphi(t) \right) \mathrm{d}t \geqslant 0, \quad \forall \varphi \in \mathscr{W}_D.$$

The final form of the finite element problem becomes

$$\Sigma^{\mathrm{v}} \in \mathscr{W}_D: \quad \widehat{B}(\Sigma^{\mathrm{v}}, v) - \int_0^T (E_1 \varepsilon, v)\,\mathrm{d}t = 0 \quad \forall v \in \mathscr{W}_D, \tag{4.6}$$

where

$$\widehat{B}(w,v) = \sum_{n=1}^{N} \int_{I_n} \left(w(t) + \int_0^t \tilde{\beta}(t,s) w(s)\,\mathrm{d}s + \delta(t) w(t), v(t) \right) \mathrm{d}t,$$

and instead of (3.2) we now solve

$$\Sigma_n^{\mathrm{v}} + \hat{q}_n(\Sigma^{\mathrm{v}}) - E_1 \bar{\varepsilon}_n = 0.$$

The additional term δ_n calls for a modification of the global quadrature error.

Theorem 4.3. *The global quadrature error is bounded by*

$$\begin{aligned} |\widehat{B}(w,v) - B(w,v)| \leqslant{}& \left(\sum_{n=1}^{N} k_n \left(\sum_{j=1}^{M_L} \epsilon_{nj} k_j \|w_j\| \right)^2 \right)^{1/2} \|v\|_{L_2(0,t_N)} \\ &+ \left(\sum_{n=1}^{N} k_n (\delta_n \|w_n\|)^2 \right)^{1/2} \|v\|_{L_2(0,t_N)}, \quad \forall w \in \mathscr{W}_D, \quad \forall v \in \mathscr{W}. \end{aligned} \tag{4.7}$$

Proof. We begin by rewriting the bilinear form according to

$$\widehat{B}(w,v) - B(w,v) = \left(\widetilde{B}(w,v) - B(w,v) \right) + \left(\widehat{B}(w,v) - \widetilde{B}(w,v) \right).$$

Hence,

$$|\widehat{B}(w,v) - B(w,v)| \leqslant |\widetilde{B}(w,v) - B(w,v)| + |\widehat{B}(w,v) - \widetilde{B}(w,v)|.$$

The first term was estimated in Theorem 4.1 and the second term is treated analogously by using that $\delta(t) = \delta_n$ and $w(t) = w_n$ for $t \in I_n$. $\square$

5. Error estimates

We define the error $e = \Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}} \in \mathscr{W}$. Then by using the first variant of (3.4) and (3.6), we have

$$B(e,v) = B(\Sigma^{\mathrm{v}}, v) - B(\sigma^{\mathrm{v}}, v) = B(\Sigma^{\mathrm{v}}, v) - \int_0^T (E_1 \varepsilon, v)\,\mathrm{d}t. \tag{5.1}$$

By introducing the residual

$$r(t) = \Sigma^{\mathrm{v}}(t) + \int_0^t \beta(t-s)\Sigma^{\mathrm{v}}(s)\,\mathrm{d}s - E_1\varepsilon(t), \tag{5.2}$$

we obtain the following equation in weak form for the error:

$$e \in \mathscr{W}: \quad B(e,v) = \int_0^T (r,v)\,\mathrm{d}t \quad \forall v \in \mathscr{W}. \tag{5.3}$$

5.1. A priori error estimates

In this section we prove a priori error estimates for the discontinuous Galerkin method with piecewise constant basis functions together with sparse quadrature. However, we begin by considering the case without sparse quadrature.

Theorem 5.1. *Let Σ^{v} and σ^{v} be the solutions to* (3.5) *and* (3.6), *respectively. Further, let $\bar{\sigma}^{\mathrm{v}} \in \mathscr{W}_D$ denote the orthogonal projection of σ^{v} onto the space of piecewise constant functions, determined by*

$$\bar{\sigma}^{\mathrm{v}}(t) = k_n^{-1}\int_{I_n} \sigma^{\mathrm{v}}(s)\,\mathrm{d}s, \quad t \in I_n. \tag{5.4}$$

Then, for all $t_N \geqslant 0$,

$$\|\Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}}\|_{L_2(0,t_N)} \leqslant C_\alpha \|\sigma^{\mathrm{v}} - \bar{\sigma}^{\mathrm{v}}\|_{L_2(0,t_N)} \leqslant C_\alpha \left(\sum_{n=1}^N k_n \min\left\{ k_n \int_{I_n} \|\sigma_t^{\mathrm{v}}\|^2\,\mathrm{d}t, 4\max_{I_n}\|\sigma^{\mathrm{v}}\|^2 \right\} \right)^{1/2},$$

where $C_\alpha = 2 + t_N^\alpha/(\alpha\Gamma(\alpha))$ and $\sigma_t^{\mathrm{v}} = \mathrm{d}\sigma^{\mathrm{v}}/\mathrm{d}t$.

Proof. We begin the proof by showing that the bilinear form B is positive. By using the first variant of B in (3.4), we get

$$B(v,v) = \int_0^{t_N} \|v(t)\|^2\,\mathrm{d}t + \int_0^{t_N}\int_0^t \beta(t-s)(v(s),v(t))\,\mathrm{d}s\,\mathrm{d}t.$$

Here the last term is non-negative, see (2.10), so that

$$B(v,v) \geqslant \|v\|^2_{L_2(0,t_N)} \quad \forall v \in \mathscr{W}. \tag{5.5}$$

Now, let $e = (\Sigma^{\mathrm{v}} - \bar{\sigma}^{\mathrm{v}}) + (\bar{\sigma}^{\mathrm{v}} - \sigma^{\mathrm{v}}) = \theta + \rho$. Note that $\theta \in \mathscr{W}_D$ and $\rho \in \mathscr{W}$. In view of (3.5) and (5.3)

$$B(\theta,v) = B(e,v) - B(\rho,v) = -B(\rho,v) \quad \forall v \in \mathscr{W}_D. \tag{5.6}$$

By choosing $v = \theta$ in (5.5) and (5.6), we get

$$\|\theta\|^2_{L_2(0,t_N)} \leqslant |B(\rho,\theta)|. \tag{5.7}$$

The right-hand side of (5.7) can be estimated as

$$\begin{aligned}|B(\rho,\theta)| &= \left| \int_0^{t_N} (\rho(t),\theta(t))\,\mathrm{d}t + \int_0^{t_N} \left(\int_0^t \beta(t-s)\rho(s)\,\mathrm{d}s, \theta(t) \right) \mathrm{d}t \right| \\ &\leqslant \int_0^{t_N} \|\rho(t)\|\|\theta(t)\|\,\mathrm{d}t + \int_0^{t_N} \left\| \int_0^t \beta(t-s)\rho(s)\,\mathrm{d}s \right\| \|\theta(t)\|\,\mathrm{d}t \\ &\leqslant \left(\int_0^{t_N} \|\rho(t)\|^2\,\mathrm{d}t \right)^{1/2} \left(\int_0^{t_N} \|\theta(t)\|^2\,\mathrm{d}t \right)^{1/2} \\ &\quad + \left(\int_0^{t_N} \left\| \int_0^t \beta(t-s)\rho(s)\,\mathrm{d}s \right\|^2 \mathrm{d}t \right)^{1/2} \left(\int_0^{t_N} \|\theta(t)\|^2\,\mathrm{d}t \right)^{1/2}. \end{aligned} \tag{5.8}$$

Upon using the following inequality

$$\left(\int_0^{t_N} \left\| \int_0^t \beta(t-s)\rho(s)\,\mathrm{d}s \right\|^2 \mathrm{d}t \right)^{1/2} \leqslant \int_0^{t_N} \|\beta(t)\|\,\mathrm{d}t \left(\int_0^{t_N} \|\rho(t)\|^2\,\mathrm{d}t \right)^{1/2}, \tag{5.9}$$

while combining (5.7) and (5.8), we arrive at

$$\|\theta\|_{L_2(0,t_N)} \leqslant (1 + \|\beta\|_{L_1(0,t_N)})\|\rho\|_{L_2(0,t_N)}.$$

Since $\|e\|_{L_2} \leqslant \|\theta\|_{L_2} + \|\rho\|_{L_2}$ this completes the proof for the first estimate.

For the intervals where σ^{v}_t exists, we can use the formula

$$\sigma^{\mathrm{v}}(t) - \bar{\sigma}^{\mathrm{v}}(t) = \frac{1}{k_n} \int_{I_n} (\sigma^{\mathrm{v}}(t) - \sigma^{\mathrm{v}}(s))\,\mathrm{d}s = \frac{1}{k_n} \int_{I_n} \int_s^t \sigma^{\mathrm{v}}_t(r)\,\mathrm{d}r\,\mathrm{d}s$$

to get

$$\int_{I_n} \|(\sigma^{\mathrm{v}} - \bar{\sigma}^{\mathrm{v}})(t)\|^2\,\mathrm{d}t \leqslant k_n^2 \int_{I_n} \|\sigma^{\mathrm{v}}_t(t)\|^2\,\mathrm{d}t.$$

We also have the alternative estimate, to be used when σ^{v}_t is unbounded or large,

$$\int_{I_n} \|(\sigma^{\mathrm{v}} - \bar{\sigma}^{\mathrm{v}})(t)\|^2\,\mathrm{d}t \leqslant k_n \max_{t \in I_n} \|(\sigma^{\mathrm{v}} - \bar{\sigma}^{\mathrm{v}})(t)\|^2 \leqslant 4k_n \max_{I_n} \|\sigma^{\mathrm{v}}\|^2,$$

and the second estimate follows. □

We will now discuss the convergence of the present approximation method. First we consider the case when $\sigma^{\mathrm{v}}_t \in L_2(0,t_N;H)$. In this case the method is of first order, $\mathrm{O}(k)$. In order to show this we let $k = \max_{1 \leqslant n \leqslant N} k_n$. Then

$$\left(\sum_{n=1}^N k_n^2 \int_{I_n} \|\sigma^{\mathrm{v}}_t\|^2\,\mathrm{d}t \right)^{1/2} \leqslant \left(k^2 \|\sigma^{\mathrm{v}}_t\|^2_{L_2(0,t_N)} \right)^{1/2} = k\|\sigma^{\mathrm{v}}_t\|_{L_2(0,t_N)}. \tag{5.10}$$

Next we consider the case when σ^{v} is discontinuous (unbounded σ^{v}_t) at a finite number of time points, resulting from discontinuities in ε. In this case the method will be of order $\mathrm{O}(k^{1/2})$, but still convergent. For an interval containing such a discontinuity, we have

$$\left(4k_n \max_{I_n} \|\sigma^{\mathrm{v}}\|^2 \right)^{1/2} \leqslant 2k^{1/2} \max_{I_n} \|\sigma^{\mathrm{v}}\|. \tag{5.11}$$

The total error estimate will then consist of a finite number of contributions of the form (5.11) plus a term of the form (5.10) with the discontinuity intervals excluded. Thus the order is $O(k^{1/2})$. This motivates the use of adaptivity in viscoelastic applications. It is favorable to take shorter time steps in the vicinity of discontinuities in the strain to maintain the original precision.

In the next theorem we include sparse quadrature.

Theorem 5.2. *Let σ^{v} and Σ^{v} be the solutions to* (3.6) *and* (4.6), *respectively, with δ_j as in Lemma* 4.2. *Let also $\bar{\sigma}^{\mathrm{v}}$ be as in* (5.4). *Then, for all $t_N \geqslant 0$,*

$$\|\Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}}\|_{L_2(0,t_N)} \leqslant C_\alpha \|\sigma^{\mathrm{v}} - \bar{\sigma}^{\mathrm{v}}\|_{L_2(0,t_N)} + E_{\mathrm{Q}} \leqslant C_\alpha \left(\sum_{n=1}^{N} k_n \min\left\{ k_n \int_{I_n} \|\sigma_t^{\mathrm{v}}\|^2 \,\mathrm{d}t, 4 \max_{I_n} \|\sigma^{\mathrm{v}}\|^2 \right\} \right)^{1/2} + E_{\mathrm{Q}},$$

where $C_\alpha = 2 + t_N^\alpha/(\alpha\Gamma(\alpha))$, and E_{Q} is a bound for the quadrature error

$$E_{\mathrm{Q}} = \left(\sum_{n=1}^{N} k_n \left(\sum_{j=1}^{M_L} \epsilon_{nj} k_j \max_{I_j} \|\sigma^{\mathrm{v}}\| \right)^2 \right)^{1/2} + \left(\sum_{n=1}^{N} k_n \left(\delta_n \max_{I_n} \|\sigma^{\mathrm{v}}\| \right)^2 \right)^{1/2}. \tag{5.12}$$

Proof. We modify the proof of Theorem 5.1. Let $e = (\Sigma^{\mathrm{v}} - \bar{\sigma}^{\mathrm{v}}) + (\bar{\sigma}^{\mathrm{v}} - \sigma^{\mathrm{v}}) = \theta + \rho$. Then

$$\widehat{B}(\theta, v) = -B(\rho, v) - (\widehat{B}(\bar{\sigma}^{\mathrm{v}}, v) - B(\bar{\sigma}^{\mathrm{v}}, v)) \quad \forall v \in \mathscr{W}_D.$$

We choose $v = \theta$ here. By Lemma 4.2 we have $\widehat{B}(\theta,\theta) \geqslant \|\theta\|^2_{L_2(0,t_N)}$ and hence

$$\|\theta\|^2_{L_2(0,t_N)} \leqslant |B(\rho,\theta)| + |\widehat{B}(\bar{\sigma}^{\mathrm{v}},\theta) - B(\bar{\sigma}^{\mathrm{v}},\theta)|,$$

which, in view of $\|\bar{\sigma}^{\mathrm{v}}(t)\| \leqslant \max_{I_j} \|\sigma^{\mathrm{v}}\|$ in I_j, (4.7), (5.8) and (5.9), proves the desired result. □

In the following we show that the first order accuracy can be preserved when sparse quadrature is included. The first terms in Theorems 5.1 and 5.2 are the same, so it remains to consider E_{Q}. Let $K = \max_{1 \leqslant l \leqslant L} K_l$. Then we have an upper bound for ϵ_{nj} in (4.4) as

$$\epsilon_{nj} = \frac{1}{8} \max_{s \in I_{nl}} |\beta''(s)| K_l^2 \leqslant \frac{1}{8} \max_{s \in I_{nl}} |\beta''(s)| K^2. \tag{5.13}$$

Further, we choose δ_j in accordance with Lemma 4.2 as

$$\delta_j = \frac{t_N}{16} \max_{s \in I_{Nl}} |\beta''(s)| K^2, \tag{5.14}$$

where β'' is given by (4.5); recall that l depends on j in (5.13) and (5.14). Upon using (5.13) and (5.14) in the quadrature error E_{Q} (5.12), we finally get

$$\begin{aligned} E_{\mathrm{Q}} \leqslant K^2 &\left(\sum_{n=1}^{N} k_n \left(\sum_{j=1}^{M_L} \frac{1}{8} \max_{s \in I_{nl}} |\beta''(s)| k_j \max_{I_j} \|\sigma^{\mathrm{v}}\| \right)^2 \right)^{1/2} \\ &+ K^2 \left(\sum_{n=1}^{N} k_n \left(\frac{t_N}{16} \max_{s \in I_{Nl}} |\beta''(s)| \max_{I_n} \|\sigma^{\mathrm{v}}\| \right)^2 \right)^{1/2}. \end{aligned} \tag{5.15}$$

The quadrature error is evidently second order accurate $O(K^2)$. So the choice $K = k^{1/2}$ (while having a suitable margin to the singularity point at $s = t$) will retain the original order of the method.

The purpose of the sparse quadrature in (4.1) is to reduce the storage requirement and hence the operation count of the algorithm. However, when a finite margin $t_N - t_{M_L} \geqslant 1$ is used, the storage is asymptotically the same as for the original method in (3.3). Nevertheless, if the time interval is large we expect a substantial reduction of the storage. We illustrate this in the case of a uniform mesh with step k, sparse step $K = \sqrt{k}$, and final time $T = Nk$. The storage requirement for the original method is $\sim N$, while the sparse part of the sum in (4.1) requires $\sim L \sim T/K = \sqrt{T}\sqrt{N}$ and the margin part $\sim k^{-1} = T^{-1}N$. Thus, for fixed T, the storage for both methods grows linearly with N, but for large T the slope is much smaller for the sparse method. This is even more important in a structural analysis calculation, where Σ^{v}_n belongs to a high-dimensional space resulting from spatial discretization.

5.2. A posteriori error estimates

In this section we develop a posteriori error estimates, i.e., errors expressed in terms of the finite element solution Σ^{v}. These estimates will be used in the numerical computations and constitute the basis for the adaptive strategy.

First we consider the case without using sparse quadrature. Recall that the error is defined as $e = \Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}}$.

Theorem 5.3. *Let σ^{v} be the exact solution of* (3.6) *and let Σ^{v} be the finite element solution of* (3.5) *with $T = t_N$. Then, for all $t_N \geqslant 0$,*

$$\|\Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}}\|_{L_2(0,t_N)} \leqslant \|r\|_{L_2(0,t_N)},$$

where the residual r is defined in (5.2).

Proof. By using $v = e = \Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}}$ in (5.3), we obtain

$$B(e,e) = \int_0^{t_N} (r(t), e(t))\,\mathrm{d}t \leqslant \|r\|_{L_2(0,t_N)}\|e\|_{L_2(0,t_N)}. \tag{5.16}$$

Then by using the positivity property (5.5) of B, we get

$$\|e\|^2_{L_2(0,t_N)} \leqslant B(e,e) \leqslant \|r\|_{L_2(0,t_N)}\|e\|_{L_2(0,t_N)},$$

and the desired result follows. □

In the case of sparse quadrature an additional error arises which will be incorporated in Theorem 5.4. The a posteriori error estimates are based on the positivity of B rather than $\widehat{B}$. This means that δ_j need not be chosen in accordance with Lemma 4.2 to prove a posteriori estimates, but the stability might be lost. However, in the present numerical examples stability is achieved with $\delta_j = 0$.

Theorem 5.4. *Let σ^{v} be the exact solution of* (3.6) *and let Σ^{v} be the finite element solution of* (4.6) *with $\delta_j \geqslant 0$. Further, denote the residual*

$$\hat{r}(t) = (1 + \delta(t))\Sigma^{\mathrm{v}}(t) + \int_0^t \tilde{\beta}(t,s)\Sigma^{\mathrm{v}}(s)\,\mathrm{d}s - E_1\varepsilon(t). \tag{5.17}$$

Then, for all $t_N \geqslant 0$,

$$\|\Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}}\|_{L_2(0,t_N)} \leqslant E_{\mathrm{G}} + E_{\mathrm{Q}}.$$

The first part is an estimate of the Galerkin error

$$E_{\mathrm{G}} = \|\hat{r}\|_{L_2(0,t_N)}, \tag{5.18}$$

and the second part is an estimate of the quadrature error

$$E_{\mathrm{Q}} = \left(\sum_{n=1}^{N} k_n \left(\sum_{j=1}^{M_L} \epsilon_{nj} k_j \|\Sigma_j^{\mathrm{v}}\|\right)^2\right)^{1/2} + \left(\sum_{n=1}^{N} k_n (\delta_n \|\Sigma_n^{\mathrm{v}}\|)^2\right)^{1/2}. \tag{5.19}$$

Proof. By adding and subtracting $\widehat{B}$ to (5.1) with $v = e$, we obtain

$$B(e,e) = \left(\widehat{B}(\Sigma^{\mathrm{v}}, e) - \int_0^{t_N} (E_1 \varepsilon, e)\,\mathrm{d}t\right) + \left(B(\Sigma^{\mathrm{v}}, e) - \widehat{B}(\Sigma^{\mathrm{v}}, e)\right).$$

Hence, using (5.17), we obtain

$$B(e,e) \leqslant \left|\int_0^{t_N} (\hat{r}, e)\,\mathrm{d}t\right| + \left|\widehat{B}(\Sigma^{\mathrm{v}}, e) - B(\Sigma^{\mathrm{v}}, e)\right|.$$

The first part is estimated in the same way as in (5.16) of Theorem 5.3 by making the replacement $r \mapsto \hat{r}$

$$\left|\int_0^{t_N} (\hat{r}, e)\,\mathrm{d}t\right| \leqslant \|\hat{r}\|_{L_2(0,t_N)} \|e\|_{L_2(0,t_N)}.$$

The last part is Theorem 4.3

$$|\widehat{B}(\Sigma^{\mathrm{v}}, e) - B(\Sigma^{\mathrm{v}}, e)| \leqslant \left(\sum_{n=1}^{N} k_n \left(\sum_{j=1}^{M_L} \epsilon_{nj} k_j \|\Sigma_j^{\mathrm{v}}\|\right)^2\right)^{1/2} \|e\|_{L_2(0,t_N)} + \left(\sum_{n=1}^{N} k_n (\delta_n \|\Sigma_n^{\mathrm{v}}\|)^2\right)^{1/2} \|e\|_{L_2(0,t_N)}.$$

Finally, combining the two estimates while using (5.5) with $v = e$, we arrive at

$$\|e\|^2_{L_2(0,t_N)} \leqslant B(e,e) \leqslant (E_{\mathrm{G}} + E_{\mathrm{Q}}) \|e\|_{L_2(0,t_N)},$$

which completes the proof. □

Note that the quadrature sum in the quadrature error (5.19) is a sparse sum. This means that the effort to compute the a posteriori error estimate and the solution Σ^{v} is equivalent.

An alternative, and possibly sharper, estimate can be formulated by means of a duality argument. For this purpose we introduce the adjoint problem to (2.7) with source term $e = \Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}}$,

$$\psi(t) + \frac{1}{\Gamma(\alpha)} \int_t^T \frac{\psi(s)}{(s-t)^{1-\alpha}}\,\mathrm{d}s = e(t), \quad t \in (0,T). \tag{5.20}$$

By the replacement $t \mapsto T - t$ the adjoint problem takes the same form as the primal problem (2.7) with $E_1 \varepsilon = e$. For this reason the adjoint solution satisfies stability estimates of the same form as the primal solution, see (2.11), and we have

$$\|\psi\|_{L_2(0,T)} \leqslant \|e\|_{L_2(0,T)}.$$

Using the second variant of $B(w,v)$ in (3.4) we have the weak form of (5.20),

$$\psi \in \mathscr{W}: \quad B(w,\psi) = \int_0^T (w,e)\,\mathrm{d}t \quad \forall w \in \mathscr{W}. \tag{5.21}$$

Then by letting $w = e$ in (5.21) and $v = \psi$ in (5.3), we obtain a representation of the error in the L_2-norm,

$$\|e\|^2_{L_2(0,T)} = \int_0^T (r, \psi)\,\mathrm{d}t. \tag{5.22}$$

In the theorem below we present increasingly larger a posteriori error estimates.

Theorem 5.5. *Let σ^{v} be the exact solution of* (3.6), *let Σ^{v} be the finite element solution of* (4.6) *with $\delta_j \geqslant 0$, and let ψ be the solution of* (5.21). *Further, let $\bar{\psi} \in \mathscr{W}_D$ denote the orthogonal projection of ψ onto the space of piecewise constant functions, determined by*

$$\bar{\psi}(t) = k_n^{-1} \int_{I_n} \psi(s)\,\mathrm{d}s, \quad t \in I_n.$$

Then, for all $t_N \geqslant 0$,

$$\|\Sigma^{\mathrm{v}} - \sigma^{\mathrm{v}}\|^2_{L_2(0,t_N)} \leqslant \left| \int_0^{t_N} (\hat{r}, \psi)\,\mathrm{d}t \right| + \sum_{n=1}^{N} E_{q,n} \|\psi\|_{L_2(I_n)} \leqslant \sum_{n=1}^{N} \|\hat{r}\|_{L_2(I_n)} \|\psi - \bar{\psi}\|_{L_2(I_n)} + \sum_{n=1}^{N} E_{q,n} \|\psi\|_{L_2(I_n)}, \tag{5.23}$$

where $\hat{r}$ is defined in (5.17) *and*

$$E_{q,n} = k_n^{1/2} \sum_{j=1}^{M_L} \epsilon_{nj} k_j \|\Sigma^{\mathrm{v}}_j\| + k_n^{1/2} \delta_n \|\Sigma^{\mathrm{v}}_n\|.$$

Proof. We begin as in the proof of Theorem 5.4,

$$|B(e, \psi)| \leqslant \left| \int_0^{t_N} (\hat{r}, \psi)\,\mathrm{d}t \right| + \left| \widehat{B}(\Sigma^{\mathrm{v}}, \psi) - B(\Sigma^{\mathrm{v}}, \psi) \right|. \tag{5.24}$$

Due to the orthogonality of (4.6) and since $\bar{\psi} \in \mathscr{W}_D$, we obtain

$$\left| \int_0^{t_N} (\hat{r}, \psi)\,\mathrm{d}t \right| = \left| \int_0^{t_N} (\hat{r}, \psi - \bar{\psi})\,\mathrm{d}t \right| \leqslant \sum_{n=1}^{N} \|\hat{r}\|_{L_2(I_n)} \|\psi - \bar{\psi}\|_{L_2(I_n)}.$$

The last part is obtained in a similar way as in Theorem 4.3,

$$|\widehat{B}(\Sigma^{\mathrm{v}}, \psi) - B(\Sigma^{\mathrm{v}}, \psi)| \leqslant \sum_{n=1}^{N} E_{q,n} \|\psi\|_{L_2(I_n)}.$$

Finally, by inserting the two estimates above in (5.24) while using (5.3) and (5.22), the proof follows. □

Note that this estimate is not straightforward to use because it requires a reasonably good approximation of the error to be used as the data of the adjoint problem. This could be achieved by solving the primal problem on a finer mesh. Moreover, the adjoint problem needs to be solved on the fine mesh and its solution inserted into (5.23). The potentially sharper estimate of Theorem 5.5 thus requires additional computational work. Therefore, we use the estimate of Theorem 5.4 as the basis for our adaptive algorithm.

6. Adaptive strategy

In this section we develop an adaptive strategy. The goal of the strategy is to produce an approximate solution to (2.4) and (2.7) with an error below a user defined tolerance, TOL. Here the L_2-norm is used to measure the error.

The adaptive strategy is based on the a posteriori error estimates and the purpose is to equidistribute the error contribution from each time step. We recall the error estimate in Theorem 5.4,

$$\|e\|_{L_2(0,t_N)} \leqslant E = E_\mathrm{G} + E_\mathrm{Q},$$

where we now have introduced E as the total estimated error. The first part of the error E is the estimated Galerkin error in (5.18)

$$E_\mathrm{G} = \left(\sum_{n=1}^{N} \int_{I_n} \|\hat{r}\|^2 \,\mathrm{d}t \right)^{1/2}, \tag{6.1}$$

while the second part of the error E is the estimated quadrature error in (5.19) (for convenience split into two parts)

$$E_\mathrm{Q} = E_\mathrm{Q1} + E_\mathrm{Q2} = \left(\sum_{n=1}^{N} k_n \left(\sum_{j=1}^{M_L} \epsilon_{nj} k_j \|\Sigma_j^\mathrm{v}\| \right)^2 \right)^{1/2} + \left(\sum_{n=1}^{N} k_n (\delta_n \|\Sigma_n^\mathrm{v}\|)^2 \right)^{1/2}.$$

We now need a stop criterion and a modification strategy for creating a new set of time steps. As a stop criterion, we choose $E \leqslant \mathrm{TOL}$. In order to obtain a strategy to adapt the mesh it is more convenient to use the squared L_2-norm. We obtain the following relations

$$\|e\|^2_{L_2(0,t_N)} \leqslant (E_\mathrm{G} + E_\mathrm{Q})^2 \leqslant 2(E_\mathrm{G}^2 + E_\mathrm{Q}^2) \leqslant 2E_\mathrm{G}^2 + 4(E_\mathrm{Q1}^2 + E_\mathrm{Q2}^2).$$

With the aim to equidistribute the error contributions from the different time steps, we introduce

$$\sum_{n=1}^{N} k_n R_n = \sum_{n=1}^{N} \frac{\mathrm{TOL}^2}{N},$$

where

$$R_n = \frac{1}{k_n} \int_{I_n} 2\|\hat{r}\|^2 \,\mathrm{d}t + 4\left(\sum_{j=1}^{M_L} \epsilon_{nj} k_j \|\Sigma_j^\mathrm{v}\| \right)^2 + 4(\delta_n \|\Sigma_n^\mathrm{v}\|)^2.$$

From this expression and the previous time step k_n, a new time step $\hat{k}_n$ can be calculated by

$$\hat{k}_n = \frac{k_n}{C_n} \approx \frac{\mathrm{TOL}^2}{N R_n}, \tag{6.2}$$

where C_n is the number of subintervals that the interval I_n should be divided into. Note here that we use N and R_n from the previous mesh since these quantities are not known a priori. In this way a new mesh is obtained, in which we choose sparse quadrature steps K_l. In order to retain the first order accuracy of the original Galerkin method, a natural choice of the sparse time steps is (see (5.15))

$$K_l = \sqrt{\bar{k}}, \quad \text{with } \bar{k} = T/N, \tag{6.3}$$

where N now is the updated number of elements. Below we outline the procedure:

(1) Start with a uniform mesh and choose K_l according to (6.3) while imposing the margin $t_n - t_{M_L} \in (1,2)$.
(2) Solve the primal problem for $\Sigma^{\mathrm{v}} \in \mathscr{W}_D$.
(3) Compute the total error estimate E and R_n for $n = 1, \ldots, N$.
(4) If $E \leqslant \mathrm{TOL}$ then stop, and if not modify the mesh where the error contribution is large, i.e., $k_n R_n \geqslant \mathrm{TOL}^2/N$, by splitting these elements according to (6.2). Create K_l as in 1 and return to 2.

7. Numerical examples

In this section we verify the numerical method by solving the non-dimensional viscoelastic problem (2.4) and (2.7) numerically for some simple but realistic loading cases. We also compute error estimates of the numerical solutions expressed in the L_2-norm. In all examples, fictitious materials with non-dimensional stiffnesses $E_1 = E_2 = 1/2$, and integration order $\alpha \in (0,1]$ are considered. The weights ω_{nj} in (3.3) and $\tilde{\omega}_{nj}$ in (4.2) are integrated analytically. The estimated Galerkin error in (6.1) needs to be integrated carefully to ensure that the error is negligible compare to the discretization error of the method. Numerical experiments indicate that the trapezoidal rule is sufficient as well as efficient for this purpose.

7.1. Uniform mesh

We begin by solving the viscoelastic equations on a uniformly distributed (non-dimensional) time mesh for $t \in (0,10)$ using the discontinuous Galerkin method in its original form, i.e., without sparse quadrature. We compute the stress response due to the following step strain

$$\varepsilon(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t < 0. \end{cases} \tag{7.1}$$

In this case we have an analytical solution, which can be used to compute the true error. The solution reads (see [19])

$$\sigma(t) = \frac{1}{2}E_\alpha(-t^\alpha) + \frac{1}{2},$$

where

$$E_\alpha(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(1+\alpha k)}$$

is the Mittag–Leffler function of order α. In Fig. 2 the numerical solutions with time steps $k_n = 0.067$ are compared to the analytical solutions for different values of α. We see that the largest error in the stress occurs in the region close to the discontinuity in the strain. To show that the numerical method is first order accurate we solve the same problem with different time steps for $\alpha = 0.67$. The discontinuity in the strain coincides with a node point which means that we expect a first order accuracy according to the a priori error estimate (5.10). From Table 1 we observe a first order error. Note that the estimated error E is in close agreement with the exact error $\|e\|_{L_2}$. Next consider the same problem but now with the step strain applied at $t = 0.0555$, which implies that the discontinuity occurs inside a time step for the meshes in consideration. According to (5.11) we now expect a square root order of the error which is seen in Table 2. In order to investigate the reliability of the error estimate in the case of a variable source term, we calculate the stress response to the harmonic strain

$$\varepsilon(t) = \begin{cases} \sin(\pi t), & t \geqslant 0, \\ 0, & t < 0 \end{cases} \tag{7.2}$$

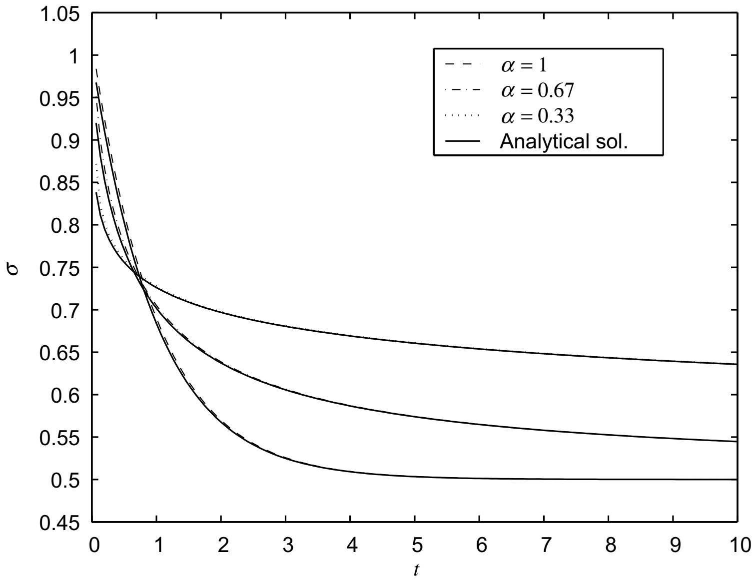

Fig. 2. The stress response due to a unit step strain imposed at time $t = 0$. Numerical solutions are compared with analytical solutions for different values of α.

Table 1
Exact errors $\|e\|_{L_2}$ and estimated errors E for different time steps with $\alpha = 0.67$

N	$\|e\|_{L_2}$	E
100	1.71×10^{-2}	1.82×10^{-2}
1000	2.04×10^{-3}	2.07×10^{-3}
10,000	2.19×10^{-4}	2.19×10^{-4}

Discontinuity in the strain coincides with a node.

Table 2
Estimated errors E for different time steps with $\alpha = 0.67$

N	E
100	8.34×10^{-2}
1000	2.71×10^{-2}
10,000	7.87×10^{-3}

Discontinuity in the strain lies inside an element.

for $\alpha = 0.67$ on $t \in (0, 10)$ with $k_n = 0.1$. In this case the mean values $\bar{\varepsilon}_n$ in (3.3) can not be exactly evaluated. Instead the right endpoint rule is used. Here we have no analytical solution. Instead a numerical solution obtained on a finer mesh $k_n = 0.001$ is used to estimate the true error. In the case of $k_n = 0.1$ the estimated error becomes $E = 2.11 \times 10^{-1}$ while the estimated true error becomes $\|e\|_{L_2} \approx 1.91 \times 10^{-1}$. This shows that the sharpness of the error estimate is well sufficient also for the harmonic source term.

7.1.1. Sparse quadrature

We now introduce sparse quadrature in the numerical algorithm and investigate its effect on the solution. The integral exponent considered is here $\alpha = 0.5$ while the time mesh is still uniform for $t \in (0, 10)$. Sparse quadrature steps are chosen uniformly as $K_l = \sqrt{T/N}$ while the margin is forced to be in the interval $t_n - t_{M_L} \in (1, 2)$, which is in accordance with the margin proposed in Section 4. Two different choices of δ_n are used. First we choose δ_n as in (5.14) which guarantees stability. Second, for simplicity, we choose $\delta_n = 0$.

First consider the harmonic strain in (7.2). The total error estimates and their different parts are given in Table 3. Both solutions converge with a first order rate of convergence as predicted in (5.10) and (5.15). We observe that the error contribution mostly arises from the Galerkin error. This means that we may consider taking longer sparse quadrature steps and/or use a shorter margin. Fig. 3 shows the stress response to the harmonic strain when $N = 1000$ and $\delta_n = 0$. For comparison the stationary solution (obtained using Fourier transformation technique)

Table 3
The total estimated errors and their different parts for different time steps in the case of the harmonic strain

N	L	E	E_G	E_{Q1}	E_{Q2}
100	28	2.02×10^{-1}	1.96×10^{-1}	2.21×10^{-3}	3.49×10^{-3}
1000	89	2.14×10^{-2}	2.09×10^{-2}	2.07×10^{-4}	2.69×10^{-4}
10,000	284	2.17×10^{-3}	2.12×10^{-3}	2.02×10^{-5}	2.39×10^{-5}
100	28	1.98×10^{-1}	1.96×10^{-1}	2.21×10^{-3}	0
1000	89	2.11×10^{-2}	2.09×10^{-2}	2.07×10^{-4}	0
10,000	284	2.14×10^{-3}	2.12×10^{-3}	2.02×10^{-5}	0

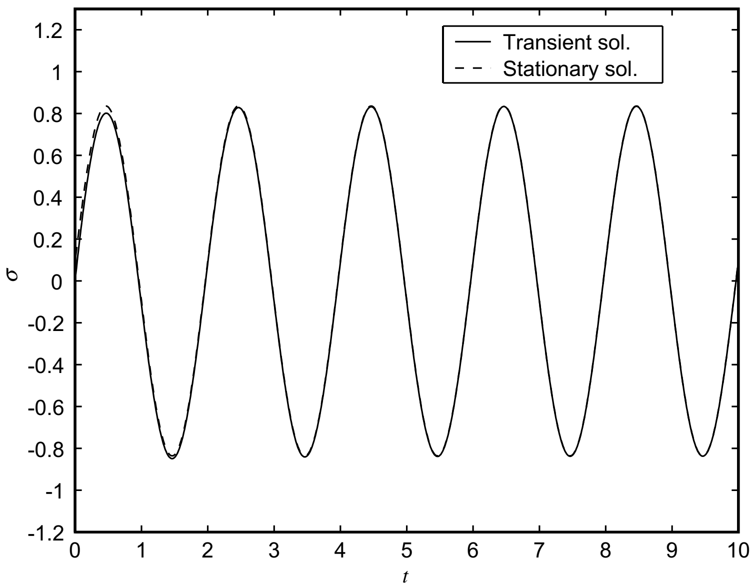

Fig. 3. The stress response due to the harmonic strain in (7.2) using sparse quadrature. For comparison the stationary solution is also shown.

$$\sigma(t) = \frac{1}{2}\left(\frac{1 + 2\pi^{2\alpha} + 3\pi^{\alpha}\cos(\alpha\pi/2)}{1 + \pi^{2\alpha} + 2\pi^{\alpha}\cos(\alpha\pi/2)}\right)\sin(\pi t) + \frac{1}{2}\left(\frac{\pi^{\alpha}\sin(\alpha\pi/2)}{1 + \pi^{2\alpha} + 2\pi^{\alpha}\cos(\alpha\pi/2)}\right)\cos(\pi t)$$

is also displayed. Note that the transient decays fast due to the high damping inherent in the viscoelastic model with the present choice of parameters.

Second consider the step strain in (7.1). In this case the history effects become stronger than in the previous harmonic case. The total error estimates and their different parts are given in Table 4. Again, we observe that the convergence rate is of first order. Fig. 4 shows the stress response to the step strain when $N = 1000$ and $\delta_n = 0$.

Tables 3 and 4 also show the maximum number of large time steps L in the calculations. We observe that the major part of the time interval is covered by the large time steps. This means that for sufficiently large times the number of data that needs to be stored is reduced from $O(n)$ to $O(L)$. It is also worth mentioning that in a structural analysis calculation the benefit of using sparse quadrature will increase significantly because the stress response is to be computed at each Gauss point of the structure.

Table 4
The total estimated errors and their different parts for different time steps in the case of the step strain

N	L	E	E_G	E_{Q1}	E_{Q2}
100	28	2.17×10^{-2}	1.83×10^{-2}	1.64×10^{-3}	1.75×10^{-3}
1000	89	2.82×10^{-3}	2.56×10^{-3}	1.40×10^{-4}	1.28×10^{-4}
10,000	284	3.48×10^{-4}	3.23×10^{-4}	1.36×10^{-5}	1.14×10^{-5}
100	28	2.00×10^{-2}	1.83×10^{-2}	1.64×10^{-3}	0
1000	89	2.69×10^{-3}	2.56×10^{-3}	1.40×10^{-4}	0
10,000	284	3.36×10^{-4}	3.23×10^{-4}	1.36×10^{-5}	0

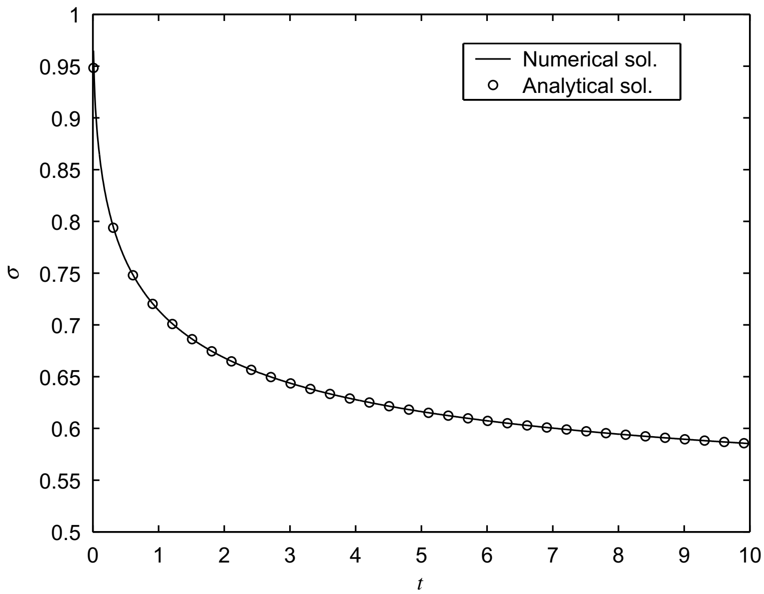

Fig. 4. The stress response due to the step strain in (7.1) using sparse quadrature.

7.2. Adaptivity

We now investigate the capability of the numerical method to handle variable time steps and to produce a solution with an error below a user defined tolerance, TOL. For this purpose we use the adaptive strategy developed in Section 6. Since no indication of lost stability has been observed with $\delta_n = 0$, we keep this choice. We compute the stress due to a unit step strain followed by a step unloading

$$\varepsilon(t) = \begin{cases} 1, & t \in (0, 2.5), \\ 0, & \text{otherwise.} \end{cases} \tag{7.3}$$

The analytical solution is given by

$$\sigma(t) = \begin{cases} \frac{1}{2}E_\alpha(-t^\alpha) + \frac{1}{2}, & t \in (0, 2.5), \\ \frac{1}{2}(E_\alpha(-t^\alpha) - E_\alpha(-(t-2.5)^\alpha)) + \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Again, we consider $t \in (0, 10)$ and $\alpha = 0.5$. We set $\mathrm{TOL} = 1 \times 10^{-2}$ and introduce a minimum step length $k_n = 0.005$ to prevent unnecessary short steps which may be suggested since information from the previous calculation is used. With an initial mesh of 100 uniform steps, two iterations (i.e., one refinement) are needed to meet the tolerance. The strategy suggests 334 time steps. Table 5 shows the result of the two iterations. Figs. 5 and 6 show the solutions after each iteration. Finally, Fig. 7 shows the time mesh sug-

Table 5
Estimated errors when using the adaptive strategy

N	L	E	E_G	E_{Q1}	E_{Q2}
100	28	2.65×10^{-2}	2.52×10^{-2}	1.36×10^{-3}	0
334	50	3.64×10^{-3}	3.28×10^{-3}	3.66×10^{-4}	0

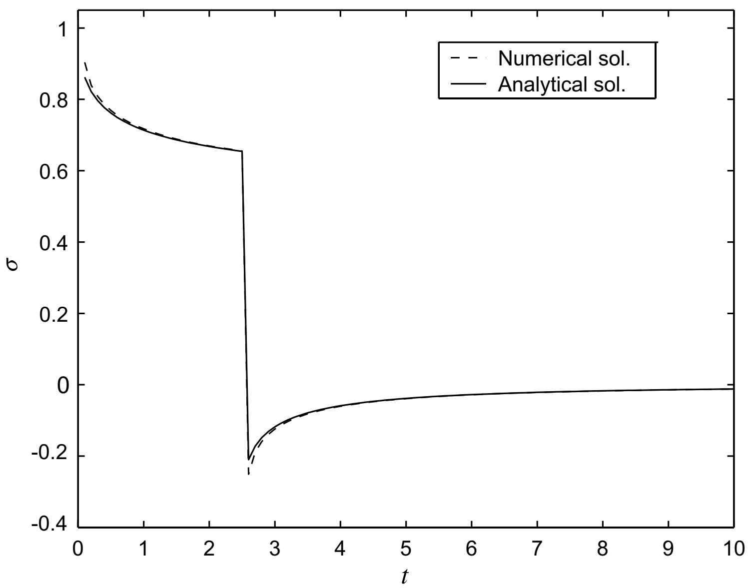

Fig. 5. The stress response to the strain in (7.3) after the first iteration.

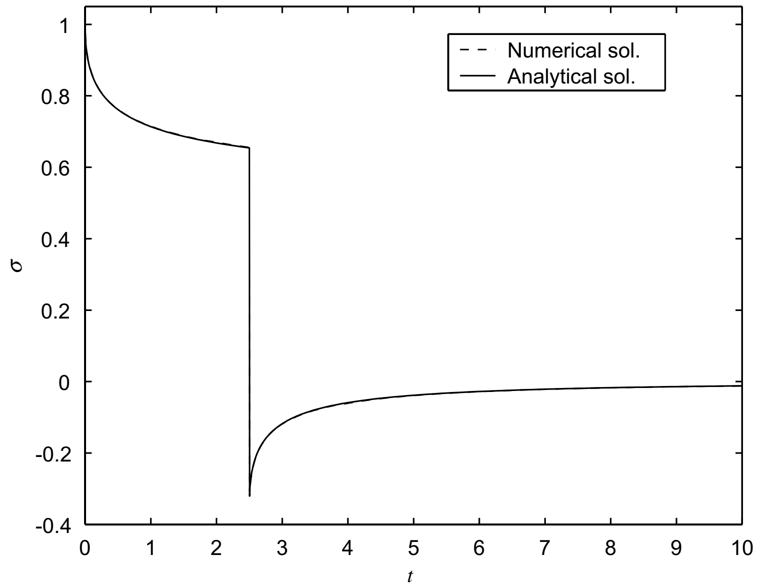

Fig. 6. The stress response to the strain in (7.3) after the second (final) iteration.

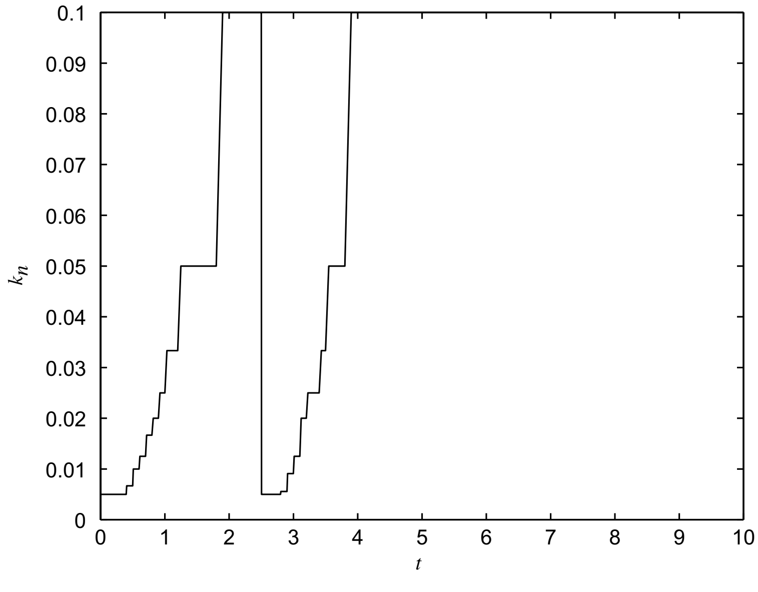

Fig. 7. The time step distribution suggested by the adaptive strategy.

gested by the adaptive strategy aiming to equidistribute the error contribution from the different steps. For comparison, to obtain the same error $E = 3.64 \times 10^{-3}$ with a uniform mesh, 1011 time steps need to be used. Having in mind that the equation contains a convolution integral the use of adaptivity results in a considerable gain in computational cost.

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