Abstract

We present an abstract framework for semilinear parabolic problems based on analytic semigroup theory. The same framework is used for numerical discretization based on the finite element method. We prove local existence of solutions and local error estimates. These are applied in the context of dynamical systems. The framework is also used to analyze the finite element method for a stochastic parabolic equation.

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Semilinear parabolic partial differential equations—theory, approximation, and application

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1 The continuous problem

We consider the following initial-boundary value problem for a reaction-diffusion equation,

$$\begin{aligned} u_t - \Delta u &= \tilde f(u), && x \in \Omega,\ t>0, \\ u &= 0, && x \in \partial\Omega,\ t>0, \\ u(\cdot,0) &= u_0, && x \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d=1,2,3$, $u=u(x,t)$, $u_t=\partial u/\partial t$, $\Delta u = \sum_{i=1}^d \partial^2 u/\partial x_i^2$, and $\tilde f : \mathbf{R} \to \mathbf{R}$ is twice continuously differentiable.

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If $d = 2,3$ we assume, in addition, that the derivatives of $\tilde{f}$ satisfy the growth condition

$$|\tilde{f}^{(l)}(\xi)| \leq C(1 + |\xi|^{\delta+1-l}), \quad \xi \in \mathbf{R},\ l = 1,2, \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}$ and the induced operator norms, and where $\delta = 2$ if $d = 3$, $\delta \in [1,\infty)$ if $d = 2$.

Example 1.1. The Allen-Cahn equation. Let $\tilde{f}$ be

$$\tilde{f}(\xi) = -V'(\xi) = -(\xi^3 - \xi),$$

where V is the quadratic potential

$$V(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2.$$

Then the differential equation in (1.1) becomes

$$u_t - \Delta u = -(u^3 - u).$$

Clearly, $\tilde{f}$ satisfies (1.2) with $\delta = 2$ so we can have $d \leq 3$. $\square$

We assume that Ω is a convex polygonal domain (a polygon if $d = 2$, or polyhedron if $d = 3$), so that we have access to the elliptic regularity theory and so that finite element meshes can be fitted exactly to the domain. This is further explained below.

In the sequel we use the Hilbert space $H = L_2(\Omega)$, with its standard norm and inner product

$$\|v\| = \Big(\int_\Omega |v|^2\,dx\Big)^{1/2}, \quad (v,w) = \int_\Omega vw\,dx. \tag{1.3}$$

The norms in the Sobolev spaces $H^m(\Omega)$, $m \geq 0$, are denoted by

$$\|v\|_m = \left(\sum_{|\alpha|\leq m} \|D^\alpha v\|^2\right)^{1/2}. \tag{1.4}$$

The space $V = H_0^1(\Omega)$, with norm $\|\cdot\|_1$, consists of the functions in $H^1(\Omega)$ that vanish on $\partial\Omega$. $V^* = H^{-1}(\Omega)$ is the dual space of V with norm

$$\|v\|_{-1} = \sup_{\chi \in V} \frac{|(v,\chi)|}{\|\chi\|_1}. \tag{1.5}$$

If X,Y are Banach spaces, then by $\mathcal{L}(X,Y)$ we denote the space of bounded linear operators from X into Y, $\mathcal{L}(X) = \mathcal{L}(X,X)$, and $B_X(x,R)$ denotes the closed ball in X with center x and radius R. In particular, we let $B_R = B_V(0,R)$ denote the the closed ball of radius R in V:

$$B_R = \{v \in V : \|v\|_1 \leq R\}. \tag{1.6}$$

We also use the notation $C([0,T],X)$ for the Banach space of continuous functions $v : [0,T] \to X$ with norm

$$\|v\|_{L_\infty([0,T],X)} = \sup_{t \in [0,T]} \|v(t)\|_X. \tag{1.7}$$

We set the problem up in the framework of [8]. We define the unbounded operator $A = -\Delta$ on H with domain of definition $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then A is a closed, densely defined, and self-adjoint positive definite operator in H with compact inverse. Moreover, our assumption (1.2) guarantees that the mapping $\tilde{f}$ induces a nonlinear operator $f : V \to H$ through $f(v)(x) = \tilde{f}(v(x))$, see Lemma 1.1 below. The initial-boundary value problem (1.1) may then be formulated as an initial value problem in V: find $u(t) \in V$ such that

$$u' + Au = f(u),\ t > 0; \quad u(0) = u_0. \tag{1.8}$$

The operator $-A$ is the infinitesimal generator of the analytic semigroup $E(t) = \exp(-tA)$ defined by

$$E(t)v = \sum_{j=1}^{\infty} e^{-t\lambda_j}(v, \varphi_j)\varphi_j, \quad v \in H, \ t \geq 0, \tag{1.9}$$

where λ_j and φ_j denote the eigenvalues and a corresponding orthonormal basis of eigenvectors of A, i.e., $A\varphi_j = \lambda_j\varphi_j$, and

$$\begin{aligned} &0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \to \infty, \quad (\varphi_j, \varphi_i) = \delta_{ij}, \\ &v = \sum_{j=1}^{\infty} (v, \varphi_j)\varphi_j, \quad \|v\| = \Big(\sum_{j=1}^{\infty} (v, \varphi_j)^2 \Big)^{1/2}, \quad v \in H. \end{aligned} \tag{1.10}$$

This is based on the spectral theorem for self-adjoint operators with compact inverse; in more general situations where the operator A is not self-adjoint the analytic semigroup $E(t) = \exp(-tA)$ can still be defined under suitable assumptions on the generator $-A$ without using the spectral theorem, see [8].

The semigroup $E(t)$ is the solution operator of the initial value problem for the homogeneous equation,

$$u' + Au = 0, \ t > 0; \quad u(0) = v. \tag{1.11}$$

The solution of (1.11) is thus given by $u(t) = E(t)v$. By Duhamel's principle it follows that solutions of (1.8) satisfy the equation

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s))\,ds, \quad t \geq 0. \tag{1.12}$$

Conversely, we will see that appropriately defined solutions of the nonlinear integral equation (1.12) are solutions of the differential equation (1.8); this is Theorem 1.3 below. We shall mainly work with (1.12) and discretized variants of it.

An important ingredient in our framework is that f can be controlled by fractional powers of A. We define the fractional powers of A by means of the spectral theorem and we have, for any exponent $\alpha \in \mathbf{R}$,

$$
\begin{aligned}
A^{\alpha} v &= \sum_{j=1}^{\infty} \lambda_j^{\alpha} (v, \varphi_j) \varphi_j, \\
\|A^{\alpha} v\| &= \Big(\sum_{j=1}^{\infty} \Big(\lambda_j^{\alpha} (v, \varphi_j) \Big)^2 \Big)^{1/2}, \\
\mathcal{D}(A^{\alpha}) &= \Big\{ v : \|A^{\alpha} v\| < \infty \Big\}.
\end{aligned} \tag{1.13}
$$

We also need the elliptic regularity estimate,

$$\|v\|_2 \le C\|Av\|, \quad v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{1.14}$$

This is true for any domain Ω with smooth boundary, but also under our present assumption that Ω is a convex polygonal domain. It means that, for any $f \in L_2(\Omega)$ the solution of the elliptic problem

$$-\Delta v = f \text{ in } \Omega; \quad v = 0 \text{ on } \partial\Omega,$$

belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ and obeys the inequality

$$\|v\|_2 \le C\|f\|, \quad f \in L_2(\Omega).$$

Since $f = -\Delta v = Av$ we obtain (1.14). For the proof you must consult an advanced book on partial differential equations. Using also the trace inequality,

$$\|v\|_{L_2(\partial\Omega)} \le C\|v\|_1, \quad v \in H^1(\Omega),$$

and the Poincaré inequality

$$\|v\| \le C\|\nabla v\|, \quad v \in V = H_0^1(\Omega), \tag{1.15}$$

we obtain $\mathcal{D}(A^{l/2}) = H^l(\Omega) \cap H_0^1(\Omega)$, $l = 1,2$, with the equivalence of norms

$$c\|v\|_l \le \|A^{l/2}v\| \le C\|v\|_l, \quad v \in \mathcal{D}(A^{l/2}),\ l = 1,2. \tag{1.16}$$

The tricky part is to show that the spaces are equal, $\mathcal{D}(A^{1/2}) = H_0^1(\Omega)$ and $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, we refer to [11, Theorem 6.4] or [12, Chapt. 3] for a proof. The norm equivalences follow more or less directly from (1.15) and (1.14).

A simple exercise using the special case $l = 1$ of (1.16) shows that $\mathcal{D}(A^{-1/2}) = H^{-1}(\Omega)$ and

$$c\|v\|_{-1} \le \|A^{-1/2}v\| \le C\|v\|_{-1}; \tag{1.17}$$

cf. (3.7) below.

The analyticity of the semigroup $E(t)$ is reflected in the inequalities (where $D_t = \partial/\partial t$)

$$\|D_t^l E(t)v\| = \|A^l E(t)v\| \le C_l t^{-l}\|v\|, \quad t > 0,\ v \in H,\ l \ge 0. \tag{1.18}$$

These follow easily from (1.9) and Parseval's identity. For example, we prove (1.18) as follows:

$$\begin{aligned}\|A^l E(t)v\|^2 &= \sum_{j=1}^{\infty} \lambda_j^{2l} e^{-2t\lambda_j}(v,\varphi_j)^2 = t^{-2l}\sum_{j=1}^{\infty}\big((t\lambda_j)^l e^{-t\lambda_j}\big)^2(v,\varphi_j)^2\\ &\le C_l t^{-2l}\sum_{j=1}^{\infty}(v,\varphi_j)^2 = C_l t^{-2l}\|v\|^2.\end{aligned}$$

Combining (1.18) with the norm equivalences (1.16), (1.17) we obtain the smoothing property

$$\begin{aligned}\|D_t^l E(t)v\|_\beta \le C t^{-l-(\beta-\alpha)/2}\|v\|_\alpha, \quad t > 0,\ v \in \mathcal{D}(A^{\alpha/2}),\\ -1 \le \alpha \le \beta \le 2,\ l = 0,1.\end{aligned} \tag{1.19}$$

This means that the solution $u(t) = E(t)v$ of the linear homogeneous problem (1.11) has β spatial derivatives and l temporal derivatives in $L_2(\Omega)$ even if the initial value v is only in $\mathcal{D}(A^{\alpha/2})$. Note however that the corresponding norm may blow up as t approaches 0.

We need the fact that the operator $f : V \to H$ satisfies a local Lipschitz condition. This is contained in our first lemma. We also prove that the Fréchet derivative $f' : V \to \mathcal{L}(V,H)$ satisfies a local Lipschitz condition. The proof is based on the assumption (1.2) and Sobolev's inequality, where $p = 6$ if $d = 3$, $p < \infty$ if $d = 2$, and $p = \infty$ if $d = 1$,

$$\|v\|_{L_p} \le C\|v\|_1, \tag{1.20}$$

and on Hölder's inequality in the form

$$\|v^\delta w\|_{L_r} \le \|v\|_{L_q}^\delta \|w\|_{L_p}, \quad \tfrac{\delta}{q} + \tfrac{1}{p} = \tfrac{1}{r},\ \delta > 0. \tag{1.21}$$

Lemma 1.1. *For each nonnegative number R there is a constant $C(R)$ such that, for all $u, v, w \in B_R$, $l = 0, 1$,*

$$\|f'(u)\|_{\mathcal{L}(V,H)} \le C(R), \tag{1.22}$$

$$\|f'(u)\|_{\mathcal{L}(H,V^*)} \le C(R), \tag{1.23}$$

$$\|f(u) - f(v)\| \le C(R)\|u - v\|_1, \tag{1.24}$$

$$\|f(u) - f(v)\|_{-1} \le C(R)\|u - v\|, \tag{1.25}$$

$$\|f'(u) - f'(v)\|_{\mathcal{L}(V,H)} \le C(R)\|u - v\|_1, \tag{1.26}$$

$$\|f'(u) - f'(v)\|_{\mathcal{L}(H,V^*)} \le C(R)\|u - v\|_1, \tag{1.27}$$

$$\begin{aligned}\|f(u) - f(v) - f'(w)(u - v)\|_{l-1} \le C(R)\big(&\|u - w\|_1 \\ &+ \|v - w\|_1\big)\|u - v\|_l.\end{aligned} \tag{1.28}$$

If, in addition, $u \in H^2(\Omega)$ and $z \in H^2(\Omega) \cap V$, then

$$\|A^{1/2}(f'(u)z)\| \leq C(R)\big(\|z\|_2 + \|u\|_2\|z\|_1\big). \tag{1.29}$$

Note that the Lipschitz constant $C(R)$ depends on the size of u, v, w in the H^1-norm through the assumption $u, v, w \in B_R$, see (1.6). This is why we say local Lipschitz condition.

Proof. We have in view of (1.2) and the Hölder and Sobolev inequalities, for $z \in V$,

$$\begin{aligned} \|f'(u)z\|_{L_2} &\leq C\big(1 + \|u\|_{L_q}^{\delta}\big)\|z\|_{L_p} \\ &\leq C\big(1 + \|u\|_1^{\delta}\big)\|z\|_1, \end{aligned}$$

where $\frac{1}{p} + \frac{\delta}{q} = \frac{1}{2}$ with $p = q = 6$ if $d = 3$, and with arbitrary $p \in (1, \infty)$ if $d \leq 2$. This proves (1.22) and (1.24) follows. Similarly, for any $z \in V$,

$$\begin{aligned} \|(f'(u) - f'(v))z\|_{L_2} &\leq C\big(1 + \|u\|_{L_q}^{\delta-1} + \|v\|_{L_q}^{\delta-1}\big)\|u - v\|_{L_p}\|z\|_{L_q} \\ &\leq C\big(1 + \|u\|_1^{\delta-1} + \|v\|_1^{\delta-1}\big)\|u - v\|_1\|z\|_1, \end{aligned}$$

with the same p and q as before. This proves (1.26).

Moreover, for any $z, \chi \in V$,

$$\begin{aligned} (f'(u)z, \chi) &\leq C\big(1 + \|u\|_{L_q}^{\delta}\big)\|z\|_{L_2}\|\chi\|_{L_p} \\ &\leq C\big(1 + \|u\|_1^{\delta}\big)\|z\|\,\|\chi\|_1, \end{aligned}$$

and

$$\begin{aligned} ((f'(u) - f'(v))z, \chi) &\leq C\big(1 + \|u\|_{L_q}^{\delta-1} + \|v\|_{L_q}^{\delta-1}\big)\|u - v\|_{L_2}\|z\|_{L_q}\|\chi\|_{L_p} \\ &\leq C\big(1 + \|u\|_1^{\delta-1} + \|v\|_1^{\delta-1}\big)\|u - v\|\,\|z\|_1\|\chi\|_1, \end{aligned}$$

where $\frac{\delta}{q}+\frac{1}{2}+\frac{1}{p}=1$, i.e., with the same p and q as before. This proves (1.23) and (1.27); (1.25) follows from (1.23).

Next, (1.28) is obtained by applying (1.26) and (1.27) to the identity

$$f(u)-f(v)-f'(w)(u-v)=\int_0^1\Big(f'(su+(1-s)v)-f'(w)\Big)\,ds\,(u-v).$$

Finally, (1.29) is proved in a similar way, by using the equivalence of norms (1.16) and computing the first order partial derivatives of $f'(u)z$. Note that this uses the fact that $f'(u)z\in V$, i.e., $f'(u)z=0$ on $\partial\Omega$. $\square$

We may now prove local existence of solutions of (1.12) and hence of (1.8).

Theorem 1.2. *For any $R_0>0$ there is $\tau=\tau(R_0)$ such that* (1.12) *has a unique solution $u\in C([0,\tau],V)$ for any initial value $u_0\in V$ with $\|u_0\|_1\le R_0$. Moreover, there is c such that $\|u\|_{L_\infty([0,\tau],V)}\le cR_0$.*

Proof. Let $u_0\in B_{R_0}$, define

$$\mathcal{S}(u)(t)=E(t)u_0+\int_0^t E(t-s)f(u(s))\,ds,$$

and note that (1.12) is a fixed point equation, $u=\mathcal{S}(u)$. We shall choose τ and R such that we can apply Banach's fixed point theorem (the contraction mapping theorem) in the closed ball

$$\mathcal{B}=\{u\in C([0,\tau],V):\|u\|_{L_\infty([0,\tau],V)}\le R\}$$

in the Banach space $C([0,\tau],V)$, cf. (1.7).

We must show (i) that $\mathcal{S}$ maps $\mathcal{B}$ into itself, (ii) that $\mathcal{S}$ is a contraction on $\mathcal{B}$. In order to prove (i) we take $u\in\mathcal{B}$ and first note that the Lipschitz condition (1.24) implies that

$$\begin{aligned}\|f(u(t))\| &\le \|f(0)\| + \|f(u(t)) - f(0)\| \\ &\le \|f(0)\| + C(R)\|u(t)\|_1 \\ &\le \|f(0)\| + C(R)R, \quad 0 \le t \le \tau.\end{aligned} \tag{1.30}$$

Hence, using also (1.19), we get

$$\begin{aligned}\|\mathcal{S}(u)(t)\|_1 &\le \|E(t)u_0\|_1 + \int_0^t \|E(t-s)f(u(s))\|_1\,ds \\ &\le c_0\|u_0\|_1 + c_1\int_0^t (t-s)^{-1/2}\|f(u(s))\|\,ds \\ &\le c_0R_0 + 2c_1\tau^{1/2}\big(\|f(0)\| + C(R)R\big), \quad 0 \le t \le \tau.\end{aligned}$$

This implies

$$\|\mathcal{S}(u)\|_{L_\infty([0,\tau],V)} \le c_0R_0 + 2c_1\tau^{1/2}\big(\|f(0)\| + C(R)R\big).$$

Choose $R = 2c_0R_0$ and $\tau = \tau(R_0)$ so small that

$$2c_1\tau^{1/2}\big(\|f(0)\| + C(R)R\big) \le \tfrac{1}{2}R. \tag{1.31}$$

Then $\|\mathcal{S}(u)\|_{L_\infty([0,\tau],V)} \le R$ and we conclude that $\mathcal{S}$ maps $\mathcal{B}$ into itself.

To show (ii) we take $u, v \in \mathcal{B}$ and note that

$$\|f(u(t)) - f(v(t))\| \le C(R)\|u-v\|_{L_\infty([0,\tau],V)}, \quad 0 \le t \le \tau.$$

Hence

$$\begin{aligned}\|\mathcal{S}(u)(t) - \mathcal{S}(v(t))\|_1 &\le \int_0^t \|E(t-s)(f(u(s)) - f(v(s)))\|_1\,ds \\ &\le c_1\int_0^t (t-s)^{-1/2}\|f(u(s)) - f(v(s))\|\,ds \\ &\le 2c_1\tau^{1/2}C(R)\|u-v\|_{L_\infty([0,\tau],V)}, \quad 0 \le t \le \tau,\end{aligned}$$

so that

$$\|\mathcal{S}(u)-\mathcal{S}(v)\|_{L_\infty([0,\tau],V)} \leq 2c_1\tau^{1/2}C(R)\|u-v\|_{L_\infty([0,\tau],V)}.$$

It follows from (1.31) that $2c_1\tau^{1/2}C(R) \leq \frac{1}{2}$ and we conclude that $\mathcal{S}$ is a contraction on $\mathcal{B}$. Hence $\mathcal{S}$ has a unique fixed point $u \in \mathcal{B}$. $\square$

The integral equation (1.12) thus has a unique local solution for any initial datum $u_0 \in V$. We denote by $S(t,\cdot)$ the corresponding (local) solution operator, so that $u(t) = S(t,u_0)$ is the solution of (1.12). By uniqueness of solutions it is clear that S satisfies the semigroup property:

$$S(t+s,u_0) = S(t,S(s,u_0)), \quad t,s \geq 0,\ t+s \leq \tau. \tag{1.32}$$

The following theorem provides regularity estimates for solutions u of (1.12). These will be used in our error analysis, but the theorem also shows that $u'(t) \in H$ and $Au(t) \in H$ for $t > 0$, so that the solution of the integral equation (1.12) is also a solution of the differential equation (1.8).

Theorem 1.3. *Let $R \geq 0$ and $\tau > 0$ be given and let $u \in C([0,\tau],V)$ be a solution of* (1.12). *If $\|u(t)\|_1 \leq R$ for $t \in [0,\tau]$, then*

$$\|u(t)\|_2 \leq C(R,\tau)t^{-1/2}, \qquad t \in (0,\tau], \tag{1.33}$$

$$\|u'(t)\|_s \leq C(R,\tau)t^{-1-(s-1)/2}, \qquad t \in (0,\tau],\ s = 0,1,2. \tag{1.34}$$

Proof. We shall not present a complete proof here but refer to [8, Theorem 3.5.2] for the missing parts. The argument is based on a generalization of Gronwall's lemma, Lemma 1.4 below. The tricky part is to show that u is differentiable with respect to t and that $u'(t)$ belongs to H, V

and $H^2(\Omega)$. In order to illustrate the techniques involved, we only show that $u'(t) \in H$. A simple calculation shows, assuming for simplicity that $h > 0$,

$$
\begin{aligned}
u(t+h) - u(t) &= \big(E(t+h) - E(t)\big)u_0 \\
&\quad + \int_0^{t+h} E(t+h-s) f(u(s))\,ds - \int_0^t E(t-s) f(u(s))\,ds \\
&= \big(E(h) - I\big)E(t)u_0 \\
&\quad + \int_0^{t+h} E(s) f(u(t+h-s))\,ds - \int_0^t E(s) f(u(t-s))\,ds \\
&= \big(E(h) - I\big)E(t)u_0 + \int_t^{t+h} E(s) f(u(t+h-s))\,ds \\
&\quad + \int_0^t E(s)\Big(f(u(t+h-s)) - f(u(t-s))\Big)\,ds \\
&= \big(E(h) - I\big)E(t)u_0 + \int_0^h E(t+h-s) f(u(s))\,ds \\
&\quad + \int_0^t E(t-s)\Big(f(u(s+h)) - f(u(s))\Big)\,ds.
\end{aligned}
$$

We will take norms in the above identity. In the first term we use the identity $\big(E(h) - I\big)E(t)u_0 = A^{-1}\big(E(h) - I\big)AE(t)u_0$ and

$$
\|A^{-1}\big(E(h) - I\big)v\| \le Ch\|v\|,
$$

which is proved in a similar way as (1.18). Hence,

$$
\|\big(E(h) - I\big)E(t)u_0\| \le Ch\|AE(t)u_0\| \le Cht^{-1/2}\|u_0\|_1 \le C(R)ht^{-1/2}.
$$

In view of (1.24) we have $\|f(u(s))\| \le C(R)$, $s \in [0, \tau]$, cf. (1.30). Also, from (1.25) follows

$$
\|f(u(s+h)) - f(u(s))\|_{-1} \le C(R)\|u(s+h) - u(s)\|.
$$

Using the smoothing property (1.19) with $\alpha=\beta=0$ and $\alpha=-1,\ \beta=0$, we obtain

$$\begin{aligned}\|u(t+h)-u(t)\| &\le C(R)ht^{-1/2}+C(R)h\\ &\qquad +C(R)\int_0^t (t-s)^{-1/2}\|u(s+h)-u(s)\|\,ds\\ &\le C(R)\Big(ht^{-1/2}+\int_0^t (t-s)^{-1/2}\|u(s+h)-u(s)\|\,ds\Big),\end{aligned}$$

for $0<t\le\tau$. By using Lemma 1.4 we conclude

$$\|u(t+h)-u(t)\|\le C(R,\tau)ht^{-1/2},$$

which (essentially) proves that $u'(t)\in H$ together with the bound in (1.34) with $s=0$. □

Lemma 1.4. (Generalized Gronwall lemma.) *Let* $A,B\ge 0$, $\alpha,\beta>0$, *be constants and* $0\le t_0<t\le T$. *There is a constant* $C=C(B,T,\alpha,\beta)$ *such that, if the function* $\varphi(t,t_0)\ge 0$ *is continuous and*

$$\varphi(t,t_0)\le A\,(t-t_0)^{-1+\alpha}+B\int_{t_0}^t (t-s)^{-1+\beta}\varphi(s,t_0)\,ds,\quad 0\le t_0<t\le T,$$

then

$$\varphi(t,t_0)\le CA\,(t-t_0)^{-1+\alpha},\quad 0\le t_0<t\le T.$$

Proof. Iterating the given inequality $N-1$ times, using the identity

$$\int_{t_0}^t (t-s)^{-1+\alpha}(s-t_0)^{-1+\beta}\,ds=C(\alpha,\beta)\,(t-t_0)^{-1+\alpha+\beta},\quad \alpha,\beta>0,$$

(Abel's integral) and estimating $(t-t_0)^{\beta}$ by T^{β}, we obtain

$$\varphi(t,t_0)\le C_1A\,(t-t_0)^{-1+\alpha}+C_2\int_{t_0}^t (t-s)^{-1+N\beta}\varphi(s,t_0)\,ds,$$

where $C_1 = C_1(B,T,\alpha,\beta,N)$, $C_2 = C_2(B,\beta,N)$. We now choose the smallest N so that $-1+N\beta \geq 0$, and estimate $(t-s)^{-1+N\beta}$ by $T^{-1+N\beta}$. If $-1+\alpha \geq 0$ we obtain the desired conclusion by the standard version of Gronwall's lemma. Otherwise we set $\psi(t,t_0) = (t-t_0)^{1-\alpha}\varphi(t,t_0)$ to obtain

$$\psi(t,t_0) \leq C_1 A + C_3 \int_{t_0}^{t} (s-t_0)^{-1+\alpha}\psi(s,t_0)\,ds, \qquad 0 \leq t_0 < t \leq T,$$

and the standard Gronwall lemma yields $\psi(t,t_0) \leq CA$ for $0 \leq t_0 < t \leq T$, which is the desired result. $\square$

Note that the constant in Gronwall's lemma grows exponentially with the length T of the time interval. Hence, results derived by means of this lemma are often useful only for short time intervals. There is also a discrete version of Lemma 1.4; see [4, Lemma 7.1].

We finish this lecture by discussing global existence of solutions, i.e., existence of solutions over some predetermined long time interval $[0,T]$, not just some sufficiently short time interval $[0,\tau]$.

Assume that we can provide a global a priori bound: there is R such that if $u \in C([0,T],V)$ is a solution, then

$$\|u(t)\|_1 \leq R, \quad t \in [0,T]. \tag{1.35}$$

Note that what we assume is that if a solution exists then we can estimate its size globally in terms of the data of the evolution problem (such as u_0, f, T, or Ω). That is why we say "a priori bound"; the solution is estimated before we know if it really exists.

Then by repeated application of the local existence theorem with $\tau = \tau(R)$ we can prove the solution actually exists for $t \in [0,T]$. More precisely, since by (1.35) we have $\|u_0\|_1 \leq R$, we conclude that $u(t) = S(t,u_0)$ exists on $[0,\tau]$ with $\tau = \tau(R)$. Again by (1.35) we have $\|u(\tau)\|_1 \leq R$, and so by application of the local existence theorem with initial value $u(\tau)$ at time $t = \tau$, we conclude that $u(t) = S(t-\tau, u(\tau)) = S(t,u_0)$ exists on $[\tau, 2\tau]$. After a finite numbers of steps we reach the final time T. The a priori bound guarantees that we can use the same τ all the time.

Example 1.2. We recall the Allen-Cahn equation,

$$u_t - \Delta u = -(u^3 - u) = -V'(u)$$

where the potential $V(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$ is bounded from below:

$$V(\xi) \geq -K,$$

actually $K = 1/4$. Assume now that a solution exists and multiply the equation by u_t and integrate over Ω,

$$(u_t,u_t) - (\Delta u, u_t) = -(V'(u), u_t).$$

Here

$$-(\Delta u, u_t) = (\nabla u, \nabla u_t) = \frac{1}{2} D_t \|\nabla u\|^2$$

and

$$-(V'(u), u_t) = -D_t \int_\Omega V(u)\,dx,$$

so that

$$\|u_t\|^2 + \frac{1}{2} D_t \|\nabla u\|^2 = -D_t \int_\Omega V(u)\,dx.$$

After integration with respect to t,

$$\begin{aligned}\int_0^t \|u_t\|^2\,ds + \frac{1}{2}\|\nabla u(t)\|^2 &= \frac{1}{2}\|\nabla u_0\|^2 - \int_\Omega V(u(t))\,dx + \int_\Omega V(u_0)\,dx \\ &\leq \frac{1}{2}\|\nabla u_0\|^2 + K\int_\Omega dx + \int_\Omega V(u_0)\,dx.\end{aligned}$$

We conclude

$$\|u(t)\|_1 \le R, \quad t \in [0,\infty),$$

with

$$R = \frac{1}{2}\|\nabla u_0\|^2 + K\int_\Omega dx + \int_\Omega V(u_0)\,dx.$$

Hence we have an a priori bound for any time interval $[0,T]$ and we get global existence for all time. Thus, the solution operator $S(t,\cdot)$ is defined for $t \in [0,\infty)$. Note the form of the a priori bound: R depends only on the size of $\|u_0\|_1$ (with f and Ω being fixed). This is because, by the Sobolev inequality (1.20),

$$\int_\Omega V(u_0)\,dx = \frac{1}{4}\|u_0\|_{L_4}^4 - \frac{1}{2}\|u_0\|^2 \le C\|u_0\|_1^4 + \frac{1}{2}\|u_0\|^2 \le C(R_0),$$

if $\|u_0\|_1 \le R_0$. $\square$

2 The finite element method

In this lecture we introduce spatial discretization by the finite element method in the context of a linear elliptic boundary value problem. The presentation follows [11], see also [5], [9], for other elementary presentations. For more detailed treatments of the finite element method we refer to [2] and [1].

We consider the linear elliptic problem to find $u = u(x)$ such that

$$\begin{aligned} -\Delta u &= f, \qquad x \in \Omega, \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{2.1}$$

where $f \in H = L_2(\Omega)$, which can be seen as the linear stationary case of the evolution problem (1.1). In our abstract framework this equation is $Au = f$.

The weak formulation of (2.1) is: find $u \in V$ such that

$$a(u,v) = (f,v), \quad \forall v \in V, \tag{2.2}$$

where $a(u,v) = (\nabla u, \nabla v) = (-\Delta u, v) = (Au, v)$ is the bilinear form associated with A. From (1.15) it follows that $a(u,v)$ is a scalar product on V which is equivalent to the the standard scalar product $(u,v) + (\nabla u, \nabla v)$. The existence of a unique solution $u \in V$ of (2.2) now follows from the Riesz representation theorem. By the elliptic regularity (1.14) we conclude that $u \in H^2(\Omega) \cap V$.

Let $\{V_h\}_{0<h<1}$ be a family of finite dimensional subspaces of V, where each V_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to a triangulation $\mathcal{T}_h$ of Ω with maximal mesh size h. In other words, we divide the polygonal domain Ω into simplices (intervals if $d = 1$, triangles if $d = 2$, and tetrahedra if $d = 3$). More precisely, let $\mathcal{T}_h = \{K\}$ be a set of closed simplices K, a triangulation of Ω, such that

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K, \quad h_K = \operatorname{diam}(K), \quad h = \max_{K \in \mathcal{T}_h} h_K.$$

The vertices P of the simplices $K \in \mathcal{T}_h$ are called the nodes of the triangulation $\mathcal{T}_h$. We require that the intersection of any two simplices of $\mathcal{T}_h$ is either empty, a node, or a common edge or face, and that no node is located in the interior of an edge or face of $\mathcal{T}_h$. Since Ω is assumed to be a polygonal domain, the mesh (triangulation) can be made to fit exactly as described above. For domains with curved boundary there is an additional difficulty concerning the approximation of the boundary, which we do not address here.

With each $\mathcal{T}_h$ we associate the function space

$$V_h = \big\{ v \in C(\bar{\Omega}) : v \text{ linear in } K \text{ for each } K \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega \big\}.$$

Using our above assumptions on $\mathcal{T}_h$ it is not difficult to verify that $V_h \subset V = H_0^1(\Omega)$. Let $\{P_i\}_{i=1}^{M_h}$ be the set of interior nodes, i.e., those that do not lie on Γ. A function in V_h is then uniquely determined by its values at the P_j, and the set of pyramid functions $\{\Phi_i\}_{i=1}^{M_h} \subset V_h$, defined by

$$\Phi_i(P_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

forms a basis for V_h. If $v \in V_h$ we thus have $v(x) = \sum_{i=1}^{M_h} v_i \Phi_i(x)$, where the $v_i = v(P_i)$ are the nodal values of v. It follows that V_h is a finite-dimensional subspace of the Hilbert space V.

The approximate solution $u_h \in V_h$ of (2.1) is defined by

$$a(u_h, \chi) = (f, \chi), \quad \forall \chi \in V_h. \tag{2.3}$$

Our task is now to estimate the error $u_h - u$. In order to do so we define the interpolation operator $I_h : C(\bar{\Omega}) \cap H_0^1(\Omega) \to V_h$ by

$$(I_h v)(x) = \sum_{i=1}^{M_h} v_i \Phi(x), \quad \text{where } v_i = v(P_i). \tag{2.4}$$

The interpolant $I_h v$ thus agrees with v at the nodes P_j, i.e.,

$$(I_h v)(P_i) = v(P_i), \quad \text{for } i = 1, \ldots, M_h.$$

One can prove the local error estimates, with $\|v\|_K = \|v\|_{L_2(K)}$, $\|v\|_{2,K} = \|v\|_{H^2(K)}$,

$$\|I_h v - v\|_K \leq C_K h_K^2 \|v\|_{2,K}, \quad \forall K \in \mathcal{T}_h, \tag{2.5}$$

and

$$\|\nabla(I_h v - v)\|_K \leq C_K h_K \|v\|_{2,K}, \quad \forall K \in \mathcal{T}_h. \tag{2.6}$$

In what follows we impose the restriction on the family $\{\mathcal{T}_h\}_{0<h<1}$ of triangulations that the angles of all triangles K belonging to all members of the family $\{\mathcal{T}_h\}$ are bounded below, independently of h. It is then possible to prove that the constants C_K are uniformly bounded, so that we have the global estimates

$$\begin{aligned}\|I_h v - v\| &= \Big(\sum_K \|I_h v - v\|_K^2\Big)^{1/2} \le \Big(\sum_K C_K^2 h_K^4 \|v\|_{2,K}^2\Big)^{1/2} \\ &\le Ch^2\|v\|_2, \quad \forall v \in H^2(\Omega)\cap H_0^1(\Omega),\end{aligned} \tag{2.7}$$

and similarly

$$\|I_h v - v\|_1 \le Ch\|v\|_2, \quad \forall v \in H^2(\Omega)\cap H_0^1(\Omega). \tag{2.8}$$

We can now prove error estimates. We begin with the H^1-norm.

Theorem 2.1. *Let u_h and u be the solutions of* (2.3) *and* (2.2). *Then*

$$\|u_h - u\|_1 \le Ch\|u\|_2. \tag{2.9}$$

Proof. Since $V_h \subset H_0^1$ we may take $v = \chi \in V_h$ in (2.2) and subtract it from (2.3) to obtain

$$a(u_h - u, \chi) = 0, \quad \forall \chi \in V_h, \tag{2.10}$$

which means that u_h is the orthogonal projection of u onto V_h with respect to the inner product $a(\cdot,\cdot) = (\nabla\cdot, \nabla\cdot)$. The projection theorem then yields

$$\|\nabla(u_h - u)\| = \min_{\chi\in V_h} \|\nabla(\chi - u)\| \le \|\nabla(I_h u - u)\|$$

and by norm equivalence and the interpolation error estimate (2.8),

$$\|u_h - u\|_1 \le C\|I_h u - u\|_1 \le Ch\|u\|_2. \tag{2.11}$$

This proves (2.9). $\square$

Our next result concerns the L_2-norm of the error.

Theorem 2.2. *Let u_h and u be the solutions of* (2.3) *and* (2.2). *Then*

$$\|u_h - u\| \le Ch^2 \|u\|_2. \tag{2.12}$$

Proof. We use a duality argument based on the auxiliary problem

$$-\Delta\phi = e \text{ in } \Omega; \quad \phi = 0 \text{ on } \partial\Omega, \quad \text{where } e = u_h - u. \tag{2.13}$$

Its weak formulation is to find $\phi \in V$ such that

$$a(w, \phi) = (w, e), \quad \forall w \in V. \tag{2.14}$$

By the regularity estimate (1.14) we have

$$\|\phi\|_2 \le C\|A\phi\| = C\|e\|. \tag{2.15}$$

Taking $w = e$ in (2.14) and using (2.10) and (2.8), we therefore obtain

$$\begin{aligned} \|e\|^2 = a(e, \phi) = a(e, \phi - I_h\phi) &\le C\|e\|_1 \, \|\phi - I_h\phi\|_1 \\ &\le Ch\|e\|_1 \, \|\phi\|_2 \le Ch\|e\|_1 \, \|e\|. \end{aligned}$$

Canceling one factor $\|e\|$ we see that we have gained one factor h over the error estimate for $\|e\|_1$,

$$\|e\| \le Ch\|e\|_1, \tag{2.16}$$

and (2.12) follows if we use (2.8) again. □

Let $R_h : V \to V_h$ be the orthogonal projection with respect to the energy inner product, so that

$$a(R_h v - v, \chi) = 0, \quad \forall \chi \in V_h, \quad v \in V. \tag{2.17}$$

The operator R_h is called the Ritz projection (or elliptic projection). It follows from (2.10) that the finite element solution u_h is the Ritz projection of the solution u of (2.2), i.e., $u_h = R_h u$. Our previous error estimates for the finite element solution may be expressed as follows in terms of the operator R_h, which will be convenient when we discuss parabolic finite element problems later.

Theorem 2.3. *For $s = 1, 2$, we have*

$$\|R_h v - v\| \le Ch^s \|v\|_s, \quad \|R_h v - v\|_1 \le Ch^{s-1} \|v\|_s, \quad \forall v \in H^s(\Omega) \cap V.$$

Proof. The case $s = 2$ is contained in Theorems 2.1 and 2.2. For the case $s = 1$ we first note that since R_h is the orthogonal projection with respect to $a(\cdot,\cdot)$, we have $\|\nabla R_h v\| \le \|\nabla v\|$. Hence $\|R_h v\|_1 \le C\|v\|_1$ and $\|R_h v - v\|_1 \le C\|v\|_1$. Finally, using (2.16) we obtain

$$\|R_h v - v\| \le Ch\|R_h v - v\|_1 \le Ch\|v\|_1,$$

which completes the proof. □

3 Local error estimates for semilinear parabolic problems

In this section we first discretize (1.1) with respect to the spatial variables by means of a standard piecewise linear finite element method. We then briefly discuss completely discrete approximation by means of the backward Euler time-stepping. This material is taken from [10]. A general reference is [12].

3.1 The spatially semidiscrete problem

The weak formulation of (1.1) is: find $u(t) \in V$ such that

$$\begin{aligned} (u', v) + a(u, v) &= (f(u), v), \quad \forall v \in V,\ t > 0, \\ u(0) &= u_0, \end{aligned} \tag{3.1}$$

where $a(u, v) = (\nabla u, \nabla v) = (-\Delta u, v) = (Au, v)$ is the bilinear form associated with A.

Let $\{V_h\}_{0<h<1}$ be a family of finite dimensional subspaces of V, where each V_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to a triangulation $\mathcal{T}$ of Ω with maximal mesh size h. The approximate solution $u_h(t) \in V_h$ of (1.1) is defined by

$$\begin{aligned} (u_h', \chi) + a(u_h, \chi) &= (f(u_h), \chi), \quad \forall \chi \in V_h,\ t > 0, \\ u_h(0) &= u_{h,0}, \end{aligned} \tag{3.2}$$

where $u_{h,0} \in V_h$ is an approximation of u_0.

Introducing the linear operator $A_h : V_h \to V_h$ and the orthogonal projection $P_h : H \to V_h$, defined by

$$(A_h\psi, \chi) = a(\psi, \chi), \quad (P_h g, \chi) = (g, \chi) \quad \forall \psi, \chi \in V_h,\ g \in H, \tag{3.3}$$

we may write (3.2) as

$$u_h' + A_h u_h = P_h f(u_h),\ t > 0; \quad u_h(0) = u_{h,0}. \tag{3.4}$$

The operator A_h is self-adjoint positive definite (uniformly in h), i.e.,

$$(A_h v_h, v_h) = a(v_h, v_h) \geq \lambda_1 \|v_h\|^2, \quad v_h \in V_h,$$

where $\lambda_1 > 0$ is the principal eigenvalue of A, see (1.10). The corresponding semigroup $E_h(t) = \exp(-tA_h) : V_h \to V_h$ therefore satisfies inequalities analogous to (1.18) (uniformly in h):

$$\|D_t^l E_h(t)v\| = \|A_h^l E_h(t)v\| \le C_l t^{-l}\|v\|, \quad t > 0,\ v \in V_h,\ l \ge 0.$$

Moreover, in V_h we have the equivalence of norms, cf. (1.16),

$$c\|v\|_1 \le \|A_h^{1/2}v\| = \sqrt{a(v,v)} = \|A^{1/2}v\| \le C\|v\|_1, \quad v \in V_h. \tag{3.5}$$

We also have

$$\|P_h f\| \le \|f\|, \quad f \in H,$$

and, cf. (1.17),

$$\|A_h^{-1/2}P_h f\| \le C\|f\|_{-1}, \quad f \in H, \tag{3.6}$$

which is obtained by using (3.5) in the calculation

$$\begin{aligned}\|A_h^{-1/2}P_h f\| &= \sup_{v_h \in V_h} \frac{|(A_h^{-1/2}P_h f, v_h)|}{\|v_h\|} = \sup_{v_h \in V_h} \frac{|(f, A_h^{-1/2}v_h)|}{\|v_h\|} \\ &= \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\|A_h^{1/2}w_h\|} \le C \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\|w_h\|_1} \\ &\le C \sup_{w \in V} \frac{|(f, w)|}{\|w\|_1} = C\|f\|_{-1}.\end{aligned} \tag{3.7}$$

Using the above inequalities we easily prove the smoothing property of $E_h(t)P_h$, cf. (1.19):

$$\begin{aligned}\|D_t^l E_h(t)P_h f\|_\beta \le C t^{-l-(\beta-\alpha)/2}\|f\|_\alpha, \quad t > 0,\ f \in \mathcal{D}(A^{\alpha/2}), \\ -1 \le \alpha \le \beta \le 1,\ l = 0, 1.\end{aligned} \tag{3.8}$$

Note that the upper limit to β is 1, while it is 2 in the continuous case (1.19). This is because finite element functions do not admit second order derivatives.

The initial-value problem (3.4) is equivalent to the integral equation

$$u_h(t) = E_h(t)u_{h,0} + \int_0^t E_h(t-s)P_h f(u_h(s))\,ds, \quad t \geq 0. \tag{3.9}$$

Because of (3.8) the proof of Theorem 1.2 carries over verbatim to the semidiscrete case. We thus have:

Theorem 3.1. *For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that* (3.9) *has a unique solution $u_h \in C([0,\tau],V)$ for any initial value $u_{h,0} \in V_h$ with $\|u_{h,0}\|_1 \leq R_0$. Moreover, there is c such that $\|u_h\|_{L_\infty([0,\tau],V)} \leq cR_0$.*

We denote by $S_h(t,\cdot)$ the corresponding (local) solution operator, so that $u_h(t) = S_h(t,u_{h,0})$ is the solution of (3.9).

We may now estimate the difference between the local solutions $u(t) = S(t,u_0)$ and $u_h(t) = S_h(t,u_{h,0})$ that we have obtained. We refer to the following result as a local a priori error estimate. It is local because the constant $C(R,\tau)$ grows exponentially with the length τ of the time interval and also because it grows with the size R of the solutions as measured in the H^1-norm via the assumption that $u(t), u_h(t) \in B_R$. It is a priori because the error is evaluated in terms of derivatives of u, which are estimated a priori in Theorem 1.3. Note also the weak singularity $t^{-1/2}$ which is due to the fact that we only assume that the initial value u_0 is in $V = H_0^1(\Omega)$.

Theorem 3.2. *Let $R \geq 0$ and $\tau > 0$ be given. Let $u(t)$ and $u_h(t)$ be solutions of* (3.1) *and* (3.2) *respectively, such that $u(t), u_h(t) \in B_R$ for*

$t \in [0, \tau]$. *Then, for* $t \in (0, \tau]$,

$$\|u_h(t) - u(t)\|_1 \le C(R, \tau) t^{-1/2} \big(\|u_{h,0} - P_h u_0\| + h \big), \tag{3.10}$$

$$\|u_h(t) - u(t)\| \le C(R, \tau) \big(\|u_{h,0} - P_h u_0\| + h^2 t^{-1/2} \big). \tag{3.11}$$

Proof. We recall the Ritz projection operator $R_h : V \to V_h$ defined by

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in V_h, \tag{3.12}$$

and the error bounds from Theorem 2.3,

$$\|R_h v - v\| + h \|R_h v - v\|_1 \le C h^s \|v\|_s, \quad v \in H^s(\Omega) \cap V,\ s = 1, 2. \tag{3.13}$$

Following a standard practice we divide the error into two parts:

$$e(t) \equiv u_h(t) - u(t) = \big(u_h(t) - R_h u(t) \big) + \big(R_h u(t) - u(t) \big) \equiv \theta(t) + \rho(t).$$

In view of (3.13) and (1.33), (1.34) we have, for $j = 0, 1$ and $s = 1, 2$,

$$\|\rho(t)\|_j \le C h^{s-j} \|u(t)\|_s \le C(R, \tau) h^{s-j} t^{-(s-1)/2}, \qquad t \in (0, \tau], \tag{3.14}$$

$$\|\rho'(t)\| \le C h^s \|u'(t)\|_s \le C(R, \tau) h t^{-1-(s-1)/2}, \qquad t \in (0, \tau]. \tag{3.15}$$

It remains to estimate $\theta(t)$, which belongs to V_h. In view of (3.2), the identity $A_h R_h = P_h A$ (which follows easily from (3.3) and (3.12)), and (1.8), we find that

$$\begin{aligned}
\theta' + A_h \theta &= u_h' - R_h u' + A_h u_h - A_h R_h u \\
&= u_h' + A_h u_h - R_h u' - P_h A u \\
&= P_h f(u_h) - \big(P_h f(u) - u' \big) - R_h u' \\
&= P_h \big(f(u_h) - f(u) \big) + P_h \big(u' - R_h u' \big),
\end{aligned}$$

that is,

$$\theta' + A_h\theta = P_h\big(f(u_h) - f(u)\big) - P_h\rho'. \tag{3.16}$$

Hence, with $D_\sigma = \partial/\partial\sigma$,

$$\theta(t) = E_h(t)\theta(0) + \int_0^t E_h(t-\sigma)P_h\Big(f(u_h(\sigma)) - f(u(\sigma)) - D_\sigma\rho(\sigma)\Big)\,d\sigma.$$

Integration by parts yields

$$\begin{aligned} -\int_0^{t/2} E_h(t-\sigma)P_h D_\sigma\rho(\sigma)\,d\sigma = E_h(t)P_h\rho(0) - E_h(t/2)P_h\rho(t/2) \\ + \int_0^{t/2} \big(D_\sigma E_h(t-\sigma)\big)P_h\rho(\sigma)\,d\sigma. \end{aligned}$$

Hence

$$\begin{aligned} \theta(t) = {} & E_h(t)P_h e(0) - E_h(t/2)P_h\rho(t/2) \\ & + \int_0^{t/2} \big(D_\sigma E_h(t-\sigma)\big)P_h\rho(\sigma)\,d\sigma \\ & - \int_{t/2}^{t} E_h(t-\sigma)P_h D_\sigma\rho(\sigma)\,d\sigma \\ & + \int_0^t E_h(t-\sigma)P_h\Big(f(u_h(\sigma)) - f(u(\sigma))\Big)\,d\sigma. \end{aligned} \tag{3.17}$$

Using the smoothing property (3.8) of $E_h(t)P_h$ we obtain

$$\begin{aligned} \|\theta(t)\|_1 \le {} & Ct^{-1/2}\big(\|P_h e(0)\| + \|\rho(t/2)\|\big) \\ & + C\int_0^{t/2} (t-\sigma)^{-3/2}\|\rho(\sigma)\|\,d\sigma \\ & + C\int_{t/2}^{t} (t-\sigma)^{-1/2}\|D_\sigma\rho(\sigma)\|\,d\sigma \\ & + C\int_0^t (t-\sigma)^{-1/2}\|f(u_h(\sigma)) - f(u(\sigma))\|\,d\sigma. \end{aligned}$$

Using also the error estimates for the elliptic projection in (3.14), (3.15) with $j = 0$, $s = 1$, and the Lipschitz condition (1.24), we then get

$$
\begin{aligned}
\|\theta(t)\|_1 \le{}& C(R,\tau)t^{-1/2}\big(\|P_h e(0)\| + h\big)\\
&+ C(R,\tau)h\bigg(\int_0^{t/2}(t-\sigma)^{-3/2}\,d\sigma + \int_{t/2}^{t}(t-\sigma)^{-1/2}\sigma^{-1}\,d\sigma\bigg)\\
&+ C(R)\int_0^t (t-\sigma)^{-1/2}\|e(\sigma)\|_1\,d\sigma\\
\le{}& C(R,\tau)t^{-1/2}\big(\|P_h e(0)\| + h\big)\\
&+ C(R)\int_0^t (t-\sigma)^{-1/2}\|e(\sigma)\|_1\,d\sigma,
\end{aligned}
$$

for $t \in (0,\tau]$. Together with (3.14), (3.15) and $e = \theta + \rho$ this yields

$$
\|e(t)\|_1 \le C(R,\tau)t^{-1/2}\big(\|P_h e(0)\| + h\big) + C(R)\int_0^t (t-\sigma)^{-1/2}\|e(\sigma)\|_1\,d\sigma,
$$

for $t \in (0,\tau]$, and the desired bound follows by the generalized Gronwall lemma, cf. Lemma 1.4. This proves (3.10), because $P_h e(0) = u_{h,0} - P_h u_0$.

To prove (3.11) we use the error estimates for the elliptic projection in (3.14), (3.15) with $j = 0$, $s = 2$, and the Lipschitz condition (1.25) instead of (1.24). We omit the details. □

We now formulate an immediate consequence of the previous theorem, which is the form in which we will apply it later on. (It is convenient to change the notation so that the length of the local interval of existence is now 2τ.)

Theorem 3.3. *Let $R \ge 0$ and $\tau > 0$ be given. Assume $S(t,v), S_h(t,v_h) \in B_R$ for $t \in [0,2\tau]$. Then, for $l = 0,1$,*

$$
\|S_h(t,v_h) - S(t,v)\|_l \le C(R,\tau)\big(\|v_h - P_h v\| + h^{2-l}\big), \quad t \in [\tau,2\tau].
$$

3.2 A completely discrete scheme

In this subsection we show that the above program can be carried out also for a completely discrete scheme based on the backward Euler method. We replace the time derivative in (3.2) by a backward difference quotient $\partial_t U_j = (U_j - U_{j-1})/k$, where k is a time step and U_j is the approximation of $u_j = u(t_j)$ and $t_j = jk$. The discrete solution $U_j \in V_h$ thus satisfies

$$\partial_t U_j + A_h U_j = P_h f(U_j),\ t_j > 0; \quad U_0 = u_{h,0}. \tag{3.18}$$

Duhamel's principle yields

$$U_j = E_{kh}^j u_{h,0} + k \sum_{l=1}^{j} E_{kh}^{j-l-1} P_h f(U_l), \quad t_j \geq 0, \tag{3.19}$$

where $E_{kh} = (I + kA_h)^{-1}$. Since A_h is self-adjoint positive definite (uniformly in h), we have the inequality

$$\|\partial_t^l E_{kh}^j v\| = \|A_h^l E_{kh}^j v\| \leq C_l t_j^{-l} \|v\|, \quad t_j \geq t_l,\ v \in V_h,\ l \geq 0,$$

which is the discrete analogue of (1.18). In view of the inequalities (3.5) and (3.6) this leads to a smoothing property analogous to (3.8):

$$\begin{aligned} \|\partial_t^l E_{kh}^j P_h f\|_\beta \leq C t_j^{-l-(\beta-\alpha)/2} \|f\|_\alpha, \quad t_j > 0,\ f \in \mathcal{D}(A^{\alpha/2}), \\ -1 \leq \alpha \leq \beta \leq 1,\ l = 0, 1. \end{aligned} \tag{3.20}$$

Again the proof of Theorem 1.2 carries over verbatim to the discrete case. We thus have:

Theorem 3.4. *For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that* (3.19) *has a unique solution U_j, $t_j \in [0, \tau]$, for any initial value $u_{h,0} \in V_h$ with $\|u_{h,0}\|_1 \leq R_0$. Moreover, there is c such that* $\max_{t_j \in [0,\tau]} \|U_j\|_1 \leq cR_0$.

We denote by $S_{hk}(t_j,\cdot)$ the corresponding (local) solution operator, so that $U_j = S_{hk}(t_j,u_{h,0})$ is the solution of (3.19).

We may now estimate the difference between the local solutions $u(t) = S(t,u_0)$ and $U_j = S_{hk}(t_j,u_{h,0})$ that we have obtained. We refer to the following result as a local a priori error estimate. The proof can be found in [10]. It follows the lines of the proof of Theorem 3.2 but is more technical.

Theorem 3.5. *Let $R \geq 0$ and $\tau > 0$ be given. Let $u(t)$ and U_j be solutions of* (3.1) *and* (3.19) *respectively, such that $u(t), U_j \in B_R$ for $t, t_j \in [0,\tau]$. Then, for $k \leq k_0(R)$ and $t_j \in (0,\tau]$, we have*

$$\begin{aligned}
\|U_j - u(t_j)\|_1 &\leq C(R,\tau)\big(\|u_{h,0} - P_h u_0\| t_j^{-1/2} + h t_j^{-1/2} + k t_j^{-1}\big),\\
\|U_j - u(t_j)\| &\leq C(R,\tau)\big(\|u_{h,0} - P_h u_0\| + h^2 t_j^{-1/2} + k t_j^{-1/2}\big).
\end{aligned}$$

There is also an analogue of Theorem 3.3.

Theorem 3.6. *Let $R \geq 0$ and $\tau > 0$ be given. Assume $S(t,v), S_{hk}(t_j,v_h) \in B_R$ for $t, t_j \in [0,2\tau]$. Then, for $k \leq k_0(R)$ and $l = 0,1$, we have*

$$\|S_{hk}(t_j,v_h) - S(t_j,v)\|_l \leq C(R,\tau)\big(\|v_h - P_h v\| + h^{2-l} + k\big), \quad t_j \in [\tau,2\tau].$$

4 Application to dynamical systems theory

Recall that we have defined (local) nonlinear semigroups: $S(t,\cdot): V \to V$ and $S_h(t,\cdot): V_h \to V_h$, where $u(t) = S(t,v)$ is the (local) solution of

$$u' + Au = f(u),\ t > 0; \quad u(0) = v, \tag{4.1}$$

and $u_h(t) = S_h(t,v_h)$ is the (local) solution of

$$u_h' + A_h u_h = P_h f(u_h),\ t > 0; \quad u_h(0) = v_h. \tag{4.2}$$

We now assume that we have global a priori bounds of the form: for any bounded set $B \subset V$ there is R such that, for all $v \in B$ and $v_h \in B \cap V_h$,

$$\|S(t,v)\|_1 \leq R, \quad \|S_h(t,v_h)\|_1 \leq R, \quad t \in [0,\infty). \tag{4.3}$$

Then $S(t,\cdot)$ and $S_h(t,\cdot)$ are defined for all $t \in [0,\infty)$. This is true, for example, for the Allen-Cahn equation as explained above.

We assume that $S(t,\cdot)$ has a global attractor $\mathcal{A}$, i.e., $\mathcal{A}$ is a compact invariant subset of V, which attracts the bounded sets of V. Thus, for any bounded set $B \subset V$ and any $\varepsilon > 0$ there is $T > 0$ such that

$$S(t,B) \subset \mathcal{N}(\mathcal{A},\varepsilon), \qquad t \in [T,\infty),$$

where $\mathcal{N}(\mathcal{A},\varepsilon)$ denotes the ε-neighborhood of $\mathcal{A}$ in V. Or equivalently,

$$\delta(S(t,B),\mathcal{A}) \to 0 \qquad \text{as } t \to \infty, \tag{4.4}$$

where

$$\delta(A,B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|_1$$

denotes the unsymmetric semidistance between two subsets A,B of V. We assume similarly that $S_h(t,\cdot)$ has a global attractor $\mathcal{A}_h$ in V_h.

Theorem 4.1. *Assume that $S(t,\cdot)$ has a global attractor $\mathcal{A}$ in V, and that $S_h(t,\cdot)$ has a global attractor $\mathcal{A}_h$ in V_h. Then*

$$\delta(\mathcal{A}_h,\mathcal{A}) \to 0 \quad \textit{as } h \to 0. \tag{4.5}$$

In other words: for any $\varepsilon > 0$ there is $h_0 > 0$ such that $\mathcal{A}_h \subset \mathcal{N}(\mathcal{A},\varepsilon)$ if $h < h_0$. We also say that $\mathcal{A}_h$ is upper semicontinuous at $h = 0$. This type of result was first proved in [6]. The opposite conclusion

$$\delta(\mathcal{A},\mathcal{A}_h) \to 0 \quad \text{as } h \to 0, \tag{4.6}$$

lower semicontinuity, has also been proved under additional assumptions on the structure of the attractor. The proof is much more complicated, [7].

Proof. We will use the local error estimate in Theorem 3.3. Take $\varepsilon > 0$ and a bounded set $B \subset V$. Consider $u_h(t) = S_h(t, v_h)$ with $v_h \in B \cap V_h$. We must find h_0 and T so that

$$\delta(S_h(t, v_h), \mathcal{A}) \leq \varepsilon, \quad \text{for } t \geq T,\ h \leq h_0.$$

Let $B_R = \{v \in V : \|v\|_1 \leq R\}$ with $R = R(B)$ as in (4.3). According to (4.4) we have $T = T(\varepsilon, B_R)$ such that

$$\delta(S(t, v_h), \mathcal{A}) \leq \varepsilon/2, \quad \text{for } t \geq T.$$

From Theorem 3.3 and (4.3), where $R = R(B)$, we have

$$\|S_h(t, v_h) - S(t, v_h)\|_1 \leq C(R, T)h, \quad t \in [T, 2T].$$

Hence there is $h_0 = h_0(\varepsilon, B, T)$ such that

$$\delta(S_h(t, v_h), \mathcal{A}) \leq \varepsilon/2 + C(R, T)h \leq \varepsilon, \quad t \in [T, 2T],\ h \leq h_0.$$

In order to obtain a bound on $[2T, 3T]$, we note that $u_h(T) = S_h(T, v_h) \in B_R$ by (4.3). Hence, according to (4.4) we have, with the same $T = T(\varepsilon, B_R)$,

$$\delta(S(t, u_h(T)), \mathcal{A}) \leq \varepsilon/2, \quad \text{for } t \geq 2T.$$

From Theorem 3.3 and (4.3), we have with the same $C(R, T)$,

$$\|S_h(t, u_h(T)) - S(t, u_h(T))\|_1 \leq C(R, T)h, \quad t \in [2T, 3T].$$

Hence there is $h_0 = h_0(\varepsilon, B, T)$ such that

$$\delta(S_h(t, v_h), \mathcal{A}) \le \varepsilon/2 + C(R,T)h \le \varepsilon, \quad t \in [2T, 3T],\ h \le h_0.$$

Repeating this we obtain the same bound on each interval of the form $[nT, (n+1)T]$. $\square$

Clearly, one can prove a similar theorem in the completely discrete case of Subsection 3.2.

5 A stochastic parabolic problem

In this lecture we briefly present an application of the above ideas to the error analysis for a stochastic parabolic problem. This is based on [13]. The theory of stochastic partial differential equations is developed in the monograph [3].

In order to comply with the standard notation in stochastic mathematics we let Ω denote the sample space and we change the notation for the spatial domain to be $\mathcal{D}$. Thus, we set

$$H = L_2(\mathcal{D}),\ V = H_0^1(\mathcal{D}),$$

and we let $A = -\Delta$ be as before with $D(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. We consider the equation

$$\begin{aligned} &du + Au\,dt = f(u)\,dt + g(u)\,dW, \qquad t > 0,\\ &u(0) = u_0, \end{aligned} \tag{5.1}$$

which is equation (1.8) written in differential form and with a stochastic noise term $g(u)\,dW$ added. Here $u(t)$ is an H-valued random process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ and $W(t)$ is an H-valued Wiener process.

To give a meaning to this equation we apply Duhamel's principle to get

$$\begin{aligned} u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s))\,ds \\ + \int_0^t E(t-s)g(u(s))\,dW(s), \end{aligned} \tag{5.2}$$

which corresponds to (1.12). A solution of (5.2) is called a mild solution of (5.1).

In order to proceed we must give a rigorous meaning to the stochastic integral $\int_0^t E(t-s)g(u(s))\,dW(s)$. To simplify the presentation we will do this for the reduced equation (5.1) where $f(u) = 0$ and $g(u) = I$, i.e.,

$$\begin{aligned} &du + Au\,dt = dW, \qquad t > 0, \\ &u(0) = u_0, \end{aligned} \tag{5.3}$$

and the mild solution is given by

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s), \quad t \geq 0. \tag{5.4}$$

A noise term $g(u)\,dW$, where $g(u)$ depends on u, is called *multiplicative noise* while the noise term dW in (5.3) is called *added noise*.

Following [3] we assume that W is given by an orthogonal series

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} \beta_l(t) e_l, \tag{5.5}$$

where $\gamma_l > 0$ and $\{e_l\}_{l=1}\infty$ are the eigenvalues and an orthonormal basis of corresponding eigenfunctions of a self-adjoint, positive definite, bounded, linear operator $Q : H \to H$, which is called the covariance operator of W. Furthermore, $\beta_l(t)$ are independent, identically distributed, real-valued, Brownian motions. More precisely, W depends on three variables

$$W(t,x,\omega) = \sum_{l=1}^{\infty} \gamma_l^{1/2} \beta_l(t,\omega) e_l(x), \quad t > 0,\ x \in \mathcal{D},\ \omega \in \Omega. \tag{5.6}$$

We may identify two interesting cases. In the first case we assume that $\mathrm{Tr}(Q) < \infty$. Then $W(t)$ is an H-valued Wiener process in the sense that the following sum is convergent:

$$\mathbf{E}\Big\| \sum_{l=1}^{\infty} \gamma_l^{1/2} \beta_l(t) e_l \Big\|^2 = \sum_{l=1}^{\infty} \gamma_l \mathbf{E} \beta_l(t)^2 = \sum_{l=1}^{\infty} \gamma_l t = t\,\mathrm{Tr}(Q) < \infty.$$

This is called "colored noise". In the second case we assume that $Q = I$. Then $W(t)$ is not H-valued, since $\mathrm{Tr}(I) = \infty$, but $W(t)$ exists in a weaker sense. This is called "white noise". Thus, the regularity of W is governed by the decay rate $\gamma_l \to 0$; the faster the decay the smoother the noise.

We now consider the stochastic integral in (5.4). It turns out that a stochastic integral

$$\int_0^t B(s)\, dW(s)$$

can be defined provided that the operator $B(s)Q^{1/2}$ is a Hilbert-Schmidt operator on H. Recall that an operator is T is Hilbert-Schmidt if

$$\|T\|_{\mathrm{HS}}^2 = \sum_{l=1}^{\infty} \|T\varphi_l\|^2 < \infty,$$

where $\{\varphi_l\}$ is an arbitrary orthonormal basis in H. From the construction follows that the integral has the isometry property

$$\mathbf{E}\Big\| \int_0^t B(s)\, dW(s) \Big\|^2 = \mathbf{E} \int_0^t \|B(s)Q^{1/2}\|_{\mathrm{HS}}^2\, ds. \tag{5.7}$$

We can now prove an regularity estimate for the mild solution in (5.4). It is convenient to define the following norms and spaces:

$$|v|_\beta = \|A^{\beta/2}v\|, \quad \dot{H}^\beta = D(A^{\beta/2}), \quad \beta \in \mathbf{R}, \tag{5.8}$$

and

$$\|v\|^2_{L_2(\Omega,\dot{H}^\beta)} = \mathbf{E}(|v|^2_\beta) = \int_\Omega \int_{\mathcal{D}} |A^{\beta/2}v|^2 \, dx \, d\mathbf{P}(\omega), \quad \beta \in \mathbf{R}. \tag{5.9}$$

Theorem 5.1. *If* $\|A^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \geq 0$*, then the mild solution in* (5.4) *satisfies*

$$\|u(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\Big(\|u_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big). \tag{5.10}$$

Proof. We take norms in (5.4) and use the isometry in (5.7) to get

$$\begin{aligned}
\mathbf{E}|u(t)|^2_\beta &\leq C\Big(\mathbf{E}|E(t)u_0|^2_\beta + \mathbf{E}\Big|\int_0^t E(t-s)\,dW(s)\Big|^2_\beta\Big) \\
&= C\Big(\mathbf{E}\|A^{\beta/2}E(t)u_0\|^2 + \mathbf{E}\Big\|\int_0^t A^{\beta/2}E(t-s)\,dW(s)\Big\|^2\Big) \\
&= C\Big(\mathbf{E}\|A^{\beta/2}E(t)u_0\|^2 + \int_0^t \|A^{\beta/2}E(t-s)Q^{1/2}\|^2_{\mathrm{HS}}\,ds\Big) \\
&= C\Big(\mathbf{E}\|E(t)A^{\beta/2}u_0\|^2 \\
&\qquad + \sum_l \int_0^t \|A^{1/2}E(t-s)A^{(\beta-1)/2}Q^{1/2}\varphi_l\|^2\,ds\Big) \\
&\leq C\Big(\mathbf{E}\|A^{\beta/2}u_0\|^2 + \sum_{l=1}^\infty \|A^{(\beta-1)/2}Q^{1/2}\varphi_l\|^2\Big) \\
&= C\Big(\mathbf{E}|u_0|^2_\beta + \|A^{(\beta-1)/2}Q^{1/2}\|^2_{\mathrm{HS}}\Big),
\end{aligned}$$

which is (5.10). Here we also used the bounds

$$\|E(t)v\| \leq \|v\|, \quad \int_0^t \|A^{1/2}E(t-s)v\|^2\,ds \leq \|v\|^2, \tag{5.11}$$

which are easily proved as in (1.18). □

We examine the assumption $\|A^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ of the theorem in the two cases mentioned above. First, if $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then we may take $\beta = 1$. In the other case, if $Q = I$, then since $\lambda_j \approx j^{2/d}$, see [11, Chapt. 6], we have

$$\|A^{(\beta-1)/2}\|_{\mathrm{HS}}^2 = \sum_j \lambda_j^{-(1-\beta)} \approx \sum_j j^{-(1-\beta)2/d} < \infty \tag{5.12}$$

if and only if $d = 1$, $\beta < 1/2$. Thus, the assumption can only be satisfied for $d = 1$, in which case $A = -\frac{\partial^2}{\partial x^2}$.

We now discretize by the finite element method. The approximation of (5.4) is to find $u_h(t) \in V_h$ such that

$$\begin{aligned} du_h + A_h u_h\, dt &= P_h dW, \\ u_h(0) &= P_h u_0, \end{aligned} \tag{5.13}$$

cf. (3.4). More rigorously, with $E_h(t) = e^{-tA_h}$, we consider the mild solution

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)P_h\, dW(s), \tag{5.14}$$

cf. (3.9). We want to estimate the error $u_h(t) - u(t)$. We prepare by proving estimates for the error in the deterministic finite element problem.

Theorem 5.2. *Denote $F_h(t)v = E_h(t)P_h v - E(t)v$. Then we have, for $0 \le \beta \le 2$ and $t \ge 0$,*

$$\|F_h(t)v\| \le Ch^\beta |v|_\beta, \tag{5.15}$$

$$\Big(\int_0^t \|F_h(s)v\|^2\, ds\Big)^{1/2} \le Ch^\beta |v|_{\beta-1}. \tag{5.16}$$

Proof. We write temporarily $u_h(t) = E_h(t)P_h v$, $u(t) = E(t)v$ and set $e(t) = u_h(t) - u(t) = E_h(t)P_h v - E(t)v$. In order to prove (5.15) we recall the proof of Theorem 3.2, in particular, (3.17) which becomes

$$\begin{aligned}\theta(t) &= -E_h(t/2)P_h\rho(t/2)\\ &\quad + \int_0^{t/2} \big(D_\sigma E_h(t-\sigma)\big)P_h\rho(\sigma)\,d\sigma\\ &\quad - \int_{t/2}^t E_h(t-\sigma)P_h D_\sigma\rho(\sigma)\,d\sigma,\end{aligned}$$

because now $P_h e(0) = P_h(u_h(0) - u(0)) = P_h(P_h v - v) = 0$ and $f = 0$. Using the smoothing property (3.8) of $E_h(t)P_h$, the error bound (3.13), and the smoothing property (1.19) of $E(t)$, we obtain

$$\begin{aligned}\|\theta(t)\| &\le \|\rho(t/2)\| + C\int_0^{t/2}(t-\sigma)^{-1}\|\rho(\sigma)\|\,d\sigma\\ &\quad + C\int_{t/2}^t \|D_\sigma\rho(\sigma)\|\,d\sigma\\ &\le Ch^2\|E(t/2)v\|_2 + Ch^2\int_0^{t/2}(t-\sigma)^{-1}\|E(\sigma)v\|_2\,d\sigma\\ &\quad + Ch^2\int_{t/2}^t \|D_\sigma E(\sigma)v\|_2\,d\sigma\\ &\le Ch^2\|v\|_2\Big(1 + \int_0^{t/2}(t-\sigma)^{-1}\,d\sigma + \int_{t/2}^t \sigma^{-1}\,d\sigma\Big)\\ &\le Ch^2\|v\|_2.\end{aligned}$$

Together with a similar bound for $\|\rho(t)\|$ this implies the special case $\beta = 2$ of (5.15), because the norms $|\cdot|_2$ and $\|\cdot\|_2$ are equivalent on $D(A)$. The special case $\beta = 0$ is trivial because

$$\|F_h(t)v\| \le \|E_h(t)v\| + \|E(t)v\| \le 2\|v\|,$$

and the intermediate cases follow by interpolation.

In order to prove (5.16) we derive another equation for the error:

$$\begin{aligned}
A_h^{-1}P_h e' + e &= A_h^{-1}P_h u_h' - A_h^{-1}P_h u' + u_h - u \\
&= -A_h^{-1}P_h A_h u_h + A_h^{-1}P_h A u + u_h - u \\
&= A_h^{-1}P_h A u - u \\
&= R_h A u - u = \rho,
\end{aligned}$$

since $u_h' = -A_h u_h$, $u' = -Au$, and $A_h^{-1}P_h A = R_h$. Taking the scalar product of this equation with e yields

$$(A_h^{-1}P_h e', e) + \|e\|^2 = (\rho, e) \le \|\rho\|\,\|e\| \le \frac{1}{2}\|\rho\|^2 + \frac{1}{2}\|e\|^2,$$

so that

$$D_t(A_h^{-1}P_h e, e) + \|e\|^2 \le \|\rho\|^2,$$

and after integration, recall that $P_h e(0) = 0$,

$$(A_h^{-1}P_h e(t), e(t)) + \int_0^t \|e\|^2\,ds \le \int_0^t \|\rho\|^2\,ds,$$

and hence

$$\begin{aligned}
\int_0^t \|e\|^2\,ds &\le \int_0^t \|\rho\|^2\,ds \le Ch^4 \int_0^t \|E(s)v\|_2^2\,ds \\
&\le Ch^4 \int_0^t \|A^{1/2}E(s)A^{1/2}v\|^2\,ds \le Ch^4\|A^{1/2}v\|^2 = Ch^4|v|_1^2,
\end{aligned}$$

where we used (3.13) and (5.11). This is the special case $\beta = 2$ of (5.16). The special case $\beta = 0$ is proved by using (5.11) again:

$$\int_0^t \|E(s)v\|^2\,ds = \int_0^t \|A^{1/2}E(s)A^{-1/2}v\|^2\,ds \le \|A^{-1/2}v\|^2 = |v|_{-1}^2,$$

and similarly for the finite element semigroup

$$\int_0^t \|E_h(s)P_h v\|^2\,ds = \int_0^t \|A_h^{1/2}E_h(s)A_h^{-1/2}P_h v\|^2\,ds$$
$$\leq \|A_h^{-1/2}P_h v\|^2 \leq |v|_{-1}^2,$$

where we also used (3.6). The intermediate cases follow by interpolation. $\square$

We can now prove strong convergence in L_2 norm. We remind the reader that

$$\|u_h(t)-u(t)\|^2_{L_2(\Omega,H)} = \mathbf{E}(\|u_h(t)-u(t)\|^2)$$

is the expected value of the square of the $L_2(\mathcal{D})$-norm of the error.

Theorem 5.3. *If* $\|A^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,2]$*, then*

$$\|u_h(t)-u(t)\|_{L_2(\Omega,H)} \leq Ch^\beta\Big(\|u_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big).$$

Proof. We recall

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s)$$

and

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)\,P_h dW(s),$$

so that with $F_h(t) = E_h(t)P_h - E(t)$,

$$u_h(t)-u(t) = F_h(t)u_0 + \int_0^t F_h(t-s)\,dW(s) = e_1(t)+e_2(t).$$

Using (5.15) we get

$$\|e_1(t)\|_{L_2(\Omega,H)} \leq Ch^\beta \|u_0\|_{L_2(\Omega,\dot{H}^\beta)}$$

as desired. For the other term we use the isometry in (5.7) and (5.16) to get

$$
\begin{aligned}
\|e_2(t)\|^2_{L_2(\Omega,H)} &= \mathbf{E}\Big\| \int_0^t F_h(t-s)\,dW(s)\Big\|^2 = \int_0^t \|F_h(t-s)Q^{1/2}\|^2_{\mathrm{HS}}\,ds \\
&= \sum_{l=1}^{\infty} \int_0^t \|F_h(t-s)Q^{1/2}\varphi_l\|^2\,ds \le C\sum_{l=1}^{\infty} h^{2\beta}|Q^{1/2}\varphi_l|^2_{\beta-1} \\
&= Ch^{2\beta}\sum_{l=1}^{\infty} \|A^{(\beta-1)/2}Q^{1/2}\varphi_l\|^2 = Ch^{2\beta}\|A^{(\beta-1)/2}Q^{1/2}\|^2_{\mathrm{HS}}.
\end{aligned}
$$

This completes the proof. □

We recall the two cases discussed before. If $\|Q^{1/2}\|^2_{\mathrm{HS}} = \mathrm{Tr}(Q) < \infty$, then the convergence rate is $O(h)$. If $Q = I$, then the assumption can be satisfied only if $d = 1$, $A = -\frac{\partial^2}{\partial x^2}$, in which case the rate is almost $O(h^{1/2})$. We have no result for $Q = I$, $d \ge 2$. This result thus indicates that convergence rates in the multidimensional situation, $d \ge 2$, can only be proved for colored noise.

The paper [13] also contains an analogous analysis of a temporal discretization by the backward Euler method.

The numerical approximation of stochastic partial differential equations is a relatively new field of research and contains many open problems. For example:

- The noise term in (5.13) is the orthogonal projection of the infinite series (5.6). This is not directly computable, except if the eigenfunctions e_l are explicitly known. We need another representation of colored noise which is computable. This can perhaps be obtained by expanding W in a wavelet basis instead of eigenfunctions.

- The convergence in Theorem 5.3 is strong convergence, i.e., convergence in norm. A more useful convergence concept is weak convergence, or convergence in law. It is a challenge to prove weak convergence of finite element approximations.

- Extension of the analysis to various nonlinear stochastic partial differential equations.

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