DISCONTINUOUS GALERKIN METHOD FOR AN INTEGRO-DIFFERENTIAL EQUATION MODELING DYNAMIC FRACTIONAL ORDER VISCOELASTICITY

STIG LARSSON, MILENA RACHEVA, AND FARDIN SAEDPANAH

Abstract. An integro-differential equation, modeling dynamic fractional order viscoelasticity, with a Mittag-Leffler type convolution kernel is considered. A discontinuous Galerkin method, based on piecewise constant polynomials is formulated for temporal semidiscretization of the problem. Stability estimates of the discrete problem are proved, that are used to prove optimal order a priori error estimates. The theory is illustrated by a numerical example.

1. Introduction

Fractional order integral/differential operators have proved to be very suitable for modeling memory effects of various materials, [3]. In particular, for modeling viscoelastic materials, for more details and references see [11]. The basic equations of the viscoelastic dynamic problem, that is a hyperbolic type integro-differential equations, can be written in the strong form,

$$\begin{aligned}
&\rho\ddot{u}(x,t) - \nabla\cdot\sigma_0(u;x,t) \\
&\qquad + \int_0^t \beta(t-s)\nabla\cdot\sigma_0(u;x,s)\,ds = f(x,t) && \text{in } \Omega\times(0,T), \\
&u(x,t) = 0 && \text{on } \Gamma_D\times(0,T), \\
&\sigma(u;x,t)\cdot n(x) = g(x,t) && \text{on } \Gamma_N\times(0,T), \\
&u(x,0) = u_0(x) && \text{in } \Omega, \\
&\dot{u}(x,0) = v_0(x) && \text{in } \Omega,
\end{aligned} \tag{1.1}$$

(throughout this text we use '$\cdot$' to denote '$\frac{\partial}{\partial t}$') where u is the displacement vector, ρ is the (constant) mass density, f and g represent, respectively, the volume and surface loads, σ_0 is the stress according to

$$\sigma_0(t) = 2\mu\epsilon(t) + \lambda\mathrm{Tr}(\epsilon(t))\mathrm{I},$$

and

$$\sigma(t) = \sigma_0(t) - \int_0^t \beta(t-s)\sigma_0(s)\,ds,$$

Date: May 9, 2014.

1991 *Mathematics Subject Classification.* 65M60, 45K05.

Key words and phrases. integro-differential equation, fractional order viscoelasticity, discontinuous Galerkin method, weakly singular kernel, stability, a priori estimate.

where $\lambda, \mu > 0$ are elastic constants of Lamé type, ϵ is the strain which is defined by $\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T)$. Here, β is the convolution kernel

$$\beta(t) = -\gamma \frac{d}{dt}\Big(\mathrm{E}_\alpha(-(t/\tau)^\alpha)\Big) = \gamma \frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{\alpha-1} \mathrm{E}_\alpha'\Big(-\big(\frac{t}{\tau}\big)^\alpha\Big) \approx Ct^{-1+\alpha},\ t \to 0, \tag{1.2}$$

where $0 < \gamma < 1$, $\tau > 0$ is the relaxation time and $\mathrm{E}_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+\alpha k)}$ is the Mittag-Leffler function of order $\alpha \in (0, 1)$. The convolution kernel is weakly singular and $\beta \in L_1(0, \infty)$ with $\int_0^\infty \beta(t)\,dt = \gamma$.

Well-posedness of the model problem (1.1) and more general form of such equations in fractional order viscoelasticity have been studied in [11], by means of Galerkin approximation methods. Continuous Galerkin methods of order one, both in time and space variables, have been applied to similar problem in [4] and [10]. Discontinuous Galerkin and continuous Galerkin method, respectively, in time and space variables have been applied to a dynamic model problem in linear viscoelasticity (with exponential kernels) in [9]. For more references on numerical and analytical treatment of integro-differential equations, among the extensive literature, see e.g., [5], [8], [7], [13], [2], and their references.

Here, we formulate the discontinuous Galerkin method dG(0), based on piecewise constant polynomials in the time variable, for the temporal semidiscretization of the problem. We prove stability estimates for the discrete problem, that are used to prove optimal order a priori error estimates for the displacement u and velocity $\dot{u}$. Then we illustrate the theory by a numerical example. The present work extends previous works, e.g., [13] and [1] on quasi-static ($\rho\ddot{u} \approx 0$) linear and fractional order viscoelasticity, to the dynamic fractional order case.

The convolution integral in the model problem generates a growing amount of data that has to be stored and used in each time step. Lubich's convolution quadrature [6], that has been improved in [12], has been commonly used for this integration. See [1] and references therein for examples of application of this approach and a different approach, the so-called "sparse quadrature". In [13], the exponential decaying kernel, in linear viscoelasticity, has been represented as a Prony series, that resulting in a recurrence formula for history updating. This means that, in this case we do not use convolution quadrature. In general we do not have global regularity of solutions, due to, e.g., regularity of the kernel and mixed boundary conditions, which calls for adaptive methods based on a posteriori error analysis. We plan to address these issues and full discrete space-time discontinuous Galerkin and continuous Galerkin methods in future work.

In the next section, we provide some definitions and the weak formulations of the model problem. In §3 we formulate the discontinuous Galerkin method. Then in §4 we show an energy identity and stability estimates for the discrete problem, that is used in §5 to prove optimal order a priori error estimates. Finally, in §6, we illustrate that the dG(0) method capture the mechanical behavior of the model problem and we investigate the rate of convergence $O(k)$, by a numerical example.

2. Preliminaries

We let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polygonal domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are disjoint and $\mathrm{meas}(\Gamma_D) \neq 0$. We introduce the function spaces $H = L_2(\Omega)^d$, $H_{\Gamma_N} = L_2(\Gamma_N)^d$, and $V = \{v \in H^1(\Omega)^d : v|_{\Gamma_D} = 0\}$. We denote the norms in H and H_{Γ_N} by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_N}$, respectively, and we equip

V with the inner product $a(\cdot,\cdot)$ and norm $\|v\|_V^2 = a(v,v)$, where (with the usual summation convention)

$$a(v,w) = \int_\Omega \big(2\mu\epsilon_{ij}(v)\epsilon_{ij}(w) + \lambda\epsilon_{ii}(v)\epsilon_{jj}(w)\big)\,dx, \quad v,w \in V\,, \tag{2.1}$$

which is a coercive bilinear form on V. Setting $Au = -\nabla\cdot\sigma_0(u)$ with $\mathcal{D}(A) = H^2(\Omega)^d \cap V$ such that $a(u,v) = (Au,v)$ for sufficiently smooth $u,v \in V$ and homogeneous boundary conditions in (1.1) ($g=0$), we can write the weak form of the equation of motion as: Find $u(t) \in V$ such that $u(0) = u_0$, $\dot u(0) = v_0$, and

$$\begin{aligned}\rho(\ddot u(t),v) + a(u(t),v) - &\int_0^t \beta(t-s)a(u(s),v)\,ds \\ &= (f(t),v) + (g(t),v)_{\Gamma_N}, \quad \forall v \in V,\, t \in (0,T),\end{aligned} \tag{2.2}$$

with $(g(t),v)_{\Gamma_N} = \int_{\Gamma_N} g(t)\cdot v\,dS$.

Defining the new variables $u_1 = u$ and $u_2 = \dot u$ we write the velocity-displacement form of (2.2) as: Find $u_1(t)$, $u_2(t) \in V$ such that $u_1(0) = u_0$, $u_2(0) = v_0$, and

$$\begin{aligned}&a\big(\dot u_1(t),v_1\big) - a\big(u_2(t),v_1\big) = 0,\\ &\rho(\dot u_2(t),v_2) + a(u_1(t),v_2) - \int_0^t \beta(t-s)a(u_1(s),v_2)\,ds \\ &\qquad\qquad = (f(t),v_2) + (g(t),v_2)_{\Gamma_N}, \quad \forall v_1,v_2 \in V,\, t \in (0,T),\end{aligned} \tag{2.3}$$

that is used for discontinuous Galerkin formulation.

We recall that the positive convolution kernel β is weakly singular, that is, $\beta > 0$ is singular at the origin, but $\|\beta\|_{L_1(0,\infty)} = \gamma < 1$. For our analysis, we define the function

$$\eta(t) = 1 - \int_0^t \beta(s)\,ds, \tag{2.4}$$

and it is easy to see that

$$\eta(0) = 1, \quad \lim_{t\to\infty}\eta(t) = 1-\gamma < 1, \quad \dot\eta(t) = -\beta(t). \tag{2.5}$$

3. The discontinuous Galerkin method

Here we formulate the discontinuous Galerkin method, dG(0), that is based on piecewise constant polynomials, for temporal discretization of the model problem (1.1) with the weak form (2.3). To simplify the notation we consider homogeneous boundary conditions, that is $g = 0$ or $\Gamma_D = \Gamma$.

Let $0 = t_0 < t_1, \ldots < t_N = T$ be a temporal mesh, $I_n = (t_{n-1}, t_n)$ denote the time intervals and $k_n = t_n - t_{n-1}$ denote the time steps. The discrete finite element space is

$$\mathcal{W}_D = \{w = (w_1,w_2) : w_i|_{I_n} = w_{i,n} \in \mathcal{D}(A),\ n = 1,\ldots,N\}\,.$$

We note that $w \in \mathcal{W}_D$ is piecewise constant in time and in general is not continuous at the time nodes $t_n, n = 1,\ldots,N$, so we use the following notations: $w_n = w|_{I_n} = w_{n-1}^+ = w_n^-$ and $[w]_n = w_n^+ - w_n^- = w_{n+1} - w_n$ for the jump terms.

Then, recalling (2.3), the dG(0) method is to find $U=(U_1,U_2)\in\mathcal{W}_D$ such that

$$\begin{aligned}
&\int_{I_n}\Big(a(\dot U_1,V_1)-a(U_2,V_1)\Big)\,dt+a([U_1]_{n-1},V_{1,n-1}^+)=0,\\
&\int_{I_n}\Big(\rho(\dot U_2,V_2)+a(U_1,V_2)-\int_0^t\beta(t-s)a(U_1(s),V_2)\,ds\Big)\,dt\\
&\qquad\qquad+\rho([U_2]_{n-1},V_{2,n-1}^+)\\
&\qquad\qquad\qquad\qquad=(f,V_2),\quad\forall V=(V_1,V_2)\in\mathcal{W}_D,\ t\in(0,T),\\
&U_{1,0}^-=u_0,\quad U_{2,0}^-=v_0.
\end{aligned}\tag{3.1}$$

Since the functions in $\mathcal{W}_D$ are piecewise constant with respect to time, we get, with $U_{1,0}=u_0$, $U_{2,0}=v_0$,

$$\begin{aligned}
&AU_{1,n}-k_nAU_{2,n}=AU_{1,n-1},\\
&(k_n-\omega_{nn})AU_{1,n}+\rho U_{2,n}=\rho U_{2,n-1}+\sum_{j=1}^{n-1}\omega_{nj}AU_{1,j}+k_n\bar f_n,
\end{aligned}$$

where obviously for $n=1$ the sum on the right side is ignored and

$$\begin{aligned}
\omega_{nj}&=\int_{I_n}\int_{t_{j-1}}^{t_j\wedge t}\beta(t-s)\,ds\,dt,\quad t_j\wedge t=\min(t_j,t),\\
\bar f_n&=\frac{1}{k_n}\int_{I_n}f(t)\,dt.
\end{aligned}$$

This is used for computer implementations.

Now, we define the function space $\mathcal{W}$ that consists of functions that are piecewise smooth with respect to the temporal mesh with values in $\mathcal{D}(A)$. We note that $\mathcal{W}_D\subset\mathcal{W}$. Then we define the bilinear form $B:\mathcal{W}\times\mathcal{W}\to\mathbb{R}$ and the linear form $L:\mathcal{W}\to\mathbb{R}$ by

$$\begin{aligned}
B((u_1,u_2),(v_1,v_2))=\sum_{n=1}^N\int_{I_n}&\Big\{a(\dot u_1,v_1)-a(u_2,v_1)\\
&+\rho(\dot u_2,v_2)+a(u_1,v_2)-\int_0^t\beta(t-s)a(u_1(s),v_2(t))\,ds\Big\}\,dt\\
&+\sum_{n=1}^{N-1}\big\{a([u_1]_n,v_{1,n}^+)+\rho([u_2]_n,v_{2,n}^+)\big\}\\
&+a(u_{1,0}^+,v_{1,0}^+)+\rho(u_{2,0}^+,v_{2,0}^+),\\
L((v_1,v_2))=\sum_{n=1}^N\int_{I_n}&(f,v_2)\,dt+a(u_0,v_{1,0}^+)+\rho(v_0,v_{2,0}^+).
\end{aligned}$$

Then $U=(U_1,U_2)\in\mathcal{W}_D$, the solution of the discrete problem (3.1), satisfies

$$\begin{aligned}
&B(U,V)=L(V),\qquad\forall V=(V_1,V_2)\in\mathcal{W}_D,\\
&U_0^-=(U_{1,0}^-,U_{2,0}^-)=(u_0,v_0).
\end{aligned}\tag{3.2}$$

We note that the solution (u_1,u_2) of (2.3) also satisfies

$$\begin{aligned}
&B((u_1,u_2),(v_1,v_2))=L((v_1,v_2)),\qquad\forall(v_1,v_2)\in\mathcal{W},\\
&(u_1(0),u_2(0))=(u_0,v_0),
\end{aligned}\tag{3.3}$$

such that the Galerkin's orthogonality holds for the error $e = (e_1, e_2) = (U_1, U_2) - (u_1, u_2)$, that is,

$$B(e, V) = 0, \qquad \forall V = (V_1, V_2) \in \mathcal{W}_D. \tag{3.4}$$

4. Stability

Here we prove a stability identity and stability estimates that are used in a priori error analysis. To this end, we need to prove a stability identity for a slightly more general problem, that is, $U \in \mathcal{W}_D$ such that

$$\begin{aligned} &B(U, V) = \hat{L}(V), \qquad \forall V \in \mathcal{W}_D, \\ &U_0^- = (U_{1,0}^-, U_{2,0}^-) = (u_0, v_0), \end{aligned} \tag{4.1}$$

where the linear form $\hat{L} : \mathcal{W} \to \mathbb{R}$ is defined by

$$L((v_1, v_2)) = \sum_{n=1}^{N} \int_{I_n} a(f_1, v_1) + (f_2, v_2)\, dt + a(u_0, v_{1,0}^+) + \rho(v_0, v_{2,0}^+).$$

Recalling η from (2.4), we define

$$\eta_n = \frac{1}{k_n} \int_{I_n} \eta(t)\, dt = 1 - \frac{1}{k_n} \int_{I_n} \int_0^t \beta(s)\, ds\, dt, \tag{4.2}$$

with $\eta_0 = 1$. We also denote the backward difference operator, for V_n,

$$\partial_n V_n = \frac{V_n - V_{n-1}}{k_n}. \tag{4.3}$$

Obviously we have

$$\begin{aligned} k_n \partial_n (W_n V_n) &= W_n V_n - W_{n-1} V_{n-1} \\ &= W_n V_n - W_{n-1} V_n + W_{n-1} V_n - W_{n-1} V_{n-1} \\ &= k_n \partial_n W_n V_n + k_n W_{n-1} \partial_n V_n, \end{aligned} \tag{4.4}$$

that also implies

$$\begin{aligned} \partial_n (V_n V_n) + k_n (\partial_n V_n \partial_n V_n) &= V_n \partial_n V_n + V_{n-1} \partial_n V_n + \partial_n V_n k_n \partial_n V_n \\ &= \partial_n V_n (V_n + V_{n-1} + V_n - V_{n-1}) \\ &= 2 V_n \partial_n V_n. \end{aligned} \tag{4.5}$$

We also define the standard L_2-projection $P_{k,n} : L_2(I_n)^d \to \mathbb{P}_0^d(I_n)$ by

$$\int_{I_n} (P_{k,n} v - v)\ dt = 0, \quad \forall v \in L_2(I_n)^d,$$

where $\mathbb{P}_0^d$ denotes all vector valued constant polynomials on I_n. We use the obvious notation P_k over the interval $(0, T)$, i.e, $P_{k,n} = P_k|_{I_n}$. It is easy to see that

$$P_{k,n} v = \bar{v} = \frac{1}{k_n} \int_{I_n} v\ dt, \quad \forall v \in L_2(I_n)^d. \tag{4.6}$$

Theorem 1. *Let $U=(U_1,U_2)$ be a solution of (4.1). Then for any $l\in\mathbb{R}$, $T>0$, we have the equality*

(4.7)
$$
\begin{aligned}
\eta_N\|U_{1,N}\|_{l+1}^2+\rho\|U_{2,N}\|_l^2&\\
&+\sum_{n=1}^N k_n\Big\{-\partial_n\eta_n\|U_{1,n-1}\|_{l+1}^2+k_n\eta_n\|\partial_nU_{1,n}\|_{l+1}^2\Big\}\\
&+\sum_{n=2}^N\sum_{j=1}^{n-1}\int_{I_n}\int_{I_j}\beta(t-s)\,ds\,dt\Big\{\partial_n\|W_{1,n,j}\|_{l+1}^2+k_n\|\partial_nW_{1,n,j}\|_{l+1}^2\Big\}\\
&+\rho\sum_{n=0}^{N-1}\|[U_2]_n\|_l^2\\
&=\|u_0\|_{l+1}^2+\rho\|v_0\|_l^2\\
&+2\int_0^T\big\{\eta a(P_kf_1,A^lU_1)+(f_2,A^lU_2)\big\}\,dt\\
&+2\int_0^T\int_0^t\beta(t-s)a\Big(P_kf_1(t),A^l(U_1(t)-U_1(s))\Big)\,ds\,dt,
\end{aligned}
$$

where $W_{1,n,j}=U_{1,n}-U_{1,j}$. All terms on the left side are non-negative.

Moreover, for some $C=C(\gamma,\rho)$, we have the stability estimate

$$
\|U_{1,N}\|_{l+1}+\|U_{2,N}\|_l\le C\Big\{\|u_0\|_{l+1}+\|v_0\|_l+\int_0^T\|f_1\|_{l+1}+\|f_2\|_l\,dt\Big\}. \tag{4.8}
$$

Proof. We organize our proof in five steps.

1. First, we find a representation of U_2 in terms of U_1 and f_1. Setting $V_2=0$ in (4.1), we have

$$
\begin{aligned}
\sum_{n=1}^N\int_{I_n}\big\{a(\dot U_1,V_1)-a(U_2,V_1)\big\}\,dt+\sum_{n=1}^{N-1}a([U_1]_n,V_{1,n}^+)+a(U_{1,0}^+,V_{1,0}^+)&\\
=\sum_{n=1}^N\int_{I_n}a(f_1,V_1)\,dt+a(u_0,V_{1,0}^+),&
\end{aligned}
$$

that, considering the fact that $U_i,\ i=1,2$, are piecewise constant with respect to time, $\dot U_1=0$ and recalling (4.6), we have

$$
\begin{aligned}
-\sum_{n=1}^N k_na(U_{2,n},V_{1,n})+\sum_{n=2}^N a([U_1]_{n-1},V_{1,n})+a(U_{1,1},V_{1,1})&\\
=\sum_{n=1}^N k_na(P_{k,n}f_1,V_{1,n})\,dt+a(u_0,V_{1,1}).&
\end{aligned}
$$

Now, for some $n\in\{1,\dots,N\}$, we take $V_{1,n}\neq0$ and $V_1=0$ otherwise, and we have

$$
-k_na(U_{2,n},V_{1,n})+a(U_{1,n}-U_{1,n-1},V_{1,n})=k_na(P_{k,n}f_1,V_{1,n}),
$$

that implies

$$
U_{2,n}=\frac{U_{1,n}-U_{1,n-1}}{k_n}-P_{k,n}f_1=\partial_nU_{1,n}-P_{k,n}f_1. \tag{4.9}
$$

2. Now, recalling function η from (2.4), we use the representation

$$
\begin{aligned}
a(U_1, V_2) &- \int_0^t \beta(t-s) a(U_1(s), V_2(t))\, ds \\
&= \eta(t) a(U_1, V_2) + \int_0^t \beta(t-s) a(U_1(t) - U_1(s), V_2(t))\, ds,
\end{aligned}
$$

and we set $V = A^l U,\ l \in \mathbb{R}$ in (4.1), to obtain

$$
\begin{aligned}
&\sum_{n=1}^{N} \int_{I_n} \Big\{ a(\dot{U}_1, A^l U_1) - a(U_2, A^l U_1) + \rho(\dot{U}_2, A^l U_2) + \eta(t) a(U_1, A^l U_2) \\
&\qquad + \int_0^t \beta(t-s) a(U_1(t) - U_1(s), A^l U_2(t))\, ds \Big\}\, dt \\
&\qquad + \sum_{n=1}^{N-1} \big\{ a([U_1]_n, A^l U_{1,n}^+) + \rho([U_2]_n, A^l U_{2,n}^+) \big\} \\
&\qquad + a(U_{1,0}^+, A^l U_{1,0}^+) + \rho(U_{2,0}^+, A^l U_{2,0}^+) \\
&\quad = \int_0^T a(f_1, A^l U_1) + (f_2, A^l U_2)\, dt + a(u_0, A^l U_{1,0}^+) + \rho(v_0, A^l U_{2,0}^+).
\end{aligned}
\tag{4.10}
$$

Then, using (4.9) and $\dot{U}_1 = 0$ we have

$$
\begin{aligned}
&\sum_{n=1}^{N} \int_{I_n} \big\{ a(\dot{U}_1, A^l U_1) - a(U_2, A^l U_1) \big\}\, dt + \sum_{n=1}^{N-1} a([U_1]_n, A^l U_{1,n}^+) + a(U_{1,0}^+, A^l U_{1,0}^+) \\
&\quad = \sum_{n=1}^{N} \int_{I_n} -a(\partial_n U_{1,n} - P_{k,n} f_1, A^l U_{1,n})\, dt \\
&\qquad + \sum_{n=1}^{N-1} a(U_{1,n+1} - U_{1,n}, A^l U_{1,n+1}) + a(U_{1,1}, A^l U_{1,1}) \\
&\quad = -\sum_{n=1}^{N} k_n a\Big(\frac{U_{1,n} - U_{1,n-1}}{k_n} - P_{k,n} f_1, A^l U_{1,n} \Big)\, dt \\
&\qquad + \sum_{n=1}^{N-1} a(U_{1,n+1} - U_{1,n}, A^l U_{1,n+1}) + a(U_{1,1}, A^l U_{1,1}),
\end{aligned}
$$

that, recalling (4.6), we have

$$
\begin{aligned}
\sum_{n=1}^{N}\int_{I_n}&\big\{a(\dot U_1,A^lU_1)-a(U_2,A^lU_1)\big\}\,dt+\sum_{n=1}^{N-1}a([U_1]_n,A^lU_{1,n}^+)+a(U_{1,0}^+,A^lU_{1,0}^+)\\
&=-\sum_{n=2}^{N}a(U_{1,n},A^lU_{1,n})+\sum_{n=2}^{N}a(U_{1,n-1},A^lU_{1,n})\\
&\qquad-a(U_{1,1},A^lU_{1,1})+a(U_{1,0},A^lU_{1,1})\\
&\quad+\sum_{n=2}^{N}a(U_{1,n},A^lU_{1,n})-\sum_{n=2}^{N}a(U_{1,n-1},A^lU_{1,n})\\
&\qquad+a(U_{1,1},A^lU_{1,1})+\sum_{n=1}^{N}\int_{I_n}a(f_1,A^lU_{1,n})\,dt\\
&=a(U_{1,0},A^lU_{1,1})+\int_0^T a(f_1,A^lU_1)\,dt\\
&=a(u_0,A^lU_{1,0}^+)+\int_0^T a(f_1,A^lU_1)\,dt.
\end{aligned}
$$

From this and $\dot U_2=0$ we can write (4.10) as

$$
\begin{aligned}
&\sum_{n=1}^{N}\int_{I_n}\eta(t)a(U_1,A^lU_2)\,dt\\
&\quad+\sum_{n=1}^{N}\int_{I_n}\int_0^t\beta(t-s)a(U_1(t)-U_1(s),A^lU_2(t))\,ds\,dt\\
&\quad+\rho\sum_{n=1}^{N-1}([U_2]_n,A^lU_{2,n}^+)+\rho(U_{2,0}^+,A^lU_{2,0}^+)-\rho(v_0,A^lU_{2,0}^+)\\
&=\int_0^T(f_2,A^lU_2)\,dt.
\end{aligned}
\tag{4.11}
$$

Now, we need to study the three terms on the left side.

3. For the first term on the left side of (4.11), recalling (4.9) and η_n from (4.2), we have

$$
\begin{aligned}
\sum_{n=1}^{N}\int_{I_n}\eta(t)a(U_1(t),A^lU_2(t))\,dt&=\sum_{n=1}^{N}\int_{I_n}\eta(t)a(U_{1,n},A^lU_{2,n})\;dt\\
&=\sum_{n=1}^{N}k_n\eta_n a(A^{l/2}U_{1,n},\partial_nA^{l/2}U_{1,n})\\
&\quad-\sum_{n=1}^{N}\int_{I_n}\eta(t)a(U_{1,n},A^lP_{k,n}f_1(t))\;dt,
\end{aligned}
$$

that, using (4.4) and (4.5), implies

$$
\begin{aligned}
&\sum_{n=1}^{N}\int_{I_n}\eta(t)a(U_1(t),A^lU_2(t))\,dt\\
&\qquad=\frac{1}{2}\sum_{n=1}^{N}k_n\eta_n\Big\{\partial_n a(A^{l/2}U_{1,n},A^{l/2}U_{1,n})+k_na(\partial_nA^{l/2}U_{1,n},\partial_nA^{l/2}U_{1,n})\Big\}\\
&\qquad\quad-\sum_{n=1}^{N}\int_{I_n}\eta a(U_{1,n},A^lP_{k,n}f_1)\ dt\\
&\qquad=\frac{1}{2}\sum_{n=1}^{N}k_n\partial_n\Big\{\eta_n a(A^{l/2}U_{1,n},A^{l/2}U_{1,n})\Big\}\\
&\qquad\quad-\frac{1}{2}\sum_{n=1}^{N}k_na(A^{l/2}U_{1,n-1},A^{l/2}U_{1,n-1})\partial_n\eta_n\\
&\qquad\quad+\frac{1}{2}\sum_{n=1}^{N}k_n^2\eta_na(\partial_nA^{l/2}U_{1,n},\partial_nA^{l/2}U_{1,n})\\
&\qquad\quad-\sum_{n=1}^{N}\int_{I_n}\eta a(U_{1,n},A^lP_{k,n}f_1)\,dt,
\end{aligned}
$$

so we have

$$
\begin{aligned}
&\sum_{n=1}^{N}\int_{I_n}\eta(t)a(U_1(t),A^lU_2(t))\,dt\\
&\qquad=\frac{1}{2}\eta_Na(A^{l/2}U_{1,N},A^{l/2}U_{1,N})-\frac{1}{2}\eta_0a(A^{l/2}U_{1,0},A^{l/2}U_{1,0})\\
&\qquad\quad-\frac{1}{2}\sum_{n=1}^{N}k_na(A^{l/2}U_{1,n-1},A^{l/2}U_{1,n-1})\partial_n\eta_n\\
&\qquad\quad+\frac{1}{2}\sum_{n=1}^{N}k_n^2\eta_na(\partial_nA^{l/2}U_{1,n},\partial_nA^{l/2}U_{1,n})-\int_0^T\eta a(U_1,A^lP_kf_1)\,dt.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\sum_{n=1}^{N}\int_{I_n}\eta(t)a(U_1(t),A^lU_2(t))\,dt\\
&\qquad=\frac{1}{2}\eta_N\|U_{1,N}\|_{l+1}^2-\frac{1}{2}\|u_0\|_{l+1}^2\\
&\qquad\quad+\frac{1}{2}\sum_{n=1}^{N}k_n\big\{-\partial_n\eta_n\|U_{1,n-1}\|_{l+1}^2+k_n\eta_n\|\partial_nU_{1,n}\|_{l+1}^2\big\}\\
&\qquad\quad-\int_0^T\eta a(U_1,A^lP_kf_1)\,dt.
\end{aligned}
\tag{4.12}
$$

Here, we note that $\partial_n \eta_n < 0$. Indeed, changing the variable $t = t_{n-1} + k_n s$, for $t \in I_n$, $n \geq 2$, we have

$$\eta_n = \frac{1}{k_n} \int_{I_n} \eta(t)\, dt = \int_0^1 \eta(t_{n-1} + sk_n)\, ds,$$

that implies

$$\partial_n \eta_n = \frac{1}{k_n} \int_0^1 \Big(\eta(t_{n-1} + sk_n) - \eta(t_{n-2} + sk_{n-1}) \Big)\, ds < 0,$$

since η is a decreasing function by (2.5). And, for $n = 1$, we have

$$\partial_1 \eta_1 = \frac{1}{k_1}(\eta_1 - \eta_0) = -\frac{1}{k_1^2} \int_{I_1} \int_0^t \beta(s)\, ds\, dt < 0.$$

Now, we study the second term on the left side of (4.11), that is the convolution integral. Recalling (4.9), we have

$$\begin{aligned}
&\sum_{n=1}^N \int_{I_n} \int_0^t \beta(t-s) a(U_1(t) - U_1(s), A^l U_2(t))\, ds\, dt \\
&\quad = \sum_{n=1}^N \sum_{j=1}^n \int_{I_n} \int_{t_{j-1}}^{t_j \wedge t} \beta(t-s)\, ds\, dt\, a\big(U_{1,n} - U_{1,j}, A^l(\partial_n U_{1,n} - P_{k,n} f_1)\big) \\
&\quad = \sum_{n=2}^N \sum_{j=1}^{n-1} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt\, a\big(U_{1,n} - U_{1,j}, A^l(\partial_n U_{1,n} - P_{k,n} f_1)\big),
\end{aligned}$$

that, recalling $W_{1,n,j} = U_{1,n} - U_{1,j}$, and using (4.5), yields

$$\begin{aligned}
&\sum_{n=1}^N \int_{I_n} \int_0^t \beta(t-s) a(U_1(t) - U_1(s), A^l U_2(t))\, ds\, dt \\
&\quad = \sum_{n=2}^N \sum_{j=1}^{n-1} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt\, a\big(W_{1,n,j}, A^l(\partial_n W_{1,n,j} - P_{k,n} f_1)\big) \\
&\quad = \frac{1}{2} \sum_{n=2}^N \sum_{j=1}^{n-1} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt\, \partial_n a(A^{l/2} W_{1,n,j}, A^{l/2} W_{1,n,j}) \\
&\qquad + \frac{1}{2} \sum_{n=2}^N k_n \sum_{j=1}^{n-1} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt\, a(\partial_n A^{l/2} W_{1,n,j}, \partial_n A^{l/2} W_{1,n,j}) \\
&\qquad - \sum_{n=2}^N \sum_{j=1}^{n-1} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt\, a(W_{1,n,j}, A^l P_{k,n} f_1).
\end{aligned}$$

Consequently, we have

$$
\begin{aligned}
\sum_{n=1}^{N} \int_{I_n} \int_0^t & \beta(t-s) a(U_1(t)-U_1(s), A^l U_2(t))\, ds\, dt \\
&= \frac{1}{2} \sum_{n=2}^{N} \sum_{j=1}^{n-1} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt\, \partial_n \|W_{1,n,j}\|_{l+1}^2 \\
&\quad + \frac{1}{2} \sum_{n=2}^{N} k_n \sum_{j=1}^{n-1} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt\, \|\partial_n W_{1,n,j}\|_{l+1}^2 \\
&\quad - \int_0^T \int_0^t \beta(t-s) a(U_1(t)-U_1(s), A^l P_k f_1)\, ds\, dt,
\end{aligned} \tag{4.13}
$$

where the second term at the right side is non-negative, since the kernel β is a decreasing function. So we need to show that the first term is also non-negative. To this end, denoting

$$
\beta_{n,j} = \frac{1}{k_n k_j} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt,
$$

we have

$$
\begin{aligned}
\sum_{n=2}^{N} \sum_{j=1}^{n-1} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt\, \partial_n \|W_{1,n,j}\|_{l+1}^2 &= \sum_{n=2}^{N} k_n \sum_{j=1}^{n-1} k_j \beta_{n,j}\, \partial_n \|W_{1,n,j}\|_{l+1}^2 \\
&= \sum_{j=1}^{N-1} k_j \sum_{n=2}^{N} k_n \partial_n \{\beta_{n,j} \|W_{1,n,j}\|_{l+1}^2\} \\
&\quad - \sum_{j=1}^{N-1} k_j \sum_{n=2}^{N} k_n \|W_{1,n-1,j}\|_{l+1}^2\, \partial_n \beta_{n,j},
\end{aligned}
$$

where we changed the order of summation and used (4.4) for the last equality. Now it is necessary to show that both terms at the right side are non-negative. For the first term we have

$$
\begin{aligned}
\sum_{j=1}^{N-1} k_j & \sum_{n=j+1}^{N} k_n \partial_n \{\beta_{n,j} \|W_{1,n,j}\|_{l+1}^2\} \\
&= \sum_{j=1}^{N-1} k_j \sum_{n=j+1}^{N} \{\beta_{n,j} \|W_{1,n,j}\|_{l+1}^2 - \beta_{n-1,j} \|W_{1,n-1,j}\|_{l+1}^2\} \\
&= \sum_{j=1}^{N-1} k_j \{\beta_{N,j} \|W_{1,N,j}\|_{l+1}^2 - \beta_{j,j} \|W_{1,j,j}\|_{l+1}^2\} \\
&= \sum_{j=1}^{N-1} k_j \beta_{N,j} \|W_{1,N,j}\|_{l+1}^2 > 0.
\end{aligned}
$$

For the second term, we should show that $\partial_n \beta_{n,j} < 0$. Indeed, changing the variable $t = t_{n-1} + k_n \tau$, we have

$$
\beta_{n,j} = \frac{1}{k_n k_j} \int_{I_n} \int_{I_j} \beta(t-s)\, ds\, dt = \frac{1}{k_j} \int_{I_j} \int_0^1 \beta(t_{n-1} + \tau k_n - s)\, d\tau\, ds,
$$

and consequently

$$\partial_n \beta_{n,j} = \frac{1}{k_j} \int_{I_j} \int_0^1 (\beta(t_{n-1} + \tau k_n - s) - \beta(t_{n-2} + \tau k_{n-1} - s) \, d\tau \, ds < 0,$$

since the kernel β is decreasing.

Finally, it remains to study the third part on the left side of (4.11). Recalling $v_0 = U_{2,0}^-$ we have

$$\begin{aligned} \rho \sum_{n=1}^{N-1} ([U_2]_n, A^l U_{2,n}^+) &+ \rho(U_{2,0}^+, A^l U_{2,0}^+) - \rho(v_0, A^l U_{2,0}^+) \\ &= \rho \sum_{n=0}^{N-1} ([U_2]_n, A^l U_{2,n}^+) = \rho \sum_{n=0}^{N-1} k_{n+1} (\partial_n U_{2,n+1}, A^l U_{2,n+1}), \end{aligned}$$

that by (4.5) implies

$$\begin{aligned} \rho \sum_{n=1}^{N-1} & ([U_2]_n, A^l U_{2,n}^+) + \rho(U_{2,0}^+, A^l U_{2,0}^+) - \rho(v_0, A^l U_{2,0}^+) \\ &= \frac{1}{2} \rho \sum_{n=0}^{N-1} \Big\{ k_{n+1} \partial_n (A^{l/2} U_{2,n+1}, A^{l/2} U_{2,n+1}) \\ &\qquad + k_{n+1}^2 (\partial_n A^{l/2} U_{2,n+1}, \partial_n A^{l/2} U_{2,n+1}) \Big\} \\ &= \frac{1}{2} \rho (A^{l/2} U_{2,N}, A^{l/2} U_{2,N}) - \frac{1}{2} \rho (A^{l/2} U_{2,0}, A^{l/2} U_{2,0}) \\ &\qquad + \frac{1}{2} \rho \sum_{n=0}^{N-1} (A^{l/2} [U_2]_n, A^{l/2} [U_2]_n) \\ &= \frac{1}{2} \rho \|U_{2,N}\|_l^2 - \frac{1}{2} \rho \|v_0\|_l^2 + \frac{1}{2} \rho \sum_{n=0}^{N-1} \|[U_2]_n\|_l^2. \end{aligned} \tag{4.14}$$

4. Hence, putting (4.12), (4.13) and (4.14) in (4.11), we conclude the energy identity (4.7).

5. Finally, we prove the stability estimate (4.8). Recalling the fact that all terms on the left side of the stability identity (4.7) are non-negative, we have

$$\begin{aligned} \eta_N \|U_{1,N}\|_{l+1}^2 &+ \rho \|U_{2,N}\|_l^2 \\ &\le \|u_0\|_{l+1}^2 + \rho \|v_0\|_l^2 + 2 \int_0^T \big\{ \eta a(P_k f_1, A^l U_1) + (f_2, A^l U_2) \big\} \, dt \\ &\quad + 2 \int_0^T \int_0^t \beta(t-s) a\Big(P_k f_1(t), A^l (U_1(t) - U_1(s)) \Big) \, ds \, dt, \end{aligned}$$

Then, using the Cauchy-Schwarz inequality, and the facts that $\eta \le 1$, $\|\beta\|_{L_1(0,\infty)} = \gamma$ and

$$\int_0^T |P_k f| \, dt \le \int_0^T |f| \, dt,$$

in a classical way, we conclude the stability estimate (4.8), for some constant $C = C(\gamma, \rho)$. Now the proof is complete. □

5. A priori error estimates

Here, we prove optimal order a priori error estimates for the displacement $u_1 = u$ and the velocity $u_2 = \dot{u}$.

We denote the standard piecewise constant interpolation of a function v with $\tilde{v}$, corresponding to the partition $0 = t_0 < t_1, \ldots < t_N = T$ of the interval $(0, T)$. We also recall the error estimates

$$\int_{I_n} |\tilde{v} - v| \, dt \leq C k_n \int_{I_n} |\dot{v}| \, dt. \tag{5.1}$$

Theorem 2. *Let* (u_1, u_2) *and* (U_1, U_2) *be the solutions of* (3.3) *and* (3.2), *respectively. Then, with* $e = (e_1, e_2) = (U_1, U_2) - (u_1, u_2)$ *and* $C = C(\gamma, \rho)$, *we have*

$$\|e_{1,N}\|_1 + \|e_{2,N}\| \leq C \sum_{n=1}^{N} k_n \int_{I_n} \big\{ \|\dot{u}_2\|_1 + \|\dot{u}_1\|_2 \big\} \, dt, \tag{5.2}$$

$$\|e_{1,N}\| \leq C \sum_{n=1}^{N} k_n \int_{I_n} \big\{ \|\dot{u}_2\| + \|\dot{u}_1\|_1 \big\} \, dt. \tag{5.3}$$

Proof. We set

$$e = (U_1, U_2) - (u_1, u_2) = \big((U_1, U_2) - (\tilde{u}_1, \tilde{u}_2) \big) + \big((\tilde{u}_1, \tilde{u}_2) - (u_1, u_2) \big) = \theta + \omega,$$

where $\tilde{u}_i, i = 1, 2$, is the standard piecewise constant interpolation of u_i. We can estimate ω by (5.1), so we need to find estimates for θ. Recalling Galerkin's orthogonality (3.4), we have

$$\begin{aligned}
B(\theta, V) &= -B(e, V) - B(\omega, V) = -B(\omega, V) \\
&= \sum_{n=1}^{N} \int_{I_n} \Big\{ -a(\dot{\omega}_1, V_1) + a(\omega_2, V_1) \\
&\qquad - \rho(\dot{\omega}_2, V_2) - a(\omega_1, V_2) + \int_0^t \beta(t-s) a(\omega_1(s), V_2(t)) \, ds \Big\} \, dt \\
&\quad - \sum_{n=1}^{N-1} \big\{ a([\omega_1]_n, V_{1,n}^+) + \rho([\omega_2]_n, V_{2,n}^+) \big\} \\
&\quad - a(\omega_{1,0}^+, V_{1,0}^+) - \rho(\omega_{2,0}^+, V_{2,n}^+),
\end{aligned}$$

and, having the fact that $\omega_i, i = 1, 2$, vanish at the time nodes and V_i are piecewise constant functions, we have

$$B(\theta, V) = \sum_{n=1}^{N} \int_{I_n} \Big\{ a(\omega_2, V_1) + \Big(-A\big\{ \omega_1 + \int_0^t \beta(t-s) \omega_1(s) \, ds \big\}, V_2 \Big) \Big\} \, dt.$$

Therefore θ satisfies (4.1) with $f_1 = \omega_2$ and $f_2 = -A\big\{\omega_1 + \int_0^t \beta(t-s)\omega_1(s)\,ds\big\}$. Hence, applying the stability estimate (4.8) and recalling $\theta_{i,0} = \theta_i(0) = 0$, we have

$$\begin{aligned}
\|\theta_{1,N}\|_{l+1} + \|\theta_{2,N}\|_l &\le C\Big\{\|\theta_{1,0}\|_{l+1} + \|\theta_{2,0}\|_l + \int_0^T \|\omega_2\|_{l+1}\,dt \\
&\qquad + \int_0^T \|A\omega_1\|_l + \Big\|\int_0^t \beta(t-s)A\omega_1(s)\,ds\Big\|_l\,dt\Big\} \\
&\le C\Big\{\int_0^T \|\omega_2\|_{l+1}\,dt + \int_0^T \|A\omega_1\|_l \\
&\qquad + \Big\|\int_0^t \beta(t-s)A\omega_1(s)\,ds\Big\|_l\,dt\Big\}.
\end{aligned} \tag{5.4}$$

Now, we consider two choices $l = 0, -1$.

To prove the first a priori error estimate (5.2), we set $l = 0$. Then, recalling $e = \theta + \omega$ and $\omega_{i,N} = 0$, we have

$$\|e_{1,N}\|_1 + \|e_{2,N}\| \le C\Big\{\int_0^T \|\omega_2\|_1\,dt + \int_0^T \|A\omega_1\| + \Big\|\int_0^t \beta(t-s)A\omega_1(s)\,ds\Big\|\,dt\Big\}.$$

Now, using (5.1), we have

$$\begin{aligned}
\int_0^T \|\omega_2\|_1\,dt &= \sum_{n=1}^N \int_{I_n} \|\tilde u_2 - u_2\|_1\,dt \le C\sum_{n=1}^N k_n \int_{I_n} \|\dot u_2\|_1\,dt, \\
\int_0^T \|A\omega_1\|\,dt &= \sum_{n=1}^N \int_{I_n} \|A(\tilde u_1 - u_1)\|\,dt \le C\sum_{n=1}^N k_n \int_{I_n} \|A\dot u_1\|\,dt, \\
\int_0^T \Big\|\int_0^t \beta(t-s)A\omega_1(s)\,ds\Big\|\,dt &\le C\int_0^T \int_0^t \beta(t-s)\|A\omega_1(s)\|\,ds\,dt \\
&\le C\int_0^T \beta\,dt \int_0^T \|A\omega_1\|\,dt \\
&\le C\gamma \sum_{n=1}^N \int_{I_n} \|A(\tilde u_1 - u_1)\|\,dt \\
&\le C\sum_{n=1}^N k_n \int_{I_n} \|A\dot u_1\|\,dt,
\end{aligned}$$

that, having $\|Av\| \le \|v\|_2$, implies the first a priori error estimate (5.2).

For the second error estimate we choose $l = -1$ in (5.4). Then, recalling $e = \theta + \omega$ and $\omega_{i,N} = 0$, we have

$$\begin{aligned}
&\|e_{1,N}\| + \|e_{2,N}\|_{-1} \\
&\qquad \le C\Big\{\int_0^T \|\omega_2\|\,dt + \int_0^T \|A\omega_1\|_{-1} + \Big\|\int_0^t \beta(t-s)A\omega_1(s)\,ds\Big\|_{-1}\,dt\Big\}.
\end{aligned}$$

Now, using (5.1), we have

$$\int_0^T \|\omega_2\|\,dt = \sum_{n=1}^N \int_{I_n} \|\tilde{u}_2 - u_2\|\,dt \le C \sum_{n=1}^N k_n \int_{I_n} \|\dot{u}_2\|\,dt,$$
$$\int_0^T \|A\omega_1\|_{-1}\,dt = \sum_{n=1}^N \int_{I_n} \|A(\tilde{u}_1 - u_1)\|_{-1}\,dt \le C \sum_{n=1}^N k_n \int_{I_n} \|A\dot{u}_1\|_{-1}\,dt,$$
$$\begin{aligned}\int_0^T \Big\| \int_0^t \beta(t-s)A\omega_1(s)\,ds\Big\|_{-1}\,dt &\le C \int_0^T \int_0^t \beta(t-s)\|A\omega_1(s)\|_{-1}\,ds\,dt\\ &\le C \int_0^T \beta\,dt \int_0^T \|A\omega_1\|_{-1}\,dt\\ &\le C\gamma \sum_{n=1}^N \int_{I_n} \|A(\tilde{u}_1 - u_1)\|_{-1}\,dt\\ &\le C \sum_{n=1}^N k_n \int_{I_n} \|A\dot{u}_1\|_{-1}\,dt,\end{aligned}$$

that, having $\|Av\|_{-1} \le \|v\|_1$, implies the second a priori error estimate (5.3). Now the proof is complete. $\square$

6. Numerical example

In this section we illustrate that dG(0) method capture the behavior of the solution and also its rate of convergence $O(k)$, by solving an example for a two dimensional square shape structure.

We consider the domain be the two dimensional unit square and the initial conditions: $u(x,0) = 0\,\mathrm{m}$, $\dot{u}(x,0) = 0\,\mathrm{m/s}$, the boundary conditions: $u = 0$ at $x = 0$, $g = (0,-1)\,\mathrm{Pa}$ at $x = 1$ and zero on the rest of the boundary. The volume load is assumed to be $f = 0\,\mathrm{N/m^3}$. The model parameters are: $\gamma = 0.5$, $\tau = 1, \alpha = 2/3$ and $\rho = 3000\,\mathrm{kg/m^3}$. The oscillatory behavior of the the solution of the model problem is illustrated in Figure 1, for different time steps $k_n = 2^{-5}, 2^{-6}$.

We also verify numerically the temporal rate of convergence $O(k)$ for $\|e_{1,N}\|$. Lacking of an explicit solution we compare with a numerical solution with fine mesh sizes h, k. Here we consider $h = 0.089095$, $k_{\min} = 2^{-6}$. The result is displayed in Figure 2.

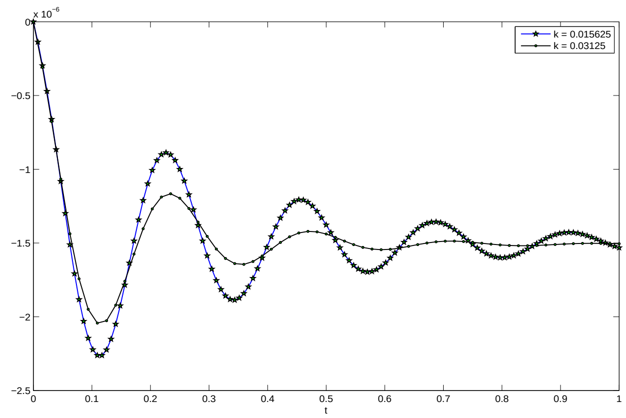

FIGURE 1. Oscilatory behavior of point $(1, 1)$ of the 2D unit square domain.

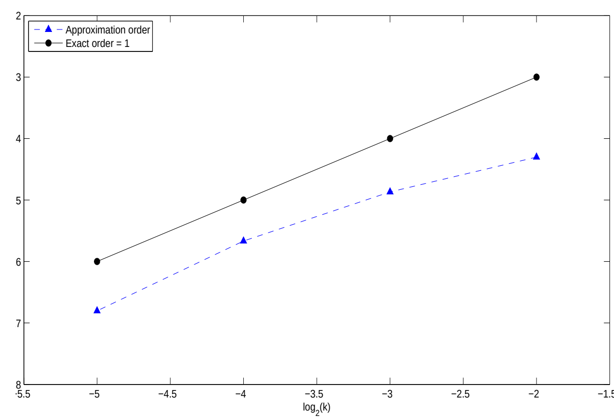

FIGURE 2. Rate of convergence of temporal discretization.

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Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE–412 96 Göteborg, Sweden
E-mail address: stig@chalmers.se

Department of Mathematics, Technical University of Gabrovo, 5300 Gabrovo, Bulgaria
E-mail address: milena@tugab.bg

Department of Mathematics, University of Kurdistan, P. O. Box 416, Sanandaj, Iran
E-mail address: f.saedpanah@uok.ac.ir, f_saedpanah@yahoo.com