# Discretization of Integro-Differential Equations Modeling Dynamic Fractional Order Viscoelasticity

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**Abstract.** We study a dynamic model for viscoelastic materials based on a constitutive equation of fractional order. This results in an integro-differential equation with a weakly singular convolution kernel. We discretize in the spatial variable by a standard Galerkin finite element method. We prove stability and regularity estimates which show how the convolution term introduces dissipation into the equation of motion. These are then used to prove a priori error estimates. A numerical experiment is included.

## 1 Introduction

Fractional order operators (integrals and derivatives) have proved to be very suitable for modeling memory effects of various materials and systems of technical interest. In particular, they are very useful when modeling viscoelastic materials, see, e.g., [3].

Numerical methods for quasistatic viscoelasticity problems have been studied, e.g., in [2] and [8]. The drawback of the fractional order viscoelastic models is that the whole strain history must be saved and included in each time step. The most commonly used algorithms for this integration are based on the Lubich convolution quadrature [5] for fractional order operators. In [1, 2], we develop an efficient numerical algorithm based on sparse numerical quadrature earlier studied in [6].

While our earlier work focused on discretization in time for the quasistatic case, we now study space discretization for the fully dynamic equations of motion, which take the form of an integro-differential equation with a weakly singular convolution kernel. A similar equation but with smooth kernel was studied in [7]. The singular kernel requires a different approach. Inspired by [4] we introduce appropriate function spaces and prove stability estimates for both the continuous and discrete problems. These are used to prove a priori error estimates. Finally, we present a numerical example for a two-dimensional viscoelastic body. Time-discretization, sparse quadrature, and a posteriori error estimates are subject to future investigations.

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## 2 Viscoelastic Equations of Motion

Assuming isothermal and isotropic conditions, the fractional order linear viscoelastic constitutive equation for the stress $\boldsymbol{\sigma}$ can be written in convolution integral form as

$$\begin{aligned}&\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_0(t) - \int_0^t \beta(t-s)\boldsymbol{\sigma}_1(s)\,\mathrm{d}s, \quad \text{with}\\ &\boldsymbol{\sigma}_0(t) = 2\mu_0\boldsymbol{\epsilon}(t) + \lambda_0 \mathrm{tr}(\boldsymbol{\epsilon}(t))\mathrm{I}, \quad \boldsymbol{\sigma}_1(t) = 2\mu_1\boldsymbol{\epsilon}(t) + \lambda_1 \mathrm{tr}(\boldsymbol{\epsilon}(t))\mathrm{I},\end{aligned} \tag{1}$$

where $\lambda_0 > \lambda_1 > 0$ and $\mu_0 > \mu_1 > 0$ are the elastic constants of Lamé type, $\boldsymbol{\epsilon}$ is the strain and $\beta$ is the convolution kernel

$$\beta(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{E}_\alpha(-(t/\tau)^\alpha)\Big) = \frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{\alpha-1}\mathrm{E}'_\alpha(-(t/\tau)^\alpha). \tag{2}$$

Here $\mathrm{E}_\alpha(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(1+\alpha k)}$ is the Mittag-Leffler function of order $\alpha$.

In the convolution kernel (2), $\tau$ is a relaxation constant and $\alpha \in (0,1)$ is the order of the fractional derivative. The convolution kernel is weakly singular and $\beta \in L_1(0,\infty)$ with $\int_0^\infty \beta(t)\,\mathrm{d}t = 1$. The fractional order model represents a fading memory because the convolution kernel in (2) is a strictly decreasing function (i.e., $\mathrm{d}\beta/\mathrm{d}t < 0$). The Lamé constants in (1) can be expressed as

$$\mu_0 = \frac{E_0}{2(1+\nu)}, \quad \mu_1 = \frac{E_1}{2(1+\nu)}, \quad \lambda_0 = \frac{E_0\nu}{(1+\nu)(1-2\nu)}, \quad \lambda_1 = \frac{E_1\nu}{(1+\nu)(1-2\nu)},$$

where $\nu$ is the Poisson ratio, $E_0$ is the instantaneous uniaxial elastic modulus, while $E_0 - E_1 > 0$ can be identified as the relaxed uniaxial modulus. For convenience we introduce $\gamma = \frac{\mu_1}{\mu_0} = \frac{\lambda_1}{\lambda_0} = \frac{E_1}{E_0} < 1$, and note that $\boldsymbol{\sigma}_1 = \gamma\boldsymbol{\sigma}_0$.

We are now in the position to formulate the viscoelastic dynamic problem. The basic equations in strong form are

$$\begin{aligned}
&\rho\ddot{\mathbf{u}}(\mathbf{x},t) - \nabla\cdot\boldsymbol{\sigma}_0(\mathbf{u};\mathbf{x},t)\\
&\qquad + \int_0^t \beta(t-s)\nabla\cdot\boldsymbol{\sigma}_1(\mathbf{u};\mathbf{x},s)\,\mathrm{d}s = \mathbf{f}(\mathbf{x},t) && \text{in } \Omega\times I,\\
&\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega,\\
&\dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) && \text{in } \Omega,\\
&\mathbf{u}(\mathbf{x},t) = \mathbf{0} && \text{on } \Gamma_\mathrm{D}\times I,\\
&\boldsymbol{\sigma}(\mathbf{u};\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = \mathbf{g}(\mathbf{x},t) && \text{on } \Gamma_\mathrm{N}\times I,
\end{aligned} \tag{3}$$

where $\rho$ is the (constant) mass density, $\mathbf{f}, \mathbf{g}$ represent the volume and surface loads, respectively, $\mathbf{u}$ is the displacement vector, $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_1$ are the stresses according to (1), and the strain is defined through the usual linear kinematic relation $\boldsymbol{\epsilon} = \frac{1}{2}\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^\mathrm{T}\Big)$.

In order to give the equations (3) a convenient mathematical formulation, we let $\Omega \subset \mathbf{R}^d$, $d = 2,3$, be a bounded domain with $\partial\Omega = \bar{\Gamma}_\mathrm{D} \cup \bar{\Gamma}_\mathrm{N}$, $\Gamma_\mathrm{D} \cap \Gamma_\mathrm{N} = \emptyset$, meas $(\Gamma_{\mathrm{D}}) > 0$, and we define $H = L_2(\Omega)^d$ with its usual inner product and norm denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. We also define $V = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{\mathrm{D}}\}$ and the bilinear form

$$a(\mathbf{v},\mathbf{w}) = \int_\Omega \big(2\mu_0\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w}) + \lambda_0\epsilon_{ii}(\mathbf{v})\epsilon_{jj}(\mathbf{w})\big)\,\mathrm{d}\mathbf{x}, \quad \mathbf{v},\mathbf{w} \in V.$$

It is well known that $a$ is coercive on $V$. The corresponding operator $A\mathbf{u} = -\nabla\cdot\boldsymbol{\sigma}_0(\mathbf{u})$, defined together with the homogeneous boundary conditions in (3) ($\mathbf{g} = \mathbf{0}$), so that $a(\mathbf{u},\mathbf{v}) = (A\mathbf{u},\mathbf{v})$ for sufficiently smooth $\mathbf{u},\mathbf{v} \in V$, can be extended to a self-adjoint, positive definite, unbounded linear operator on $H$.

The equation of motion (3) can then be written in weak form: Find $\mathbf{u}(t) \in V$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{v}_0$ and, with $\langle \mathbf{g},\mathbf{v}\rangle_{\Gamma_{\mathrm{N}}} = \int_{\Gamma_{\mathrm{N}}} \mathbf{g}\cdot\mathbf{v}\,\mathrm{d}S$,

$$\begin{aligned}\rho(\ddot{\mathbf{u}}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) - \gamma\int_0^t \beta(t-s)a(\mathbf{u}(s),\mathbf{v})\,\mathrm{d}s \\ = (\mathbf{f}(t),\mathbf{v}) + \langle \mathbf{g}(t),\mathbf{v}\rangle_{\Gamma_{\mathrm{N}}}, \quad \forall \mathbf{v} \in V.\end{aligned} \tag{4}$$

We next introduce the spatially semidiscrete finite element method. Let $V_h \subset V$ be a standard piecewise linear finite element space based on a triangulation of $\Omega$. The finite element problem is to find $\mathbf{u}_h(t) \in V_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$, $\dot{\mathbf{u}}_h(0) = \mathbf{v}_{h,0}$ and

$$\begin{aligned}\rho(\ddot{\mathbf{u}}_h(t),\mathbf{v}_h) + a(\mathbf{u}_h(t),\mathbf{v}_h) - \gamma\int_0^t \beta(t-s)a(\mathbf{u}_h(s),\mathbf{v}_h)\,\mathrm{d}s \\ = (\mathbf{f}(t),\mathbf{v}_h) + \langle \mathbf{g}(t),\mathbf{v}_h\rangle_{\Gamma_{\mathrm{N}}}, \quad \forall \mathbf{v}_h \in V_h.\end{aligned} \tag{5}$$

## 3 Stability Estimates

By adapting the analysis in [4] we can show existence and uniqueness of solutions of (3) by means of the theory of strongly continuous semigroups. We leave the details to a forthcoming paper and show only the main ingredient, namely, that the convolution term introduces dissipation into the equation. We introduce the function

$$\xi(t) = 1 - \gamma\int_0^t \beta(s)\,ds,$$

which is decreasing with $\xi(0) = 1$, $\lim_{t\to\infty}\xi(t) = 1-\gamma$, so that $\xi(t) \geq 1-\gamma > 0$. We also use the norms

$$\|\mathbf{v}\|_l = \|A^{l/2}\mathbf{v}\| = \sqrt{(\mathbf{v},A^l\mathbf{v})}, \quad l \in \mathbf{R}.$$

**Theorem 1.** *Let* $\mathbf{u}$ *be the solution of* (4) *with sufficiently smooth data* $\mathbf{u}_0,\mathbf{v}_0,\mathbf{f}$, $\mathbf{g}$, *and denote* $\mathbf{v} = \dot{\mathbf{u}}$ *and* $\mathbf{w}(t,s) = \mathbf{u}(t) - \mathbf{u}(t-s)$. *Then, for any* $l \in \mathbf{R}$, $T > 0$, *we have the identity*

$$\begin{aligned}\rho\|\mathbf{v}(T)\|_l^2 &+ \xi(T)\|\mathbf{u}(T)\|_{l+1}^2 \\ &+ \gamma\int_0^T \beta(t)\|\mathbf{u}(t)\|_{l+1}^2\,\mathrm{d}t + \gamma\int_0^T \beta(s)\|\mathbf{w}(T,s)\|_{l+1}^2\,\mathrm{d}s \\ &+ \gamma\int_0^T\int_0^t [\beta(s)-\beta(t)]\,\mathrm{D}_s\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s\,\mathrm{d}t \\ &= \rho\|\mathbf{v}_0\|_l^2 + \|\mathbf{u}_0\|_{l+1}^2 + 2\int_0^T (\mathbf{f}, A^l\mathbf{v})\,\mathrm{d}t + 2\int_0^T \langle \mathbf{g}, A^l\mathbf{v}\rangle_{\Gamma_{\mathrm{N}}}\,\mathrm{d}t,\end{aligned} \tag{6}$$

*where all terms on the left-hand side are non-negative. Moreover, for* $l=0$*,*

$$\begin{aligned}\rho^{\frac12}\|\dot{\mathbf{u}}(T)\| + (1-\gamma)^{\frac12}\|\mathbf{u}(T)\|_1 \le C\Big[\rho^{\frac12}\|\mathbf{v}_0\| + \|\mathbf{u}_0\|_1 \\ + \rho^{-\frac12}\int_0^T \|\mathbf{f}\|\,\mathrm{d}t + (1-\gamma)^{-\frac12}\Big(\max_{[0,T]}\|\mathbf{g}\|_{L_2(\Gamma_{\mathrm{N}})} + \int_0^T \|\dot{\mathbf{g}}\|_{L_2(\Gamma_{\mathrm{N}})}\,\mathrm{d}t\Big)\Big].\end{aligned} \tag{7}$$

We remark that if $\mathbf{g}=\mathbf{0}$, then we can kick back for any $l$ and estimate $\|\dot{\mathbf{u}}(T)\|_l + \|\mathbf{u}(T)\|_{l+1}$, see Theorem 2 below.

*Proof.* Equation (4) can be written in the form

$$\begin{aligned}\rho(\dot{\mathbf{v}}(t),\boldsymbol{\psi}) + \xi(t)a(\mathbf{u}(t),\boldsymbol{\psi}) + \gamma\int_0^t \beta(s)a(\mathbf{w}(t,s),\boldsymbol{\psi})\,\mathrm{d}s \\ = (\mathbf{f}(t),\boldsymbol{\psi}) + \langle \mathbf{g}(t),\boldsymbol{\psi}\rangle_{\Gamma_{\mathrm{N}}}, \quad \forall\boldsymbol{\psi}\in V.\end{aligned} \tag{8}$$

Taking $\boldsymbol{\psi} = A^l\mathbf{v}(t)$ in (8) and integrating in $t$ we get:

$$\begin{aligned}&\rho\int_0^T (\mathrm{D}_t\mathbf{v}, A^l\mathbf{v})\,\mathrm{d}t + \int_0^T \xi(t)(A\mathbf{u}, A^l\mathbf{v})\,\mathrm{d}t \\ &+ \gamma\int_0^T\int_0^t \beta(s)(A\mathbf{w}(t,s), A^l\mathbf{v}(t))\,\mathrm{d}s\,\mathrm{d}t = \int_0^T (\mathbf{f}, A^l\mathbf{v})\,\mathrm{d}t + \int_0^T \langle \mathbf{g}, A^l\mathbf{v}\rangle_{\Gamma_{\mathrm{N}}}\,\mathrm{d}t.\end{aligned}$$

We consider each term on the left-hand side. For the first term we have:

$$\rho\int_0^T (\mathrm{D}_t\mathbf{v}, A^l\mathbf{v})\,\mathrm{d}t = \frac{\rho}{2}\int_0^T \mathrm{D}_t\|\mathbf{v}\|_l^2\,\mathrm{d}t = \frac{\rho}{2}\|\mathbf{v}(T)\|_l^2 - \frac{\rho}{2}\|\mathbf{v}_0\|_l^2. \tag{9}$$

For the second one we have:

$$\begin{aligned}\int_0^T \xi(t)(A\mathbf{u}, A^l\mathbf{v})\,\mathrm{d}t &= \frac12\int_0^T \xi(t)\mathrm{D}_t\|\mathbf{u}\|_{l+1}^2\,\mathrm{d}t \\ &= \frac12\xi(T)\|\mathbf{u}(T)\|_{l+1}^2 - \frac12\|\mathbf{u}_0\|_{l+1}^2 + \frac{\gamma}{2}\int_0^T \beta(t)\|\mathbf{u}(t)\|_{l+1}^2\,\mathrm{d}t.\end{aligned} \tag{10}$$

For the third term, using $\mathbf{v}(t) = \dot{\mathbf{u}}(t) = \mathrm{D}_t\mathbf{w}(t,s) + \mathrm{D}_s\mathbf{w}(t,s)$, we get:

$$\begin{aligned}\int_0^T \int_0^t \beta(s)\left(A\mathbf{w}(t,s), A^l\mathbf{v}(t)\right)\,\mathrm{d}s\,\mathrm{d}t \\ = \frac{1}{2}\int_0^T \int_0^t \beta(s)\left(\mathrm{D}_t\|\mathbf{w}(t,s)\|_{l+1}^2 + \mathrm{D}_s\|\mathbf{w}(t,s)\|_{l+1}^2\right)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{11}$$

In the first term we change the order of integration:

$$\begin{aligned}\frac{1}{2}\int_0^T \int_s^T \beta(s)\mathrm{D}_t\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}t\,\mathrm{d}s \\ = \frac{1}{2}\int_0^T \beta(s)\|\mathbf{w}(T,s)\|_{l+1}^2\,\mathrm{d}s - \frac{1}{2}\int_0^T \beta(s)\|\mathbf{w}(s,s)\|_{l+1}^2\,\mathrm{d}s.\end{aligned}$$

Using that

$$\frac{1}{2}\int_0^T \beta(s)\|\mathbf{w}(s,s)\|_{l+1}^2\,\mathrm{d}s = \frac{1}{2}\int_0^T \int_0^t \beta(t)\mathrm{D}_s\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s\,\mathrm{d}t,$$

we can write

$$\begin{aligned}\int_0^T \int_0^t \beta(s)\left(A\mathbf{w}(t,s), A^l\mathbf{v}(t)\right)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{2}\int_0^T \beta(s)\|\mathbf{w}(T,s)\|_{l+1}^2\,\mathrm{d}s \\ -\frac{1}{2}\int_0^T \int_0^t \left[\beta(t) - \beta(s)\right]\mathrm{D}_s\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{12}$$

To show the positivity of the last term we consider, for $0 < \varepsilon < t$, the integral

$$\begin{aligned}\int_\varepsilon^t \left[\beta(t) - \beta(s)\right]\mathrm{D}_s\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s = -\left[\beta(t) - \beta(\varepsilon)\right]\|\mathbf{w}(t,\varepsilon)\|_{l+1}^2 \\ + \int_\varepsilon^t \beta'(s)\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s \le \beta(\varepsilon)\|\mathbf{w}(t,\varepsilon)\|_{l+1}^2,\end{aligned}$$

where we have taken into account that $\beta'(s) \le 0$ and $\beta(t) \ge 0$. Using

$$\mathbf{w}(t,\varepsilon) = \mathbf{w}(t,0) + \int_0^\varepsilon \mathrm{D}_s\mathbf{w}(t,s)\,\mathrm{d}s = \int_0^\varepsilon \mathrm{D}_s\mathbf{w}(t,s)\,\mathrm{d}s,$$

and the Cauchy-Schwarz inequality we get

$$\|\mathbf{w}(t,\varepsilon)\|_{l+1}^2 \le \left(\int_0^\varepsilon \|\mathrm{D}_s\mathbf{w}(t,s)\|_{l+1}\,\mathrm{d}s\right)^2 \le \int_0^\varepsilon \frac{\mathrm{d}s}{\beta(s)} \int_0^\varepsilon \beta(s)\|\mathrm{D}_s\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s,$$

and consequently

$$\int_\varepsilon^t \left[\beta(t) - \beta(s)\right]\mathrm{D}_s\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s \le \int_0^\varepsilon \frac{\beta(\varepsilon)}{\beta(s)}\,\mathrm{d}s \int_0^\varepsilon \beta(s)\|\mathrm{D}_s\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s.$$

But $\dfrac{\beta(\varepsilon)}{\beta(s)} \le 1$, which yields $\displaystyle\int_0^\varepsilon \frac{\beta(\varepsilon)}{\beta(s)}\,\mathrm{d}s \le \int_0^\varepsilon \mathrm{d}s = \varepsilon$, so that

$$\int_\varepsilon^t [\beta(t)-\beta(s)]\,\mathrm{D}_s\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s \le \varepsilon \int_0^\varepsilon \beta(s)\|\mathrm{D}_s\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s.$$

According to [4] we have $\int_0^T \beta(s)\|\mathrm{D}_s\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s < \infty$ provided that the data are sufficiently smooth ($A^{l+2}\mathbf{u}_0 \in H$, $A^{l+1}\mathbf{v}_0 \in H$, and certain conditions on $\mathbf{f}$, g). Letting $\boldsymbol{\varepsilon \longrightarrow 0}$ we get

$$\int_0^t [\beta(t)-\beta(s)]\,\mathrm{D}_s\|\mathbf{w}(t,s)\|_{l+1}^2\,\mathrm{d}s \le 0. \tag{13}$$

From (9), (10), (12), and (13) we conclude (6), from which (7) follows easily, in view of the trace inequality and the fact that the energy norm $\|\cdot\|_1$ is equivalent to the $\boldsymbol{H^1}$ norm. □

The previous proof applies also to the finite element problem (5) if we use the orthogonal projection $P_h : H \to V_h$ and the operator $A_h : V_h \to V_h$ defined by

$$a(\mathbf{v}_h,\mathbf{w}_h) = (A_h\mathbf{v}_h,\mathbf{w}_h) \quad \forall \mathbf{v}_h,\mathbf{w}_h \in V_h,$$

and use the discrete norms

$$\|\mathbf{v}_h\|_{h,l} = \|A_h^{l/2}\mathbf{v}_h\| = \sqrt{(\mathbf{v}_h, A_h^l\mathbf{v}_h)}, \quad \mathbf{v}_h \in V_h,\ l \in \mathbf{R}.$$

It is sufficient to prove the discrete stability with boundary data $\mathbf{g} = \mathbf{0}$.

**Theorem 2.** *Let $\mathbf{u}_h$ solve (5) with $\mathbf{g} = \mathbf{0}$. Then, for any $l \in \mathbf{R}$, $T > 0$, we have*

$$\begin{aligned}\rho^{\frac12}\|\dot{\mathbf{u}}_h(T)\|_{h,l} &+ (1-\gamma)^{\frac12}\|\mathbf{u}_h(T)\|_{h,l+1} \\ &\le C\Big[\rho^{\frac12}\|\mathbf{v}_{h,0}\|_{h,l} + \|\mathbf{u}_{h,0}\|_{h,l+1} + \rho^{-\frac12}\int_0^T \|P_h\mathbf{f}\|_{h,l}\,\mathrm{d}t\Big].\end{aligned} \tag{14}$$

## 4 A Priori Error Estimates

Let $R_h : V \to V_h$ be the Ritz projection defined by

$$a(R_h\mathbf{v}-\mathbf{v},\mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in V_h. \tag{15}$$

In this section we assume the elliptic regularity estimate

$$\|\mathbf{v}\|_{H^2} \le C\|A\mathbf{v}\| \quad \forall \mathbf{v} \in D(A), \tag{16}$$

so that the following error estimates can be proved (by duality)

$$\|R_h\mathbf{v}-\mathbf{v}\|_{H^l} \le Ch^{m-l}\|\mathbf{v}\|_{H^m}, \tag{17}$$

for all integers $0 \le l < m \le 2$. The elliptic regularity (16) holds, for example, for the pure Dirichlet problem ($\Gamma_\mathrm{D} = \partial\Omega$) when $\Omega$ is a convex polygonal domain. For more general boundary conditions and domains the situation is more complicated and we refrain from a discussion of this.

**Theorem 3.** *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of* (4) *and* (5)*, respectively, and denote* $\mathbf{e} = \mathbf{u}_h - \mathbf{u}$*. Then, with* $C$ *depending on* $\rho, \gamma$*,*

$$\begin{aligned}
\|\dot{\mathbf{e}}(T)\| &\le C\Big(\|\mathbf{v}_{h,0} - \mathbf{v}_0\| + \|\mathbf{u}_{h,0} - R_h\mathbf{u}_0\|_{H^1}\Big) \\
&\quad + Ch^2\Big(\|\mathbf{v}_0\|_{H^2} + \|\dot{\mathbf{u}}(T)\|_{H^2} + \int_0^T \|\ddot{\mathbf{u}}\|_{H^2}\,\mathrm{d}t\Big), \\
\|\mathbf{e}(T)\|_{H^1} &\le C\Big(\|\mathbf{v}_{h,0} - \mathbf{v}_0\| + \|\mathbf{u}_{h,0} - \mathbf{u}_0\|_{H^1}\Big) \\
&\quad + Ch\Big(\|\mathbf{v}_0\|_{H^1} + \|\mathbf{u}_0\|_{H^2} + \|\mathbf{u}(T)\|_{H^2} + \int_0^T \|\ddot{\mathbf{u}}\|_{H^1}\,\mathrm{d}t\Big), \\
\|\mathbf{e}(T)\| &\le C\Big(\|\mathbf{v}_{h,0} - \mathbf{v}_0\| + \|\mathbf{u}_{h,0} - \mathbf{u}_0\|\Big) \\
&\quad + Ch^2\Big(\|\mathbf{v}_0\|_{H^2} + \|\mathbf{u}_0\|_{H^2} + \|\mathbf{u}(T)\|_{H^2} + \int_0^T \|\ddot{\mathbf{u}}\|_{H^2}\,\mathrm{d}t\Big).
\end{aligned}$$

*Proof.* In the usual way we split the error $\mathbf{u}_h - \mathbf{u} = \boldsymbol{\theta} + \boldsymbol{\rho}$, where $\boldsymbol{\theta} = \mathbf{u}_h - R_h\mathbf{u}$, $\boldsymbol{\rho} = R_h\mathbf{u} - \mathbf{u}$. In view of (17) it is sufficient to estimate $\boldsymbol{\theta}$. From (4), (5), and (15) we have

$$\rho(\ddot{\boldsymbol{\theta}}, \mathbf{v}_h) + a(\boldsymbol{\theta}, \mathbf{v}_h) - \gamma \int_0^t \beta(s) a(\boldsymbol{\theta}(t-s), \mathbf{v}_h)\,\mathrm{d}s = -\rho(\ddot{\boldsymbol{\rho}}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h.$$

Applying the stability estimate (14) in Theorem 2 with $l = 0$, and using the fact that $\|\cdot\|_{h,0} = \|\cdot\|$ and that $\|\cdot\|_{h,1}$ is equivalent to $\|\cdot\|_{H^1}$, we get

$$\|\dot{\boldsymbol{\theta}}(T)\| + \|\boldsymbol{\theta}(T)\|_{H^1} \le C\Big(\|\dot{\boldsymbol{\theta}}(0)\| + \|\boldsymbol{\theta}(0)\|_{H^1} + \int_0^T \|P_h\ddot{\boldsymbol{\rho}}\|\,\mathrm{d}t\Big),$$

where $C$ depends on $\rho, \gamma$. Similarly, with $l = -1$, we have

$$\|\dot{\boldsymbol{\theta}}(T)\|_{h,-1} + \|\boldsymbol{\theta}(T)\| \le C\Big(\|\dot{\boldsymbol{\theta}}(0)\|_{h,-1} + \|\boldsymbol{\theta}(0)\| + \int_0^T \|P_h\ddot{\boldsymbol{\rho}}\|_{h,-1}\,\mathrm{d}t\Big).$$

Using that $\|\cdot\|_{h,-1} \le C\|\cdot\|$, $\mathbf{e} = \boldsymbol{\theta} + \boldsymbol{\rho}$, $\|\boldsymbol{\theta}(0)\| \le \|\mathbf{e}(0)\| + \|\boldsymbol{\rho}(0)\|$, we obtain the desired estimates. □

## 5 Numerical Example

The purpose of the present numerical method is demonstrated by solving the dynamic viscoelastic equations in (3) for a two-dimensional structure under plane strain condition. Initial conditions, boundary conditions and model parameters read:

$$\begin{aligned}
&\mathbf{u}(\mathbf{x},0) = \mathbf{0}\,\mathrm{m}, \quad \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{0}\,\mathrm{m/s}, \quad \mathbf{f}(\mathbf{x},t) = \mathbf{0}\,\mathrm{N/m^3}, \\
&\mathbf{u}(\mathbf{x},t) = \mathbf{0}\,\mathrm{m} \text{ at } x_1 = 0\,\mathrm{m}, \quad \mathbf{g}(\mathbf{x},t) = (0,-1)\Theta(t)\,\mathrm{Pa} \text{ at } x_1 = 1.5\,\mathrm{m}, \\
&\gamma = 0.5, \quad E_0 = 10\,\mathrm{MPa}, \quad \alpha = 0.5, \quad \nu = 0.3, \quad \rho = 40\,\mathrm{kg/m^3},
\end{aligned}$$

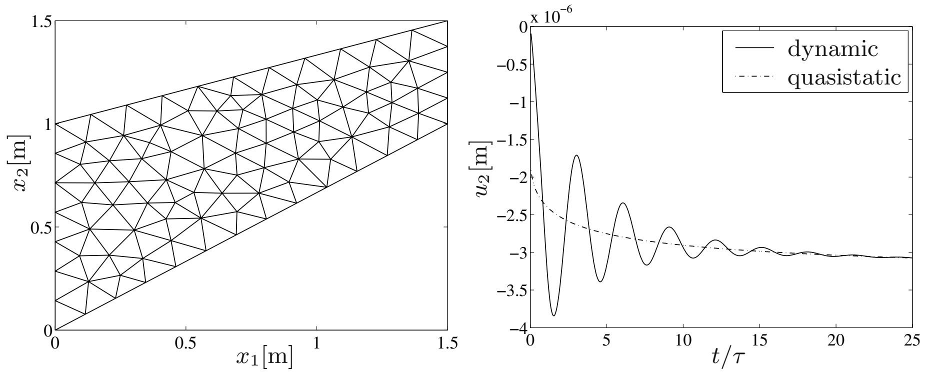


**Fig. 1.** The left figure shows the spatial discretization. The right figure shows the computed vertical displacement at the point (1.5,1.5) m.

where $\Theta$ is the Heaviside function. The geometry and the spatial discretization are shown in Figure 1. Figure 1 also shows the computed vertical displacement versus nondimensional time at the point $(1.5, 1.5)$ m. For comparison the quasistatic (i.e., neglecting inertia, $\rho\ddot{\mathbf{u}} \approx 0$) solution is included. As expected, the two solutions coincide for large times.

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