A POSTERIORI ERROR ANALYSIS FOR THE CAHN-HILLIARD EQUATION

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ABSTRACT. The Cahn-Hilliard equation is discretized by a Galerkin finite element method based on continuous piecewise linear functions in space and discontinuous piecewise constant functions in time. A posteriori error estimates are proved by using the methodology of dual weighted residuals.

1. Introduction

We consider the Cahn-Hilliard equation

$$\begin{aligned} u_t - \Delta w &= 0 \quad \text{in } \Omega \times [0,T], \\ w + \epsilon \Delta u - f(u) &= 0 \quad \text{in } \Omega \times [0,T], \\ \frac{\partial u}{\partial \nu} = 0, \ \frac{\partial w}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times [0,T], \\ u(\cdot,0) &= g_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a polygonal domain in $\mathbf{R}^d, d = 1,2,3,\ u = u(x,t), w = w(x,t)$, $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2},\ u_t = \frac{\partial u}{\partial t}$, ν is the exterior unit normal to $\partial\Omega$, and $\epsilon > 0$ is a small parameter. The Cahn-Hilliard equation is a model for phase separation and spinodal decomposition [3]. The nonlinearity f is the derivative of a double-well potential. A typical example is $f(u) = u^3 - u$.

We discretize (1.1) by a Galerkin finite element method, which is based on continuous piecewise linear functions with respect to x and discontinuous piecewise constant functions with respect to t. This numerical method is the same as the implicit Euler time stepping combined with spatial discretization by a standard finite element method.

We perform an a posteriori error analysis within the framework of dual weighted residuals [2]. If $J(u)$ is a given goal functional, this results in an

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error estimate essentially of the form

$$|J(u) - J(U)| \leq \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{\rho_{u,K}\omega_{u,K} + \rho_{w,K}\omega_{w,K}\Big\} + \mathcal{R},$$

where U denotes the numerical solution and $\mathbf{T}_n$ is the spatial mesh at time level t_n. The terms $\rho_{u,K}, \rho_{w,K}$ are local residuals from the first and second equations in (1.1), respectively. The weights $\omega_{u,K}, \omega_{w,K}$ are derived from the solution of the linearized adjoint problem. The remainder $\mathcal{R}$ is quadratic in the error.

There is an extensive literature on numerical methods for the Cahn-Hilliard equation; see, for example, [5] and [4] for a priori error estimates. Adaptive methods based on a posteriori estimates are presented in [1] and [6]. However, these estimates are restricted to spatial discretization. We are not aware of any completely discerete a posteriori error analysis.

2. Preliminaries

Here we present the methodology of dual weighted residuals [2] in an abstract form.

Let $A(\cdot;\cdot)$ be a semilinear form; that is, it is nonlinear in the first and linear in the second variable, and $J(\cdot)$ be an output functional, not necessarily linear, defined on some function space V. Consider the variational problem: Find $u \in V$ such that

$$A(u;\psi) = 0 \quad \forall\psi \in V, \tag{2.1}$$

and the corresponding finite element problem: Find $u_h \in V_h \subset V$ such that

$$A(u_h;\psi_h) = 0 \quad \forall\psi_h \in V_h. \tag{2.2}$$

We suppose that the derivatives of A and J with respect to the first variable u up to order three exist and are denoted by

$$A'(u;\varphi),\ A''(u;\psi,\varphi),\ A'''(u;\xi,\psi,\varphi),$$

and

$$J'(u;\varphi),\ J''(u;\psi,\varphi),\ J'''(u;\xi,\psi,\varphi),$$

respectively, for increments φ, ψ, $\xi \in V$. Here we use the convention that the forms are linear in the variables after the semicolon.

We want to estimate $J(u) - J(u_h)$. Introduce the dual variable $z \in V$ and define the Lagrange functional

$$\mathcal{L}(u;z) := J(u) - A(u;z)$$

and seek the stationary points $(u,z) \in V \times V$ of $\mathcal{L}(\cdot;\cdot)$; that is,

$$\mathcal{L}'(u;z,\varphi,\psi) = J'(u;\varphi) - A'(u;z,\varphi) - A(u;\psi) = 0 \quad \forall\varphi,\psi \in V. \tag{2.3}$$

By choosing $\varphi = 0$, we retrieve (2.1). By taking $\psi = 0$, we identify the linearized adjoint equation to find $z \in V$ such that

$$J'(u;\varphi) - A'(u;z,\varphi) = 0 \quad \forall\varphi \in V. \tag{2.4}$$

The corresponding finite element problem is: Find $(u_h, z_h) \in V_h \times V_h$ such that

$$\begin{aligned} \mathcal{L}'(u_h; z_h, \varphi_h, \psi_h) &= J'(u_h; \varphi_h) - A'(u_h; z_h, \varphi_h) - A(u_h; \psi_h) \\ &= 0 \quad \forall \varphi_h, \psi_h \in V_h. \end{aligned} \tag{2.5}$$

By choosing $\varphi_h = 0$, we retrieve (2.2). By taking $\psi_h = 0$, we identify the linearized adjoint equation to find $z_h \in V_h$ such that

$$J'(u_h; \varphi_h) - A'(u_h; z_h, \varphi_h) = 0 \quad \forall \varphi_h \in V_h. \tag{2.6}$$

We quote three propositions from [2, Ch. 6].

Proposition 2.1. *Let $L(\cdot)$ be a three times differentiable functional defined on a vector space X, which has a stationary point $x \in X$, that is,*

$$L'(x; y) = 0 \quad \forall y \in X.$$

Suppose that on a finite dimensional subspace $X_h \subset X$ the corresponding Galerkin approximation,

$$L'(x_h; y_h) = 0 \quad \forall y_h \in X_h,$$

has a solution, $x_h \in X_h$. Then there holds the error representation

$$L(x) - L(x_h) = \tfrac{1}{2} L'(x_h; x - y_h) + \mathcal{R} \quad \forall y_h \in X_h,$$

with a remainder term $\mathcal{R}$, which is cubic in the error $e := x - x_h$,

$$\mathcal{R} := \tfrac{1}{2} \int_0^1 L'''(x_h + se; e, e, e) s(s-1) \, \mathrm{d}s.$$

Since

$$\mathcal{L}(u; z) - \mathcal{L}(u_h; z_h) = J(u) - J(u_h),$$

at stationary points $(u, z), (u_h, z_h)$, Proposition 2.1 yields the following result for the Galerkin approximation (2.2) of the variational equation (2.1).

Proposition 2.2. *For any solutions u and u_h of equations* (2.1) *and* (2.2) *we have the error representation*

$$J(u) - J(u_h) = \tfrac{1}{2} \rho(u_h; z - \varphi_h) + \tfrac{1}{2} \rho^*(u_h; z_h, u - \psi_h) + \mathcal{R}^{(3)} \quad \forall \varphi_h, \psi_h \in V_h,$$

where z and z_h are solutions of the adjoint problems (2.4) *and* (2.6) *and*

$$\begin{aligned} \rho(u_h; \cdot) &= -A(u_h; \cdot), \\ \rho^*(u_h; z_h, \cdot) &= J'(u_h; \cdot) - A'(u_h; z_h, \cdot), \end{aligned}$$

and, with $e_u = u - u_h$, $e_z = z - z_h$, the remainder is

$$\begin{aligned} \mathcal{R}^{(3)} = \tfrac{1}{2} \int_0^1 \Big(& J'''(u_h + se_u; e_u, e_u, e_u) - A'''(u_h + se_u; z_h + se_z, e_u, e_u, e_u) \\ & - 3A''(u_h + se_u; e_u, e_u, e_z) \Big) s(s-1) \, \mathrm{d}s. \end{aligned}$$

The forms $\rho(\cdot; \cdot), \rho^*(\cdot; \cdot, \cdot)$ are the residuals of (2.1) and (2.4), respectively. The remainder $\mathcal{R}^{(3)}$ is cubic in the error. The following proposition shows that the residuals are equal up to a quadratic remainder.

Proposition 2.3. *With the notation from above, we have*

$$\rho^*(u_h; z_h, u-\psi_h) = \rho(u_h; z-\varphi_h) + \delta\rho \quad \forall \varphi_h, \psi_h \in V_h,$$

with

$$\delta\rho = \int_0^1 \Big(A''(u_h + s e_u; z_h + s e_z, e_u, e_u) - J''(u_h + s e_u; e_u, e_u) \Big) \, \mathrm{d}s.$$

Moreover, we have the simplified error representation

$$J(u) - J(u_h) = \rho(u_h; z - \varphi_h) + \mathcal{R}^{(2)} \quad \forall \varphi_h \in V_h,$$

with quadratic remainder

$$\mathcal{R}^{(2)} = \int_0^1 \Big(A''(u_h + s e_u; z, e_u, e_u) - J''(u_h + s e_u; e_u, e_u) \Big) \, \mathrm{d}s.$$

3. Galerkin discretization and dual problem

In this section, we apply the dual weighted residuals methodology to the Cahn-Hilliard equation (1.1). We denote $I = [0,T]$, $Q = \Omega \times I$, and

$$\langle v, w \rangle_{\mathcal{D}} = \int_{\mathcal{D}} v w \, \mathrm{d}z, \quad \|v\|_{\mathcal{D}}^2 = \int_{\mathcal{D}} v^2 \, \mathrm{d}z$$

for subsets $\mathcal{D}$ of Q or Ω with the relevant Lebesgue measure $\mathrm{d}z$. Let $V = H^1(\Omega)$ and $\mathcal{W} = C^1([0,T], V)$. By multiplying the first equation by $\psi_u \in V$ and the second equation by $\psi_w \in V$, integrating over Ω and using Green's formula, we obtain the weak formulation: Find $u, w \in \mathcal{W}$ such that $u(0) = g_0$ and

$$\begin{aligned} \langle u_t, \psi_u \rangle_\Omega + \langle \nabla w, \nabla \psi_u \rangle_\Omega &= 0 \quad \forall \psi_u \in V,\, t \in [0,T], \\ \langle w, \psi_w \rangle_\Omega - \epsilon \langle \nabla u, \nabla \psi_w \rangle_\Omega - \langle f(u), \psi_w \rangle_\Omega &= 0 \quad \forall \psi_w \in V,\, t \in [0,T]. \end{aligned} \tag{3.1}$$

Split the interval $I = [0,T]$ into subintervals $I_n = [t_{n-1}, t_n)$ of lengths $k_n = t_n - t_{n-1}$,

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T.$$

For each time level $t_n, n \geq 1$, let $\mathcal{V}_n$ be the space of continuous piecewise linear functions with respect to regular spatial meshes $\mathbf{T}_n = \{K\}$, which may vary from time level to time level. By extending the spatial meshes $\mathbf{T}_n$ as constant in time to the time slab $\Omega \times I_n$, we obtain meshes $\mathcal{T}_k$ of the space-time domain $Q = \Omega \times I$, which consist of $(d+1)$-dimensional prisms $Q_K^n := K \times \bar{I}_n$. Define the finite element space

$$\mathcal{V} := \Big\{ \varphi \colon \bar{Q} \to \mathbf{R} : \varphi(\cdot, t)|_{\bar{\Omega}} \in \mathcal{V}_n,\, t \in I_n,\ \varphi(x, \cdot)|_{I_n} \in \Pi_0,\, x \in \bar{\Omega} \Big\}.$$

Here Π_0 denotes the polynomials of degree 0. For functions from this space and their continuous analogues, we define

$$v_n^+ = \lim_{t \downarrow t_n} v(t), \quad v_n = v_n^- = \lim_{t \uparrow t_n} v(t), \quad [v]_n = v_n^+ - v_n^-.$$

For all $u, w, \psi_u, \psi_w \in \mathcal{V}$ or $\mathcal{W}$, consider the semilinear form

$$\begin{aligned} A(u,w;\psi_u,\psi_w) = \sum_{n=1}^{N} \int_{I_n} \Big\{ &\langle u_t, \psi_u\rangle_\Omega + \langle \nabla w, \nabla \psi_u\rangle_\Omega + \langle w, \psi_w\rangle_\Omega \\ &- \epsilon\langle \nabla u, \nabla \psi_w\rangle_\Omega - \langle f(u), \psi_w\rangle_\Omega \Big\}\,\mathrm{d}t \\ &+ \sum_{n=2}^{N} \langle [u]_{n-1}, \psi_{u,n-1}^+\rangle_\Omega + \langle u_0^+ - g_0, \psi_{u,0}^+\rangle_\Omega. \end{aligned}$$

Solutions $u, w \in \mathcal{W}$ of (1.1) satisfy the variational problem

$$A(u,w;\psi_u,\psi_w) = 0 \quad \forall \psi_u, \psi_w \in \mathcal{W} \tag{3.2}$$

and the finite element problem can formulated: Find $U, W \in \mathcal{V}$ such that

$$A(U,W;\psi_u,\psi_w) = 0 \quad \forall \psi_u, \psi_w \in \mathcal{V}. \tag{3.3}$$

We now show that this is a standard time-stepping method. Since $U(t) = U_n = U_n^- = U_{n-1}^+$, $W(t) = W_n$ for $t \in I_n$, we have

$$\begin{aligned} A(U,W;\psi_u,\psi_w) = \sum_{n=1}^{N} \int_{I_n} \Big\{ &\langle \nabla W_n, \nabla \psi_u\rangle_\Omega + \langle W_n, \psi_w\rangle_\Omega \\ &- \epsilon\langle \nabla U_n, \nabla \psi_w\rangle_\Omega - \langle f(U_n), \psi_w\rangle_\Omega \Big\}\,\mathrm{d}t \\ &+ \sum_{n=2}^{N} \langle U_n - U_{n-1}, \psi_{u,n-1}^+\rangle_\Omega + \langle U_1 - g_0, \psi_{u,0}^+\rangle_\Omega. \end{aligned} \tag{3.4}$$

By taking

$$\psi_u(t) = \begin{cases} \chi_u \in \mathcal{V}_n, & t \in I_n, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi_w(t) = \begin{cases} \chi_w \in \mathcal{V}_n, & t \in I_n, \\ 0, & \text{otherwise}, \end{cases}$$

we see that (3.3) amounts to the implicit Euler time-stepping,

$$\begin{aligned} \langle U_0 - g_0, \chi_u\rangle_\Omega &= 0 \quad \forall \chi_u \in \mathcal{V}_1, \\ k_n\langle \nabla W_n, \nabla \chi_u\rangle_\Omega + \langle U_n - U_{n-1}, \chi_u\rangle_\Omega &= 0 \quad \forall \chi_u \in \mathcal{V}_n, n \ge 1, \\ \langle W_n, \chi_w\rangle_\Omega - \epsilon\langle \nabla U_n, \nabla \chi_w\rangle_\Omega - \langle f(U_n), \chi_w\rangle_\Omega &= 0 \quad \forall \chi_w \in \mathcal{V}_n, n \ge 1. \end{aligned}$$

Now take a goal functional $J(u)$, which depends only on u, and set

$$\mathcal{L}(v;z) = J(u) - A(v;z),$$

where $v = (u, w), z = (z_u, z_w)$. With $\varphi = (\varphi_u, \varphi_w), \psi = (\psi_u, \psi_w)$, stationary points are given by

$$\mathcal{L}'(v;z,\varphi,\psi) = J'(u;\varphi_u) - A'(v;z,\varphi) - A(v;\psi) = 0 \quad \forall \varphi, \psi \in \mathcal{W} \times \mathcal{W}.$$

With $\psi = 0$ we obtain $A'(v; z, \varphi) = J'(u; \varphi_u)$, the adjoint problem. So we should compute $A'(u, w; z_u, z_w, \varphi_u, \varphi_w)$ and $J'(u; \varphi_u)$. To this end we write

$$\begin{aligned} A(u,w;\psi_u,\psi_w) = \langle u_t, \psi_u\rangle_Q + \langle \nabla w, \nabla \psi_u\rangle_Q + \langle w, \psi_w\rangle_Q - \epsilon\langle \nabla u, \nabla \psi_w\rangle_Q \\ - \langle f(u), \psi_w\rangle_Q + \langle u(0) - g_0, \psi_u(0)\rangle_\Omega. \end{aligned}$$

Hence,

$$\begin{aligned}A'(u,w;z_u,z_w,\varphi_u,\varphi_w) &= \langle \varphi_{u,t}, z_u\rangle_Q + \langle \nabla\varphi_w, \nabla z_u\rangle_Q + \langle \varphi_w, z_w\rangle_Q \\ &\quad - \epsilon\langle \nabla\varphi_u, \nabla z_w\rangle_Q - \langle \varphi_u, z_w\rangle_Q + \langle \varphi_u(0), z_u(0)\rangle_\Omega.\end{aligned}$$

By integration by parts in t,

$$\langle \varphi_{u,t}, z_u\rangle_Q = -\langle \varphi_u, z_{u,t}\rangle_Q + \langle \varphi_u(T), z_u(T)\rangle_\Omega - \langle \varphi_u(0), z_u(0)\rangle_\Omega,$$

we obtain

$$\begin{aligned}A'(u,w;z_u,z_w,\varphi_u,\varphi_w) = &-\langle \varphi_u, z_{u,t}\rangle_Q + \langle \nabla\varphi_w, \nabla z_u\rangle_Q \\ &+ \langle \varphi_w, z_w\rangle_Q + \epsilon\langle \nabla\varphi_u, \nabla z_w\rangle_Q \\ &- \langle \varphi_u, f'(u)z_w\rangle_Q + \langle \varphi_u(T), z_u(T)\rangle_\Omega.\end{aligned}$$

The adjoint problem is thus to find $z_u, z_w \in \mathcal{W}$ such that

$$\begin{aligned}&\langle \varphi_u, -z_{u,t}\rangle_Q - \epsilon\langle \nabla\varphi_u, \nabla z_w\rangle_Q \\ &\quad - \langle \varphi_u, f'(u)z_w\rangle_Q + \langle \varphi_u(T), z_u(T)\rangle_\Omega \\ &\quad + \langle \nabla\varphi_w, \nabla z_w\rangle_Q + \langle \varphi_w, z_w\rangle_Q = J'(u;\varphi_u) \quad \forall \varphi_u, \varphi_w \in \mathcal{W}.\end{aligned} \tag{3.5}$$

We now specialize to the case of a linear goal functional of the form

$$J(u) = \langle u, g\rangle_Q + \langle u(T), g_T\rangle_\Omega,$$

for some $g \in L_2(Q),\ g_T \in L_2(\Omega)$. Then

$$J'(u;\varphi_u) = \langle \varphi_u, g\rangle_Q + \langle \varphi_u(T), g_T\rangle_\Omega. \tag{3.6}$$

The adjoint problem then becomes: Find $z_u, z_w \in \mathcal{W}$ such that

$$\begin{aligned}&\langle \varphi_u, -z_{u,t} - f'(u)z_w - g\rangle_Q - \epsilon\langle \nabla\varphi_u, \nabla z_w\rangle_Q \\ &\qquad + \langle \varphi_u(T), z_u(T) - g_T\rangle_\Omega = 0 \quad \forall \varphi_u \in \mathcal{W}, \\ &\langle \varphi_w, z_w\rangle_Q + \langle \nabla\varphi_w, \nabla z_u\rangle_Q = 0 \quad \forall \varphi_w \in \mathcal{W}.\end{aligned} \tag{3.7}$$

The strong form of this is

$$\begin{aligned}-\partial_t z_u + \epsilon\Delta z_w - f'(u)z_w &= g \quad \text{in } Q, \\ z_w - \Delta z_u &= 0 \quad \text{in } Q, \\ \frac{\partial z_u}{\partial \nu} = 0,\ \frac{\partial z_w}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times I, \\ z_u(T) &= g_T \quad \text{in } \Omega.\end{aligned} \tag{3.8}$$

4. A posteriori error estimates

From Proposition 2.3 we have the error representation

$$J(u) - J(U) = -A(U,W; z_u - \pi z_u, z_w - \pi z_w) + \mathcal{R}^{(2)}, \tag{4.1}$$

where $z = (z_u, z_w)$ is the solution of the adjoint problem (3.5) and $\pi z_u, \pi z_w \in \mathcal{V}$ are appropriate approximations to be defined below. The remainder is quadratic in the error.

In order to write this as a sum of local contributions we must rewrite $A(U,W;\psi_u,\psi_w)$ in (3.4). First we compute $\int_{I_n}\langle\nabla W,\nabla\psi_u\rangle_\Omega\,\mathrm{d}t$. By using Green's formula elementwise, we have

$$\begin{aligned}\int_{I_n}\langle\nabla W,\nabla\psi_u\rangle_\Omega\,\mathrm{d}t &= \int_{I_n}\sum_{K\in\mathbf{T}_n}\langle\nabla W,\nabla\psi_u\rangle_K\,\mathrm{d}t\\ &= \int_{I_n}\sum_{K\in\mathbf{T}_n}-\langle\Delta W,\psi_u\rangle_K\,\mathrm{d}t+\int_{I_n}\sum_{K\in\mathbf{T}_n}\langle\partial_\nu W,\psi_u\rangle_{\partial K}\,\mathrm{d}t,\end{aligned}$$

where $\partial_\nu W=\nu\cdot\nabla W$. We divide the boundary $\partial K\in\mathbf{T}_n$ into two parts: internal edges, denoted by $\mathcal{E}_I^n$, and edges on the boundary $\partial\Omega$, denoted by $\mathcal{E}_{\partial\Omega}^n$. So we get, with [ ] denoting the jump across the edge,

$$\begin{aligned}&\int_{I_n}\sum_{K\in\mathbf{T}_n}\langle\partial_\nu W,\psi_u\rangle_{\partial K}\,\mathrm{d}t\\ &\quad=\int_{I_n}\sum_{E\in\mathcal{E}_I^n}\langle\partial_\nu W,\psi_u\rangle_E\,\mathrm{d}t+\int_{I_n}\sum_{E\in\mathcal{E}_{\partial\Omega}^n}\langle\partial_\nu W,\psi_u\rangle_E\,\mathrm{d}t\\ &\quad=\int_{I_n}\sum_{K\in\mathbf{T}_n}-\tfrac{1}{2}\langle[\partial_\nu W],\psi_u\rangle_{\partial K\setminus\partial\Omega}\,\mathrm{d}t+\int_{I_n}\sum_{K\in\mathbf{T}_n}\langle\partial_\nu W,\psi_u\rangle_{\partial K\cap\partial\Omega}\,\mathrm{d}t.\end{aligned}$$

Let ∂_x denote the spatial boundary and define $\partial_x Q=\partial\Omega\times I$ and $\partial_x Q_K^n=\partial K\times I_n$. Hence,

$$\begin{aligned}\int_{I_n}\langle\nabla W,\nabla\psi_u\rangle_\Omega\,\mathrm{d}t=\sum_{K\in\mathbf{T}_n}\Big\{&-\langle\Delta W,\psi_u\rangle_{Q_K^n}-\tfrac{1}{2}\langle[\partial_\nu W],\psi_u\rangle_{\partial_x Q_K^n\setminus\partial_x Q}\\ &+\langle\partial_\nu W,\psi_u\rangle_{\partial_x Q_K^n\cap\partial_x Q}\Big\},\end{aligned}$$

and in the same way

$$\begin{aligned}\epsilon\int_{I_n}\langle\nabla U,\nabla\psi_w\rangle_\Omega\,\mathrm{d}t=\sum_{K\in\mathbf{T}_n}\Big\{&-\epsilon\langle\Delta U,\psi_w\rangle_{Q_K^n}-\tfrac{1}{2}\epsilon\langle[\partial_\nu U],\psi_w\rangle_{\partial_x Q_K^n\setminus\partial_x Q}\\ &+\epsilon\langle\partial_\nu U,\psi_w\rangle_{\partial_x Q_K^n\cap\partial_x Q}\Big\}.\end{aligned}$$

Note that $\Delta W=\Delta U=0$ on Q_K^n for piecewise linear functions, but we find it instructive to keep these terms. Inserting this into (3.4) and noting that

$$\int_{I_n}\langle W,\psi_w\rangle_\Omega\,\mathrm{d}t=\sum_{K\in\mathbf{T}_n}\langle W,\psi_w\rangle_{Q_K^n},$$

and

$$\int_{I_n}\langle f(U),\psi_w\rangle_\Omega\,\mathrm{d}t=\sum_{K\in\mathbf{T}_n}\langle f(U),\psi_w\rangle_{Q_K^n},$$

gives

$$\begin{aligned} A(U,W;\psi_u,\psi_w) = \sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n} \Big\{ & -\langle \Delta W, \psi_u\rangle_{Q_K^n} \\ & + \langle \epsilon\Delta U + W - f(U), \psi_w\rangle_{Q_K^n} - \tfrac12\langle[\partial_\nu W],\psi_u\rangle_{\partial_x Q_K^n\setminus\partial_x Q} \\ & + \tfrac12\epsilon\langle[\partial_\nu U],\psi_w\rangle_{\partial_x Q_K^n\setminus\partial_x Q} + \langle\partial_\nu W,\psi_u\rangle_{\partial_x Q_K^n\cap\partial_x Q} \\ & - \epsilon\langle\partial_\nu U,\psi_w\rangle_{\partial_x Q_K^n\cap\partial_x Q} + \langle[U]_{n-1},\psi_{u,n-1}^{+}\rangle_K \Big\}, \end{aligned}$$

where we have set $U_0^- = g_0$ for simplicity. Hence (4.1) becomes

$$\begin{aligned} J(u)-J(U) = & \sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n} \Big\{ \langle R_u, z_u-\pi z_u\rangle_{Q_K^n} + \langle R_w, z_w-\pi z_w\rangle_{Q_K^n} \\ & + \langle r_u, z_u-\pi z_u\rangle_{\partial_x Q_K^n} + \langle r_w, z_w-\pi z_w\rangle_{\partial_x Q_K^n} \\ & - \langle [U]_{n-1}, (z_u-\pi z_u)_{n-1}^{+}\rangle_K \Big\} + \mathcal{R}^{(2)}, \end{aligned} \tag{4.2}$$

with the interior residuals

$$R_u = \Delta W, \quad R_w = -\epsilon\Delta U - W + f(U),$$

the edge residuals

$$r_w|_\Gamma = \begin{cases} -\frac12\epsilon[\partial_\nu U], & \Gamma\subset\partial_x Q_K^n\setminus\partial_x Q, \\ 0, & \text{otherwise}, \end{cases}$$

$$r_u|_\Gamma = \begin{cases} \frac12[\partial_\nu W], & \Gamma\subset\partial_x Q_K^n\setminus\partial_x Q, \\ 0, & \text{otherwise}, \end{cases}$$

and the boundary residuals

$$r_w|_\Gamma = \begin{cases} \epsilon\partial_\nu U, & \Gamma\subset\partial_x Q_K^n\cap\partial_x Q, \\ 0, & \text{otherwise}, \end{cases}$$

$$r_u|_\Gamma = \begin{cases} -\partial_\nu W, & \Gamma\subset\partial_x Q_K^n\cap\partial_x Q, \\ 0, & \text{otherwise}. \end{cases}$$

Here the subscript u refers to residuals from the first equation in (3.1) and the subscript w to residuals from the second equation.

We now define $\pi z_u, \pi z_w \in \mathcal{V}$. Let

$$(P_n v)(t) = \frac{1}{k_n}\int_{I_n} v(s)\,\mathrm{d}s$$

be the orthogonal projector onto constants. Let $\pi_n\colon C(\bar\Omega)\to\mathcal{V}_n$ be the nodal interpolator; that is, it is defined by

$$(\pi_n v)(a) = v(a),$$

for all nodal points a in $\mathbf{T}_n$. Then we define $\pi\colon C(\bar{Q}) \to \mathcal{V}$ by $\pi v|_{I_n} = P_n \pi_n v$. Since R_u, R_w, r_u, and r_w are piecewise constant in t, we have

$$
\begin{aligned}
&J(u) - J(U)\\
&= \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ \langle R_u, P_n(z_u - \pi_n z_u)\rangle_{Q_K^n} + \langle R_w, P_n(z_w - \pi_n z_w)\rangle_{Q_K^n} \\
&\quad + \langle r_u, P_n(z_u - \pi_n z_u)\rangle_{\partial_x Q_K^n} + \langle r_w, P_n(z_w - \pi_n z_w)\rangle_{\partial_x Q_K^n} \\
&\quad - \langle [U]_{n-1}, (z_u - \pi z_u)^+_{n-1}\rangle_K \Big\} + \mathcal{R}^{(2)}.
\end{aligned}
\tag{4.3}
$$

Applying the Cauchy-Schwartz inequality to each term gives

$$
\begin{aligned}
|J(u) - J(U)| \le \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} & \Big\{ \|R_u\|_{Q_K^n} \|P_n(z_u - \pi_n z_u)\|_{Q_K^n} \\
&+ h_K^{-\frac12} \|r_u\|_{\partial_x Q_K^n} h_K^{\frac12} \|P_n(z_u - \pi_n z_u)\|_{\partial_x Q_K^n} \\
&+ \|R_w\|_{Q_K^n} \|P_n(z_w - \pi_n z_w)\|_{Q_K^n} \\
&+ h_K^{-\frac12} \|r_w\|_{\partial_x Q_K^n} h_K^{\frac12} \|P_n(z_w - \pi_n z_w)\|_{\partial_x Q_K^n} \\
&+ k_n^{-\frac12} \|[U]_{n-1}\|_K k_n^{\frac12} \|(z_u - \pi z_u)^+_{n-1}\|_K \Big\} + |\mathcal{R}^{(2)}|.
\end{aligned}
$$

Here $h_K = \operatorname{diam}(K)$. For $a, b, c, d \ge 0$ we have

$$
(ab + cd) \le (a^2 + c^2)^{\frac12} (b^2 + d^2)^{\frac12}.
$$

We use this inequality for each term in the previous inequality and set

$$
\begin{aligned}
\rho_{u,K} &= \Big(\|R_u\|^2_{Q_K^n} + h_K^{-1} \|r_u\|^2_{\partial_x Q_K^n} \Big)^{\frac12}, \\
\omega_{u,K} &= \Big(\|P_n(z_u - \pi_n z_u)\|^2_{Q_K^n} + h_K \|P_n(z_u - \pi_n z_u)\|^2_{\partial_x Q_K^n} \Big)^{\frac12}, \\
\rho_{w,K} &= \Big(\|R_w\|^2_{Q_K^n} + h_K^{-1} \|r_w\|^2_{\partial_x Q_K^n} \Big)^{\frac12}, \\
\omega_{w,K} &= \Big(\|P_n(z_w - \pi_n z_w)\|^2_{Q_K^n} + h_K \|P_n(z_w - \pi_n z_w)\|^2_{\partial_x Q_K^n} \Big)^{\frac12}, \\
\rho_K &= \Big(k_n^{-1} \|[U]^{n-1}\|^2_K \Big)^{\frac12}, \\
\omega_K &= \Big(k_n \|(z_u - \pi z_u)^+_{n-1}\|^2_K \Big)^{\frac12}.
\end{aligned}
$$

Note that, since $R_u = \Delta W = 0$ for piecewise linear functions, the first term in $\rho_{u,K}$ and $\omega_{u,K}$ can actually be removed. So we have

$$
|J(u) - J(U)| \le \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ \rho_{u,K}\omega_{u,K} + \rho_{w,K}\omega_{w,K} + \rho_K \omega_K \Big\} + |\mathcal{R}^{(2)}|.
$$

We have proved the following theorem:

Theorem 4.1. *We have the a posteriori error estimate*

$$|J(u)-J(U)| \le \sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\Big\{\rho_{u,K}\omega_{u,K}+\rho_{w,K}\omega_{w,K}+\rho_K\omega_K\Big\}+|\mathcal{R}^{(2)}|. \tag{4.4}$$

Note that on each space-time cell Q_K^n, the terms $\rho_{u,K}\omega_{u,K}$ and $\rho_{w,K}\omega_{w,K}$ can be used to control the spatial mesh and the term $\rho_K\omega_K$ to control the time step k_n in an adaptive algorithm; see [2]. We do not pursue this here.

In the following we want to obtain a weight free a posteriori error estimate where the weights in (4.4) are replaced by a global stability constant. We need the following interpolation error estimate, see [2, Lemma 9.4].

Lemma 4.2. *With π and π_n as defined as before, there holds*

$$\|P_n(z-\pi_n z)\|_{Q_K^n}+h_K^{\frac12}\|P_n(z-\pi_n z)\|_{\partial_x Q_K^n}\le Ch_K^2\|\mathrm{D}^2 z\|_{Q_K^n}, \tag{4.5}$$

$$\|z(t_{n-1})-P_n z\|_K\le Ck_n^{\frac12}\|\partial_t z\|_{Q_K^n}. \tag{4.6}$$

Here $\|\mathrm{D}^2 z\|_{Q_K^n}$ denotes the seminorm $\Big(\sum_{|\alpha|=2}\|D^\alpha z\|_{Q_K^n}^2\Big)^{\frac12}$.

In the following we assume that $J(\cdot)$ is a linear functional given by (3.6) and Ω is such that we have the elliptic regularity estimate

$$\|\mathrm{D}^2 v\|_\Omega\le C\|\Delta v\|_\Omega\quad \forall v\in H^2(\Omega)\text{ with }\frac{\partial v}{\partial\nu}\Big|_\Gamma=0. \tag{4.7}$$

We also assume a global bound for $f'(u)$, which is reasonable since it is known that $\|u\|_{L_\infty(Q)}\le C$ (c.f. [5]).

In particular, with

$$g=(u-U)/\|u-U\|_Q \text{ and } g_T=(u_N-U_N)/\|u_N-U_N\|_\Omega$$

the following theorem provides bounds for the norms of the error, $\|u-U\|_Q$ and $\|u_N-U_N\|_\Omega$.

Theorem 4.3. *Assume that $\|f'(u)\|_{L_\infty}\le\beta$ and that (4.7) holds. Let z_u, z_w be the solutions of (3.8). Then there is $C=C(\beta)$ such that the following a posteriori error estimates hold.*

(i) Let $g\in L_2(Q)$ with $\|g\|_Q=1$ and $g_T=0$. Then

$$\begin{aligned}&|\langle u-U,g\rangle_Q|\\ &\le CC_S\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\Big\{h_K^4(\rho_{u,K}^2+\rho_{w,K}^2)+(h_K^4+k_n^2)\rho_K^2\Big\}^{\frac12}+|\mathcal{R}^{(2)}|,\end{aligned} \tag{4.8}$$

where

$$C_S=\sup_{g\in L_2(Q)}\frac{\Big(\|\mathrm{D}^2 z_u\|_Q^2+\|\partial_t z_u\|_Q^2+\|\mathrm{D}^2 z_w\|_Q^2\Big)^{\frac12}}{\|g\|_Q}.$$

(ii) Let $g_T \in L_2(\Omega)$ with $\|g_T\|_\Omega = 1$ and $g = 0$. Then

$$
\begin{aligned}
&|\langle u - U, g_T\rangle_\Omega| \\
&\le CC_S \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ h_K^4(\rho_{u,K}^2 + \sigma_n^{-1}\rho_{w,K}^2 + \sigma_n^{-1}\rho_K^2) + k_n^2\sigma^{-1}\rho_K^2 \Big\}^{\frac{1}{2}} \\
&\quad + |\mathcal{R}^{(2)}|,
\end{aligned} \tag{4.9}
$$

where $\sigma(t) = T - t$,

$$
\sigma_n = \begin{cases} \sigma(t_n) = T - t_n, & n = 1, \cdots, N-1, \\ k_N, & n = N, \end{cases}
$$

and

$$
\begin{aligned}
C_S = \sup_{g_T\in L_2(\Omega)} \Big(&\epsilon^{-1} \max_I \|z_u\|_\Omega^2 + \epsilon^{-1}\|z_w\|_Q^2 \\
&+ \|\mathrm{D}^2 z_u\|_Q^2 + \|\sigma^{\frac{1}{2}}\partial_t z_u\|_Q^2 + \epsilon^2\|\sigma^{\frac{1}{2}}\mathrm{D}^2 z_w\|_Q^2 \Big)^{\frac{1}{2}} / \|g_T\|_\Omega.
\end{aligned}
$$

Proof. Part (i). From Theorem 4.2 we have

$$
\begin{aligned}
\omega_{u,K} &= \Big(\|P_n(z_u - \pi_n z_u)\|_{Q_K^n}^2 + h_K \|P_n(z_u - \pi_n z_u)\|_{\partial_x Q_K^n}^2 \Big)^{\frac{1}{2}} \\
&\le Ch_K^2 \|\mathrm{D}^2 z_u\|_{Q_K^n},
\end{aligned}
$$

$$
\begin{aligned}
\omega_{w,K} &= \Big(\|P_n(z_w - \pi_n z_w)\|_{Q_K^n}^2 + h_K \|P_n(z_w - \pi_n z_w)\|_{\partial_x Q_K^n}^2 \Big)^{\frac{1}{2}} \\
&\le Ch_K^2 \|\mathrm{D}^2 z_w\|_{Q_K^n},
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_K &= k_n^{\frac{1}{2}} \|(z_u - \pi_n z_u)_{n-1}^+\|_K \\
&\le k_n^{\frac{1}{2}} \|P_n(z_u - \pi_n z_u)\|_K + k_n^{\frac{1}{2}} \|z_u(t_{n-1}) - P_n z_u\|_K \\
&\le Ch_K^2 \|\mathrm{D}^2 z_u\|_{Q_K^n} + Ck_n \|\partial_t z_u\|_{Q_K^n} + |\mathcal{R}^{(2)}|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\langle u - U, g\rangle_Q| &\le \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ \rho_{u,K}\omega_{u,K} + \rho_{w,K}\omega_{w,K} + \rho_K\omega_K \Big\} \\
&\le \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ Ch_K^2 \rho_{u,K} \|\mathrm{D}^2 z_u\|_{Q_K^n} + Ch_K^2 \rho_{w,K} \|\mathrm{D}^2 z_w\|_{Q_K^n} \\
&\quad + \rho_K (Ch_K^2 \|\mathrm{D}^2 z_u\|_{Q_K^n} + Ck_n \|\partial_t z_u\|_{Q_K^n}) \Big\}
\end{aligned}
$$

and the desired estimate (4.8) follows by the Cauchy-Schwartz inequality

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n} h_K^2\rho_{u,K}\|\mathrm{D}^2 z_u\|_{Q_K^n}\\
&\quad\le\Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n} h_K^4\rho_{u,K}^2\Big)^{\frac12}\Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n} h_K^2\rho_{u,K}\|\mathrm{D}^2 z_u\|_{Q_K^n}^2\Big)^{\frac12}\\
&\quad=\Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n} h_K^4\rho_{u,K}^2\Big)^{\frac12}\|\mathrm{D}^2 z_u\|_Q\le C_S\Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n} h_K^4\rho_{u,K}^2\Big)^{\frac12}\|g\|_Q,
\end{aligned}$$

and similarly for the other terms.

Part (ii). The previous bound for $\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\rho_{u,K}\omega_{u,K}$ applies here also. Consider then

$$\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\rho_{w,K}\omega_{w,K}\le\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\rho_{w,K}Ch_K^2\|\mathrm{D}^2 z_w\|_{Q_K^n}+\sum_{K\in\mathbf{T}_N}\rho_{w,K}\omega_{w,K}.$$

Here,

$$\begin{aligned}
&\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\rho_{w,K}Ch_K^2\|\mathrm{D}^2 z_w\|_{Q_K^n}\\
&\quad=\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\rho_{w,K}Ch_K^2\|\sigma^{-\frac12}\sigma^{\frac12}\mathrm{D}^2 z_w\|_{Q_K^n}\\
&\quad\le C\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\rho_{w,K}\sigma_n^{-\frac12}h_K^2\|\sigma^{\frac12}\mathrm{D}^2 z_w\|_{Q_K^n}\\
&\quad\le C\Big(\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\sigma_n^{-1}h_K^4\rho_{w,K}^2\Big)^{\frac12}\Big(\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\|\sigma^{\frac12}\mathrm{D}^2 z_w\|_{Q_K^n}^2\Big)^{\frac12}\\
&\quad\le C\Big(\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\sigma_n^{-1}h_K^4\rho_{w,K}^2\Big)^{\frac12}\|\sigma_n^{\frac12}\mathrm{D}^2 z_w\|_Q\\
&\quad\le C_S C\Big(\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\sigma_n^{-1}h_K^4\rho_{w,K}^2\Big)^{\frac12}\|g_T\|_\Omega.
\end{aligned}$$

The term with $n=N$ is special. We go back to (4.3) and replace it by

$$\begin{aligned}
\sum_{K\in\mathbf{T}_N}\langle R_w, z_w-\pi_N z_w\rangle_{Q_K^N}&=\sum_{K\in\mathbf{T}_N}\Big\langle R_w,(I-\pi_N)\int_{I_N} z_w\,\mathrm{d}t\Big\rangle_K\\
&\le\sum_{K\in\mathbf{T}_N}\|R_w\|_K Ch_K^2\Big\|\mathrm{D}^2\int_{I_N} z_w\,\mathrm{d}t\Big\|_K.
\end{aligned}$$

Here, by the regularity estimate (4.7), $\epsilon\Delta z_w = \partial_t z_u + f'(u)z_w$ from the first equation in (3.8), and $\|f'(u)\|_{L_\infty} \le \beta$, we have

$$\begin{aligned}\Big\|\mathrm{D}^2\int_{I_N} z_w\,\mathrm{d}t\Big\|_K &\le C\Big\|\int_{I_N}\Delta z_w\,\mathrm{d}t\Big\|_K \\ &= C\epsilon^{-1}\Big\|\int_{I_N}(\partial_t z_u + f'(u)z_w)\,\mathrm{d}t\Big\|_K \\ &\le C\epsilon^{-1}\Big(\|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + \beta k_N^{\frac12}\|z_w\|_{Q_K^N}\Big).\end{aligned}$$

Hence, since $\rho_{w,K} = \|R_w\|_{Q_K^N} = k_N^{\frac12}\|R_w\|_K$, we have

$$\begin{aligned}&\sum_{K\in\mathbf{T}_N}\langle R_w, z_w - \pi_N z_w\rangle_{Q_K^N} \\ &\quad\le \sum_{K\in\mathbf{T}_N}\|R_w\|_K C h_K^2\epsilon^{-1}\Big(\|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + k_N^{\frac12}\|z_w\|_{Q_K^N}\Big) \\ &\quad= C\epsilon^{-1}\sum_{K\in\mathbf{T}_N} k_N^{-\frac12}h_K^2\rho_{w,K}\Big(\|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + k_N^{\frac12}\|z_w\|_{Q_K^N}\Big) \\ &\quad\le C\epsilon^{-1}\Big(\sum_{K\in\mathbf{T}_N} k_N^{-1}h_K^4\rho_{w,K}^2\Big)^{\frac12}\Big(\|z_u(t_N)\|_\Omega + \|z_u(t_{N-1})\|_\Omega + k_N^{\frac12}\|z_w\|_Q\Big) \\ &\quad\le C\epsilon^{-1}C_S\|g_T\|_\Omega\Big(\sum_{K\in\mathbf{T}_N}\sigma_N^{-1}h_K^4\rho_{w,K}^2\Big)^{\frac12},\end{aligned}$$

where we have used $\sigma_N = k_N$. So we have

$$\sum_{n=1}^N\sum_{K\in\mathbf{T}_n}\rho_{w,K}\omega_{w,K} \le CC_S\|g_T\|_\Omega\Big(\sum_{n=1}^N\sum_{K\in\mathbf{T}_n}\sigma_n^{-1}h_K^4\rho_{w,K}^2\Big)^{\frac12}. \tag{4.10}$$

Now we compute $\sum_{n=1}^N\sum_{K\in\mathbf{T}_n}\rho_K\omega_K$. For $K\in\mathbf{T}_N$ we use

$$\begin{aligned}\omega_K &= k_N^{\frac12}\|(z_u - \pi z_u)_{N-1}^+\|_K \\ &\le k_N^{\frac12}\|P_N(z_u - \pi_N z_u)\|_K + k_N^{\frac12}\|z_u(t_{N-1}) - P_N z_u\|_K \\ &= \|P_N(z_u - \pi_N z_u)\|_{Q_K^N} + k_N^{\frac12}\|z_u(t_{N-1}) - P_N z_u\|_K \\ &\le Ch_K^2\|\mathrm{D}^2 z_u\|_{Q_K^N} + k_N^{\frac12}\|z_u(t_{N-1}) - P_N z_u\|_K.\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \rho_K\omega_K \\
&\quad = C\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \rho_K h_K^2 \|\mathrm{D}^2 z_u\|_{Q_K^n} + C\sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \rho_K k_n \sigma_n^{-\frac{1}{2}} \|\sigma^{\frac{1}{2}}\partial_t z_u\|_{Q_K^n} \\
&\qquad + \sum_{K\in\mathbf{T}_N} \rho_K k_N^{\frac{1}{2}} \|z_u(t_{N-1}) - P_N z_u\|_K \\
&\quad \le C\Big(\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} h_K^4\rho_K^2\Big)^{\frac{1}{2}} \|\mathrm{D}^2 z_u\|_Q + C\Big(\sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \rho_K^2 k_n^2 \sigma_n^{-1}\Big)^{\frac{1}{2}} \|\sigma^{\frac{1}{2}}\partial_t z_u\|_Q \\
&\qquad + C\Big(\sum_{K\in\mathbf{T}_N} k_N\rho_K^2\Big)^{\frac{1}{2}} \|z_u(t_{N-1}) - P_N z_u\|_\Omega .
\end{aligned}$$

Using $\sigma_N = k_N$ and

$$\|z_u(t_{N-1}) - P_N z_u\|_\Omega \le 2\max_I \|z_u\|_\Omega \le 2C_S\|g_T\|_\Omega,$$

gives

$$\begin{aligned}
\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \rho_K\omega_K &\le C\Big(\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} h_K^4\rho_K^2\Big)^{\frac{1}{2}} C_S\|g_T\|_\Omega \\
&\quad + C\Big(\sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \rho_K^2 k_n^2\sigma_n^{-1}\Big)^{\frac{1}{2}} C_S\|g_T\|_\Omega + C\Big(\sum_{K\in\mathbf{T}_N} k_N\rho_K^2\Big)^{\frac{1}{2}} C_S\|g_T\|_\Omega \\
&= CC_S\Big(\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} h_K^4\rho_K^2\Big)^{\frac{1}{2}} \|g_T\|_\Omega + CC_S\Big(\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \rho_K^2 k_n^2\sigma_n^{-1}\Big)^{\frac{1}{2}} \|g_T\|_\Omega .
\end{aligned}$$

This completes the proof. □

Finally, we prove a priori bounds for the stability constants C_S.

Theorem 4.4. *Assume that* $\|f'(u)\|_{L_\infty(Q)} \le \beta$ *and* $\epsilon \in (0,1]$ *and that* (4.7) *holds. Then the solution of* (3.8) *admits the following a priori bounds, where* $C = C(\beta)$. *If* $g_T = 0$, *then*

$$\|\mathrm{D}^2 z_u\|_Q^2 + \|\partial_t z_u\|_Q^2 + \epsilon^2\|\mathrm{D}^2 z_w\|_Q^2 \le C\|g\|_Q^2 \mathrm{e}^{C\epsilon^{-1}T}. \tag{4.11}$$

If $g = 0$, *then, with* $\sigma(t) = T - t$,

$$\begin{aligned}
&\epsilon^{-1}\max_I \|z_u\|_\Omega^2 + \|z_w\|_Q^2 + \|\mathrm{D}^2 z_u\|_Q^2 + \|\sigma^{\frac{1}{2}}\partial_t z_u\|_Q^2 + \epsilon^2\|\sigma^{\frac{1}{2}}\mathrm{D}^2 z_w\|_Q^2 \\
&\qquad \le C\epsilon^{-1}\|g_T\|_\Omega^2 \mathrm{e}^{C\epsilon^{-1}T}.
\end{aligned} \tag{4.12}$$

Proof. We first estimate $\|z_w\|_Q^2$. To this end we use $\Delta z_u = z_w$ from the second equation of (3.8) to get

$$\langle \Delta z_w, z_u\rangle_\Omega = \langle z_w, \Delta z_u\rangle_\Omega = \|z_w\|_\Omega^2.$$

Then we multiply the first equation of (3.8) by z_u, and integrate over $[t,T]$,

$$\int_t^T \langle -\partial_t z_u, z_u\rangle_\Omega \,\mathrm{d}s + \epsilon \int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s - \int_t^T \langle f'(u) z_w, z_u\rangle_\Omega \,\mathrm{d}s = \int_t^T \langle g, z_u\rangle_\Omega \,\mathrm{d}s.$$

By assumption we know that $\|f'(u)\|_{L_\infty(Q)} \le \beta$, so we have

$$\begin{aligned}
&\tfrac{1}{2}\|z_u(t)\|_\Omega^2 - \tfrac{1}{2}\|z_u(T)\|_\Omega^2 + \epsilon \int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s \\
&\quad\le \int_t^T \|f'(u)\|_{L_\infty(Q)} \|z_w\|_\Omega \|z_u\|_\Omega \,\mathrm{d}s + \int_t^T \|g\|_\Omega \|z_u\|_\Omega \,\mathrm{d}s \\
&\quad\le \int_t^T \big(\tfrac{\beta^2}{2\epsilon}\|z_u\|_\Omega^2 + \tfrac{\epsilon}{2}\|z_w\|_\Omega^2\big) \,\mathrm{d}s + \int_t^T \big(\tfrac{c}{2}\|g\|_\Omega^2 + \tfrac{1}{2c}\|z_u\|_\Omega^2\big) \,\mathrm{d}s \\
&\quad\le \tfrac{\beta^2}{\epsilon} \int_t^T \|z_u\|_\Omega^2 \,\mathrm{d}s + \tfrac{\epsilon}{2} \int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s + \int_t^T \big(\tfrac{c}{2}\|g\|_\Omega^2 + \tfrac{1}{2c}\|z_u\|_\Omega^2\big) \,\mathrm{d}s.
\end{aligned}$$

Hence, with $z_u(T) = g_T$ and $c = \frac{\epsilon}{\beta^2}$,

$$\begin{aligned}
&\|z_u(t)\|_\Omega^2 + \epsilon \int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s \\
&\quad\le \frac{\epsilon}{\beta^2}\|g\|_Q^2 + \|g_T\|_\Omega^2 + 2\beta^2\epsilon^{-1} \int_t^T \|z_u\|_\Omega^2 \,\mathrm{d}s \\
&\quad\le \tfrac{C}{\epsilon}\|g\|_Q^2 + \|g_T\|_\Omega^2 + C\epsilon^{-1} \int_t^T \|z_u\|_\Omega^2 \,\mathrm{d}s.
\end{aligned}$$

Define

$$\Phi(t) = \|z_u(t)\|_\Omega^2 + \epsilon \int_t^T \|z_w(s)\|_\Omega^2 \,\mathrm{d}s.$$

Obviously we have $\|z_u(s)\|_\Omega^2 \le \Phi(s)$, so that

$$\Phi(t) \le C\epsilon\|g\|_Q^2 + \|g_T\|_\Omega^2 + C\epsilon^{-1} \int_t^T \Phi(s) \,\mathrm{d}s.$$

We apply Gronwall's lemma to get

$$\Phi(t) \le C(\epsilon\|g\|_Q^2 + \|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}(T-t)}.$$

This means

$$\|z_u(t)\|_\Omega^2 + \epsilon \int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s \le C(\epsilon\|g\|_Q^2 + \|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}(T-t)}.$$

We conclude

$$\max_I \|z_u\|_\Omega^2 \le C(\epsilon\|g\|_Q^2 + \|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}T}.$$

$$\|z_w\|_Q^2 \le C(\|g\|_Q^2 + \epsilon^{-1}\|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}T}. \tag{4.13}$$

From the second equation we know $z_w = \Delta z_u$. So, by (4.7) and (4.13),

$$\|\mathrm{D}^2 z_u\|_Q^2 \le C\|\Delta z_u\|_Q^2 = C\|z_w\|_Q^2 \le C(\|g\|_Q^2 + \epsilon^{-1}\|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}T}. \tag{4.14}$$

This takes care of the first terms in (4.11) and (4.12).

Now assume that $g_T = 0$. Consider the dual problem (3.8) and multiply the first equation by $-\partial_t z_u$ and integrate over Q to get

$$\langle \partial_t z_u, \partial_t z_u\rangle_Q - \epsilon\langle \Delta z_w, \partial_t z_u\rangle_Q - \langle f'(u)z_w, \partial_t z_u\rangle_Q = -\langle g, \partial_t z_u\rangle_Q. \tag{4.15}$$

So, by using $z_w = \Delta z_u$ from the second equation, we get

$$\langle \Delta z_w, \partial_t z_u\rangle_Q = \langle z_w, \partial_t \Delta z_u\rangle_Q = \langle \Delta z_u, \partial_t \Delta z_u\rangle_Q = \tfrac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\|\Delta z_u\|_\Omega^2 \,\mathrm{d}t.$$

By putting this in (4.15) and using that $\|f'(u)\|_{L_\infty(Q)} \le \beta$, we have

$$\begin{aligned}
&\|\partial_t z_u\|_Q^2 - \tfrac{\epsilon}{2}\|\Delta z_u(T)\|_\Omega^2 + \tfrac{\epsilon}{2}\|\Delta z_u(0)\|_\Omega^2 \\
&\quad\le \|f'(u)\|_{L_\infty(Q)}\|z_w\|_Q\|\partial_t z_u\|_Q + \|g\|_Q\|\partial_t z_u\|_Q \\
&\quad\le \tfrac{c\beta^2}{2}\|z_w\|_Q^2 + \tfrac{1}{2c}\|\partial_t z_u\|_Q^2 + \tfrac{c}{2}\|g\|_Q^2 + \tfrac{1}{2c}\|\partial_t z_u\|_Q^2.
\end{aligned}$$

Put $c = 2$ and kick back $\|\partial_t z_u\|_Q^2$ to get, with $z_u(T) = g_T = 0$,

$$\tfrac{1}{2}\|\partial_t z_u\|_Q^2 + \tfrac{\epsilon}{2}\|\Delta z_u(0)\|_\Omega^2 \le \beta^2\|z_w\|_Q^2 + \|g\|_Q^2.$$

Hence, by (4.13) with $C = C(\beta)$,

$$\|\partial_t z_u\|_Q^2 \le C\|z_w\|_Q^2 + C\|g\|_Q^2 \le C\|g\|_Q^2\mathrm{e}^{-C\epsilon^{-1}T}. \tag{4.16}$$

It remains to bound $\|\mathrm{D}^2 z_w\|_Q^2$. From the first equation of (3.8) we get $\epsilon\Delta z_w = g + \partial_t z_u + f'(u)z_w$. Taking norms and using (4.7), (4.13), and (4.16) gives

$$\begin{aligned}
\epsilon^2\|\mathrm{D}^2 z_w\|_Q^2 &\le \epsilon^2 C\|\Delta z_w\|_Q^2 = C\|g + \partial_t z_u + f'(u)z_w\|_Q^2 \\
&\le C\Big(\|g\|_Q^2 + \|\partial_t z_u\|_Q^2 + \|f'(u)\|_{L_\infty(Q)}^2\|z_w\|_Q^2\Big) \\
&\le C\|g\|_Q^2\mathrm{e}^{C\epsilon^{-1}T}.
\end{aligned}$$

This completes the proof of (4.11)

Now let $g = 0$ and set $\sigma(t) = T - t$. Multiply the first equation of (3.8) by $-\sigma\partial_t z_u$ to get

$$\langle \partial_t z_u, \sigma\partial_t z_u\rangle_Q - \epsilon\langle \Delta z_w, \sigma\partial_t z_u\rangle_Q - \langle f'(u)z_w, \sigma\partial_t z_u\rangle_Q = 0.$$

Here, since $z_w = \Delta z_u$ and $\sigma'(t) = -1$,

$$
\begin{aligned}
\langle \Delta z_w, \sigma \partial_t z_u \rangle_Q &= \langle z_w, \sigma \Delta \partial_t z_u \rangle_Q \\
&= \langle \Delta z_u, \sigma \Delta \partial_t z_u \rangle_Q \\
&= \tfrac{1}{2} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} (\sigma \|\Delta z_u\|_\Omega^2) \, \mathrm{d}t - \tfrac{1}{2} \int_0^T \sigma' \|\Delta z_u\|_\Omega^2 \, \mathrm{d}t \\
&= \tfrac{1}{2} \sigma(T) \|\Delta z_u(T)\|_\Omega^2 - \tfrac{1}{2} \sigma(0) \|\Delta z_u(0)\|_\Omega^2 + \tfrac{1}{2} \int_0^T \|z_w\|_\Omega^2 \, \mathrm{d}t \\
&= -\tfrac{1}{2} T \|\Delta z_u(0)\|_\Omega^2 + \tfrac{1}{2} \|z_w\|_Q^2 .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\sigma^{\frac{1}{2}} \partial_t z_w\|_Q^2 + \|\Delta z_u(0)\|_\Omega^2 &\le \tfrac{\epsilon}{2} \|z_w\|_Q^2 + \|f'(u)\|_{L_\infty} \|\sigma^{\frac{1}{2}} z_w\|_Q \|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q \\
&\le \tfrac{1}{2} (\epsilon + \beta^2 T) \|z_w\|_Q^2 + \tfrac{1}{2} \|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q^2 .
\end{aligned}
$$

So by (4.13) we have

$$
\|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q \le (\epsilon + \beta^2 T) \|z_w\|_Q^2 C \epsilon^{-1} \|g_T\|_\Omega^2 \mathrm{e}^{C\epsilon^{-1} T} .
$$

Finally, from (4.7) and $\epsilon \Delta z_w = \partial_t z_u + f'(u) z_w$ we get

$$
\begin{aligned}
\epsilon^2 \|\sigma^{\frac{1}{2}} \mathrm{D}^2 z_w\|_Q^2 &\le \epsilon^2 C \|\sigma^{\frac{1}{2}} \Delta z_w\|_Q^2 = C \|\sigma^{\frac{1}{2}} (\partial_t z_u + f'(u) z_w)\|_Q^2 \\
&\le C \Big(\|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q^2 + T \|z_w\|_Q^2 \Big) \\
&\le C \epsilon^{-1} \|g_T\|_\Omega^2 \mathrm{e}^{C\epsilon^{-1} T} .
\end{aligned}
$$

This completes the proof of (4.12). □

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