ADAPTIVE DISCRETIZATION OF AN INTEGRO-DIFFERENTIAL EQUATION MODELING QUASI-STATIC FRACTIONAL ORDER VISCOELASTICITY

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Abstract: We study a quasi-static model for viscoelastic materials based on a constitutive equation of fractional order. In the quasi-static case this results in a Volterra integral equation of the second kind with a weakly singular kernel in the time variable involving also partial derivatives of second order in the spatial variables. We discretize by means of a discontinuous Galerkin finite element method in time and a standard continuous Galerkin finite element method in space. To overcome the problem of the growing amount of data that has to be stored and used in each time step, we introduce sparse quadrature in the convolution integral. We prove a priori and a posteriori error estimates, and develop an adaptive strategy based on the a posteriori error estimate.

Keywords: integro-differential equation, viscoelasticity, quasi-static, finite element, discontinous Galerkin, weakly singular kernel, error estimate, a priori, a posteriori, sparse quadrature, error estimate, adaptivity

1. INTRODUCTION

The fractional order viscoelastic model, *i.e.*, the linear viscoelastic model with fractional order operators in the constitutive equations, is capable of describing the behavior of many viscoelastic materials by using only a few parameters. The drawback of using fractional order operators in the constitutive equations is that they increase the mathematical complexity in the sense that the operators are nonlocal in time. This means that, when computing the fractional order derivative or integral, all function values from the previous time points need to be stored and used at each new time point. This leads to to an excessive use of memory and high computational cost. To make the fractional order models more practical to use in the analysis of complex viscoelastic structures, efficient algorithms that employ the discontinuous Galerkin method in time together with sparse quadratures have been developed in Adolfsson et al. (2003, 2004). Goal-oriented error estimates and adaptivity for the time integration are included in the algorithms.

It is important to be able to investigate the capability of the numerical model to produce simulations with high accuracy. For this reason estimates of the error due to discretization in both space and time, as well as adaptive strategies based on these estimations, need to be included. Our previous work emphasized the temporal discretization. Here we develop a space-time finite element formulation in the quasi-static case (*i.e.*, inertia effects are neglected). The formulation includes error estimates and an adaptive strategy. We use a convolution integral formulation of the fractional order viscoelastic model. The convolution kernel is the weakly singular and of Mittag-Leffler type. The resulting equation of motion is then a Volterrra integral equation of second kind with a weakly singular kernel in time and it involves partial derivatives of second order in space.

2. FRACTIONAL ORDER LINEAR VISCOELASTICITY

Let σ_{ij} and u_i denote the usual stress tensor and displacement vector and define the linear strain tensor:

$$\epsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

With the decompositions

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}, \quad e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij},$$

we formulate the constitutive equations, Bagley and Torvik (1983),

$$\begin{aligned}
&s_{ij}(t) + \tau_1^{\alpha_1} D_t^{\alpha_1} s_{ij}(t) \\
&\qquad = 2G_\infty e_{ij}(t) + 2G\tau_1^{\alpha_1} D_t^{\alpha_1} e_{ij}(t), \\
&\sigma_{kk}(t) + \tau_2^{\alpha_2} D_t^{\alpha_2} \sigma_{kk}(t) \\
&\qquad = 3K_\infty \epsilon_{kk}(t) + 3K\tau_2^{\alpha_2} D_t^{\alpha_2} \epsilon_{kk}(t),
\end{aligned}$$

with initial conditions

$$s_{ij}(0+) = 2Ge_{ij}(0+), \quad \sigma_{kk}(0+) = 3K\epsilon_{kk}(0+),$$

meaning that the initial response follows Hooke's elastic law. Note that we have two relaxation times, $\tau_1, \tau_2 > 0$, and fractional orders of differentiation, $\alpha_1, \alpha_2 \in (0,1)$, where the fractional order derivative is defined by

$$\begin{aligned}
D_t^\alpha f(t) &= D_t D_t^{-(1-\alpha)} f(t) \\
&= D_t \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s)\, ds.
\end{aligned}$$

We solve for σ by means of Laplace transformation, Enelund and Olsson (1999):

$$\begin{aligned}
s_{ij}(t) &= 2G\Big(e_{ij}(t) \\
&\qquad - \frac{G - G_\infty}{G} \int_0^t f_1(t-s) e_{ij}(s)\, ds\Big), \\
\sigma_{kk}(t) &= 3K\Big(\epsilon_{kk}(t) \\
&\qquad - \frac{K - K_\infty}{K} \int_0^t f_2(t-s) \epsilon_{kk}(s)\, ds\Big),
\end{aligned}$$

where

$$f_i(t) = -\frac{d}{dt} E_{\alpha_i}\Big(-\Big(\frac{t}{\tau_i}\Big)^{\alpha_i}\Big)$$

and

$$E_\alpha(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma(1+\alpha n)}$$

is the Mittag-Leffler function. We make the simplifying assumption (synchronous viscoelasticity):

$$\alpha = \alpha_1 = \alpha_2, \quad \tau = \tau_1 = \tau_2, \quad f = f_1 = f_2.$$

Then we may define a parameter γ, a kernel β, and the Lamé constants μ, λ,

$$\begin{aligned}
&\gamma = \frac{G - G_\infty}{G} = \frac{K - K_\infty}{K}, \\
&\beta(t) = \gamma f(t), \quad \mu = G, \quad \lambda = K - \tfrac{2}{3}G,
\end{aligned}$$

and the constitutive equations become

$$\begin{aligned}
\sigma_{ij}(t) &= \Big(2\mu\epsilon_{ij}(t) + \lambda\epsilon_{kk}(t)\delta_{ij}\Big) \\
&\quad - \int_0^t \beta(t-s)\Big(2\mu\epsilon_{ij}(s) + \lambda\epsilon_{kk}(s)\delta_{ij}\Big)\, ds.
\end{aligned}$$

Note that the viscoelastic part of the model contains only three parameters:

$$0 < \gamma < 1, \quad 0 < \alpha < 1, \quad \tau > 0.$$

The kernel is weakly singular:

$$\begin{aligned}
\beta(t) &= -\gamma \frac{d}{dt} E_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) \\
&= \gamma \frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{-1+\alpha} E'_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) \\
&\approx Ct^{-1+\alpha}, \quad t \to 0,
\end{aligned}$$

and we note the properties

$$\begin{aligned}
&\beta(t) \geq 0, \\
&\|\beta\|_{L_1(\mathbf{R}^+)} = \int_0^\infty \beta(t)\, dt \\
&\qquad = \gamma\Big(E_\alpha(0) - E_\alpha(\infty)\Big) = \gamma < 1.
\end{aligned}$$

The equations of motion now become:

$$\begin{aligned}
&\rho u_{i,tt} - \sigma_{ij,j} = f_i, && \text{in } \Omega, \\
&u_i = 0, && \text{on } \Gamma_{\mathrm{D}}, \\
&\sigma_{ij} n_j = g_i, && \text{on } \Gamma_{\mathrm{N}}.
\end{aligned} \tag{1}$$

We consider quasi-static motion, $\rho u_{i,tt} \approx 0$, in a domain $\Omega \subset \mathbf{R}^d$, $d = 1, 2, 3$. In the analysis below we consider only the displacement boundary condition, $\Gamma = \Gamma_{\mathrm{D}}$.

3. ABSTRACT FORMULATION

We introduce the L_2 norm and scalar product:

$$\|v\| = \Big(\int_\Omega v_i v_i\, dx\Big)^{1/2}, \ (f, v) = \int_\Omega f_i v_i\, dx,$$

and the function space:

$$V = \Big[H_0^1(\Omega)\Big]^3,$$

and a bilinear form on V:

$$a(u,v) = \int_\Omega \Big(2\mu\epsilon_{ij}(u)\epsilon_{ij}(v) + \lambda\epsilon_{ii}(u)\epsilon_{jj}(v)\Big)\,dx.$$

Recalling the constitutive equations we obtain the following weak formulation of the quasi-static equations of motion: find $u(t) \in V$ such that

$$\begin{aligned} a(u(t),v) = \int_0^t \beta(t-s)a(u(s),v)\,ds \\ + (f(t),v) \quad \forall v \in V. \end{aligned} \tag{2}$$

This corresponds to the strong formulation:

$$\begin{aligned} Au(t) &= \int_0^t \beta(t-s)Au(s)\,ds + f(t), \\ (Au)_i &= -(2\mu\epsilon_{ij}(u) + \lambda\epsilon_{kk}(u)\delta_{ij})_{,j}. \end{aligned} \tag{3}$$

4. REGULARITY OF SOLUTIONS

We assume that Ω is smooth, or a convex polyhedron, so that

$$\|v\|_{H^2(\Omega)} \le C_S\|Av\| \quad \forall v \in H^2(\Omega) \cap V.$$

Recalling

$$\beta(t) \ge 0, \quad \|\beta\|_{L_1(\mathbf{R}^+)} = \gamma < 1$$

it is easy to prove the spatial regularity for solutions of (3):

$$\|u\|_{L_\infty(0,T;H^2)} \le \frac{C_S}{1-\gamma}\|f\|_{L_\infty(0,T;L_2)}.$$

In order to prove temporal regularity we differentiate the equation with respecto to t and use Grönwall's inequality to obtain

$$\|u_t(t)\| = C(T)t^{-1+\alpha}\|f\|_{W^1_\infty(0,T;L_2)}, \ 0 < t \le T.$$

5. SPATIAL APPROXIMATION

We introduce a standard finite element space $V_h \subset V$ consisting of continuous piecewise linear functions on a triangulation of Ω. The spatially semidiscrete finite element problem is: find $u_h(t) \in V_h$ such that

$$\begin{aligned} a(u_h(t),v) = \int_0^t \beta(t-s)a(u_h(s),v)\,ds \\ + (f(t),v) \quad \forall v \in V_h. \end{aligned} \tag{4}$$

We begin by proving an a priori error estimate in the energy norm, defined by $\|v\|_V = \sqrt{a(v,v)}$. We use the Ritz projection $R_h : V \to V_h$ defined by

$$a(R_hv - v, v_h) = 0, \quad \forall v_h \in V_h.$$

Theorem 1. Let u and u_h denote the solutions of (2) and (4), respectively, and let $e(t) = u_h(t) - u(t)$ the denote the error. Then

$$\begin{aligned} \|e\|_{L_\infty(0,T;V)} &\le \frac{1+\gamma}{1-\gamma}\|R_hu - u\|_{L_\infty(0,T;V)} \\ &\le C\frac{1+\gamma}{1-\gamma}\|hD^2u\|_{L_\infty(0,T;L_2)} \\ &\le CC_S\frac{1+\gamma}{(1-\gamma)^2}h_{\max}\|f\|_{L_\infty(0,T;L_2)}. \end{aligned}$$

The next result is an a priori error estimate in the L_2-norm. It is proved by a duality argument based on the stationary adjoint problem with arbitrary data g:

$$\begin{cases} \psi \in V \\ a(w,\psi) = (w,g) \quad \forall w \in V. \end{cases}$$

The result is

Theorem 2. Let u and u_h denote the solutions of (2) and (4), respectively, and let $e(t) = u_h(t) - u(t)$ the denote the error. Assume the usual mesh condition:

$$|\nabla h(x)| \le c, \quad \text{with } c \text{ sufficiently small.}$$

Then

$$\begin{aligned} \|e\|_{L_\infty(0,T;L_2)} &\le \frac{1+\gamma}{1-\gamma}\|R_hu - u\|_{L_\infty(0,T;L_2)} \\ &\le C\frac{1+\gamma}{1-\gamma}\|h^2D^2u\|_{L_\infty(0,T;L_2)} \\ &\le CC_S\frac{1+\gamma}{(1-\gamma)^2}h^2_{\max}\|f\|_{L_\infty(0,T;L_2)}. \end{aligned}$$

A priori error estimates for equations with smooth kernel were proved by Pani et al. (1992).

We next turn to a posteriori error estimates. We introduce the residual:

$$\begin{aligned} \langle \mathcal{R}(t), v\rangle = a(u_h(t),v) - \int_0^t \beta(t-s)a(u_h(s),v)\,ds \\ - (f(t),v) \quad \forall v \in V, \end{aligned}$$

and note that it satisfies the orthogonality relation:

$$\langle \mathcal{R}(t), v\rangle = 0 \quad \forall v \in V_h.$$

We obtain an equation for the error:

$$\begin{cases} e(t) \in V \\ a(e(t),v) - \displaystyle\int_0^t \beta(t-s)a(e(s),v)\,ds \\ \qquad = \langle \mathcal{R}(t), v\rangle \quad \forall v \in V. \end{cases}$$

We prove

Theorem 3. Let u and u_h denote the solutions of (2) and (4), respectively, and let $e(t) = u_h(t) - u(t)$ the denote the error. Then

$$\|e\|_{L_\infty(0,T;V)} \le \frac{C}{1-\gamma}\|hR\|_{L_\infty(0,T;L_2)},$$

where the estimator is divided into three parts:

$$\begin{aligned}\|hR\|_{L_\infty(0,T;L_2)} = \|hR_1\|_{L_\infty(0,T;L_2)} \\ + \|hR_2\|_{L_\infty(0,T;L_2)} \\ + \|hR_3\|_{L_\infty(0,T;L_2)}.\end{aligned}$$

The computational residuals are defined piecewise, i.e., for each mesh simplex K

$$\begin{aligned}R_1(t) = -\nabla\cdot\sigma(u_h(t)) \\ + \int_0^t \beta(t-s)\nabla\cdot\sigma(u_h(s))\,ds - f(t),\end{aligned}$$

$$R_2(t) = \frac{1}{2}h^{-1/2}|K|^{-1/2}\|\big[\sigma(u_h(t))\cdot n\big]\|_{L_2(\partial K)},$$

$$\begin{aligned}R_3(t) = \frac{1}{2}h^{-1/2}|K|^{-1/2}\int_0^t \beta(t-s) \\ \times \|\big[\sigma(u_h(s))\cdot n\big]\|_{L_2(\partial K)}\,ds,\end{aligned}$$

and

$$\sigma(u) = 2\mu\epsilon(u) + \lambda\nabla\cdot uI.$$

We also have an estimate in the L_2-norm.

Theorem 4. Let u and u_h denote the solutions of (2) and (4), respectively, and let $e(t) = u_h(t) - u(t)$ the denote the error. Then

$$\|e\|_{L_\infty(0,T;L_2)} \le \frac{CC_S}{1-\gamma}\|h^2R\|_{L_\infty(0,T;L_2)}.$$

6. TEMPORAL DISCRETIZATION – DISCONTINUOUS GALERKIN

We introduce a temporal mesh, $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$, with intervals $I_n = (t_{n-1}, t_n)$ and steps $k_n = t_n - t_{n-1}$, and discrete function space:

$$\mathcal{W}_D = \Big\{w : w(t) = w_n \text{ for } t \in I_n, \\ w_n \in V_h,\ n = 1,\dots,N\Big\}.$$

The completely discrete finite element problem is: find $U \in \mathcal{W}_D$, such that for $n = 1,\dots,N$

$$\begin{aligned}\int_{I_n}\Big(a(U(t),v(t)) - \int_0^t \beta(t-s)a(U(s),v(t))\,ds \\ - (f(t),v(t))\Big)\,dt = 0 \quad \forall v \in \mathcal{W}_D.\end{aligned}$$

Writing $U_n = U|_{I_n} \in V_h$, $v|_{I_n} = \chi \in V_h$, $(A_hU_n, v_h) = a(U_n, v_h)\ \forall v_h \in V_h$, we note that this is a time-stepping method, where in each step we solve the equation

$$A_hU_n - q_n(A_hU) - P_h\bar{f}_n = 0,$$

with

$$\begin{aligned}\bar{f}_n &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t, \\ q_n(A_hU) &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)A_hU(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j\wedge t}\beta(t-s)A_hU_j\,\mathrm{d}s\,\mathrm{d}t \\ &= \sum_{j=1}^{n} k_j\omega_{nj}A_hU_j, \\ \omega_{nj} &= \frac{1}{k_nk_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j\wedge t}\beta(t-s)\,\mathrm{d}s\,\mathrm{d}t, \\ & t_j\wedge t = \min(t_j,t).\end{aligned}$$

Thus, in each step, we have to solve

$$(I - k_n\omega_{nn})A_hU_n = \sum_{j=1}^{n-1}k_j\omega_{nj}A_hU_j + P_h\bar{f}_n,$$

where, for k_n small,

$$\begin{aligned}k_n\omega_{nn} &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t}\beta(t-s)\,\mathrm{d}s\,\mathrm{d}t \\ &\approx \frac{\gamma}{(1+\alpha)\Gamma(1+\alpha)}\Big(\frac{k_n}{\tau}\Big)^{\alpha} < 1.\end{aligned}$$

Therefore the equation is solvable. Note that the right-hand side of the above equation is a convolution sum, which requires that the whole history is stored and which must be re-computed in each time step. This leads to an excessive use of memory and high computational cost. This can be alleviated by means of sparse quadrature as shown in Adolfsson et al. (2003, 2004).

7. ERROR ESTIMATES

We prove an a priori error estimate:

$$\begin{aligned}\|e\|_{L_\infty(0,T;L_2)} &\le \frac{1+\gamma}{1-\gamma}\|R_hu - u\|_{L_\infty(0,T;L_2)} \\ &\quad + \frac{1}{1-\gamma}\|\bar{u} - u\|_{L_\infty(0,T;L_2)} \\ &\le C\frac{1+\gamma}{1-\gamma}\|h^2D^2u\|_{L_\infty(0,T;L_2)} \\ &\quad + \frac{1}{1-\gamma}\max_{1\le n\le N}\|u_t\|_{L_1(I_n;L_2)},\end{aligned}$$

where

$$\begin{aligned}&\max_{1\le n\le N}\|u_t\|_{L_1(I_n;L_2)} \\ &\quad \le \max_{1\le n\le N}\min\Big(\|u_t\|_{L_1(I_n;L_2)}, k_n\|u_t\|_{L_\infty(I_n;L_2)}\Big),\end{aligned}$$

so that

$$\|u_t\|_{L_1(I_1;L_2)} \approx C\int_0^{k_1} t^{-1+\alpha}\,dt = Ck_1^{\alpha}.$$

The following a posteriori error estimate is based on a time-dependent adjoint problem.

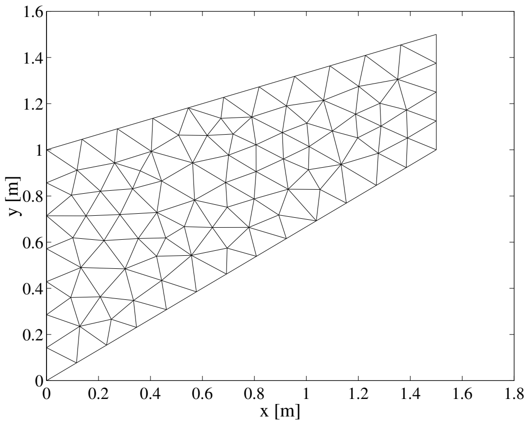

Fig. 1. Undeformed mesh.

$$\|e\|_{L_\infty(0,T;L_2)} \leq \frac{C_S}{1-\gamma}\Big(C\|h^2R\|_{L_\infty(0,T;L_2)} + 2\|R_4\|_{L_\infty(0,T;L_2)}\Big),$$

where the computational residual R is as in the previous theorems and the new residual is defined in each mesh simplex by

$$R_4(t) = A_hU(t) - \int_0^t \beta(t-s)A_hU(s)\,ds - P_hf(t).$$

8. NUMERICAL EXPERIMENT

We illustrate the theory by a numerical experiment: Cooke's membrane in two dimensions, see Fig. 1. We use the boundary conditions: $u = (0,0)$ at $x = 0$, $g = (0,-1)$ at $x = 1.5$, and $g = (0,0)$ on the remaining boundaries, cf. (1). We use the model parameters: $\gamma = 0.5$, $\tau = 0.5$, $\alpha = 0.5$. The deformed mesh at $t/\tau = 20$ is displayed in Fig. 2 with the displacement magnified by the factor 10^5. The time evolution of the node displacement at the point (1.5,1.5) is shown in Fig. 3.

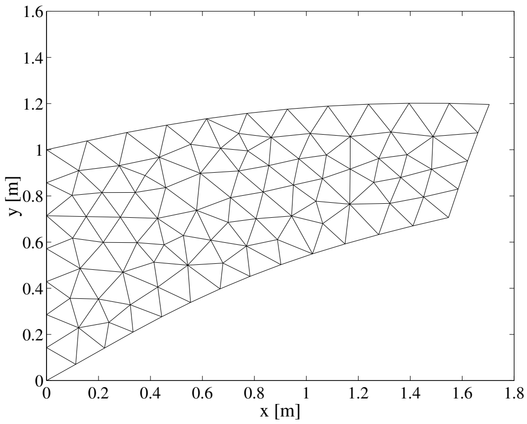

Fig. 2. Deformed mesh.

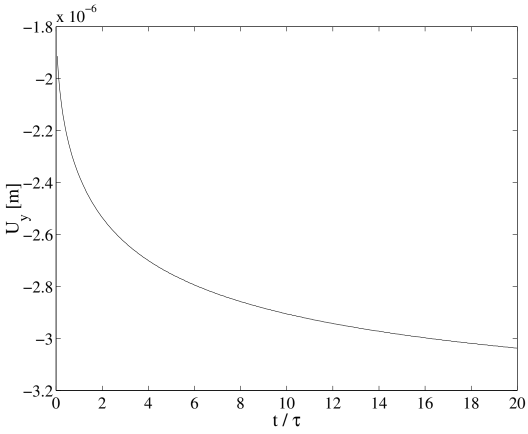

Fig. 3. Time evolution.

9. CONCLUSIONS

We have surveyed our results on a priori and a posteriori error estimates for completely discrete finite element method for the quasi-static motion of a viscoelastic material. We have also analyzed a time stepping method where the convolution integral is evaluated by a sparse quadrature rule, which reduces the storage requirement. The proofs and other details will appear in a forthcoming article.

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