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Finite Element Approximation of the Deterministic and the Stochastic Cahn-Hilliard Equation

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Abstract

This thesis consists of three papers on numerical approximation of the Cahn-Hilliard equation. The main part of the work is concerned with the Cahn-Hilliard equation perturbed by noise, also known as the Cahn-Hilliard-Cook equation.

In the first paper we consider the linearized Cahn-Hilliard-Cook equation and we discretize it in the spatial variables by a standard finite element method. Strong convergence estimates are proved under suitable assumptions on the covariance operator of the Wiener process, which is driving the equation. The analysis is set in a framework based on analytic semigroups. The main part of the work consists of detailed error bounds for the corresponding deterministic equation. Time discretization by the implicit Euler method is also considered.

In the second paper we study the nonlinear Cahn-Hilliard-Cook equation. We show almost sure existence and regularity of solutions. We introduce spatial approximation by a standard finite element method and prove error estimates of optimal order on sets of probability arbitrarily close to 1. We also prove strong convergence without known rate.

In the third paper the deterministic Cahn-Hilliard equation is considered. A posteriori error estimates are proved for a space-time Galerkin finite element method by using the methodology of dual weighted residuals. We also derive a weight-free a posteriori error estimate in which the weights are condensed into one global stability constant.

Keywords: finite element, a priori error estimate, stochastic integral, mild solution, dual weighted residuals, a posteriori error estimate, additive noise, Wiener process, Cahn-Hilliard equation, existence, regularity, Lyapunov functional, stochastic convolution.

Dissertation

This thesis consists of a short review and three papers:

Paper I: *Finite element approximation of the linearized Cahn-Hilliard-Cook equation.*
Preprint 2009:20, Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, accepted for publication in IMA Journal of Numerical Analysis (with Stig Larsson).

Paper II: *Finite element approximation of the Cahn-Hilliard-Cook equation.*
Preprint 2010:18, Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg (with Mihály Kovács and Stig Larsson).

Paper III: *A posteriori error analysis for the Cahn-Hilliard equation.*
Preprint 2010:19, Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg.

Contributions to co-authored papers

Paper I:
Took part in the theoretical developments.
Did a large part of the writing.

Paper II:
Took part in the theoretical developments.
Did a large part of the writing.

Paper III:
Took part in the theoretical developments.
Did a large part of the writing.

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Ali Mesforush
Gothenburg, April 2010

Contents

1 Introduction

The Cahn-Hilliard equation is an equation of mathematical physics which describes the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component.

In this thesis we study numerical approximation of the Cahn-Hilliard equation. We consider both the original equation and the equation perturbed by noise. The stochastic Cahn-Hilliard equation also called the Cahn-Hilliard-Cook equation. This work involves several mathematical topics:

- Semigroup theory
- Cahn-Hilliard equation
- Stochastic analysis in Hilbert space
- Finite element method
- A posteriori error analysis based on the calculus of variations

In the following we give a brief survey of these topics and finally a summary of the appended papers.

2 Semigroup approach

Semigroup theory is the abstract study of first order ordinary differential equations with values in Banach space, driven by linear, but possibly unbounded operators. This approach has a wide applications in different branches of analysis, such as harmonic analysis, approximation theory and many other subjects. In this section we outline the basics of the theory, without proof. For more complete and advanced details of the theory and its applications the partial differential equations, one may refer to Evans [8] and Pazy [17].

Definition 2.1 (Semigroup)**.** A family $\{E(t)\}_{t\geq 0}$ of bounded linear operators from Banach space X to X is called a *semigroup of bounded linear operators* if

1. $E(0) = I$, (identity operator)
2. $E(t+s) = E(t)E(s), \quad \forall s, t \geq 0.$ (semigroup property)

The semigroup is called *strongly continuous* if

$$\lim_{t\to 0^+} E(t)x = x \quad \forall x \in X.$$

The *infinitesimal generator* of the semigroup is the linear operator G defined by

$$Gx = \lim_{t\to 0^+} \frac{E(t)x - x}{t},$$

its domain of definition $D(G)$ being the space of all $x \in X$ for which the limit exists. The semigroup can be denoted by $E(t) = \mathrm{e}^{tG}$.

A strongly continuous semigroups of bounded linear operators on X is often called a C_0 semigroup. If, moreover, $\|E(t)\| \le 1$ for $t \ge 0$, it is called *a semigroup of contractions.*

In this work we consider $-\Delta$ with the homogeneous Neumann boundary condition as an unbounded linear operator on $L_2 = L_2(\mathcal{D})$ with standard scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. It has eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_j \le \cdots \quad \le \lambda_j \to \infty,$$

and corresponding orthonormal eigenfunctions $\{\varphi_j\}_{j=0}^{\infty}$. The first eigenfunction φ_0 is constant. Also we let $\dot{H}$ be the subspace of H, which is orthogonal to the constants,

$$\dot{H} = \Big\{ v \in L_2 : \langle v, 1\rangle = 0 \Big\},$$

and let P be the orthogonal projection of H onto $\dot{H}$. Define the linear operator $A = -\Delta$ with domain of definition

$$D(A) = \Big\{ v \in H^2 \cap H : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D} \Big\}.$$

By spectral theory we define $\dot{H}^s = \mathcal{D}(A^{s/2})$ with norms $|v|_s = \|A^{s/2}v\|$ for real $s \ge 0$. Then the semigroup e^{-tA^2} generated by $G = -A^2$ can be written as

$$\mathrm{e}^{-tA^2} v = \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j\rangle \varphi_j.$$

This is a strongly continuous semigroup. Moreover, it is analytic, meaning that e^{-tA^2} can be extended to be a holomorphic function of t. This leads to the important properties in the following lemma. For the proof and more details about properties of semigroups we refer to [17].

Lemma 2.2. *If $\{\mathrm{e}^{-tA^2}\}_{t\ge 0}$ is the semigroup generated by $-A^2$, then the following hold:*

$$\|A^\beta \mathrm{e}^{-tA^2} v\| \le C t^{-\beta/2} \|v\|, \quad t > 0,\ \beta \ge 0,$$
$$\int_0^t \|A \mathrm{e}^{-sA^2} v\|^2 \,\mathrm{d}s \le C\|v\|^2.$$

3 Cahn-Hilliard equation

When a homogeneous molten binary alloy is rapidly cooled, the resulting solid is usually found to be not homogeneous, but instead has a fine grained structure consisting of just two materials, which differs only in the mass fraction of the components of the alloy. The development of a fine grained structure from a homogeneous state is referred to as spinodal decomposition.

In 1958, J. Cahn and J. Hilliard [4] derived an expression for the free energy of a sample V of binary alloy with concentration field $c(x)$ of one of two species. They assumed that the free energy density depends not only on $c(x)$ but also on the derivative of c. The expression for the total free energy has the form,

$$\mathcal{E} = N_V \int_V \left(F(c) + \kappa|\nabla c|^2\right) \, \mathrm{d}V, \tag{3.1}$$

where N_V is the number of molecules per unit volume, F is the free energy per molecule of an alloy of uniform composition, and κ is a material constant which is typically very small. The function F has two wells with minima located at the two coexistent concentration states, labeled c_α and $c_\beta > c_\alpha$.

With the given average concentration τ, the equilibrium configurations satisfy the Cahn-Hilliard equation

$$2\kappa\Delta c - F'(c) = \lambda \quad \text{in } V, \tag{3.2}$$

$$\frac{\partial c}{\partial n} = 0 \quad \text{on } \partial V, \tag{3.3}$$

where Δ is the Laplacian, λ is a Lagrange multiplier associated with the constraint τ, and n is the normal to ∂V. In [4], equations (3.2), (3.3) together with the constraint are used to predict the profile and thickness of one-dimensional transitions between concentration phases c_α and c_β.

The general equation governing the evolution of a non-equilibrium state $c(x,t)$ is put forth in [3] and this is what is now referred to as the Cahn-Hilliard equation

$$\frac{\partial c}{\partial t} = \nabla \cdot \{M\nabla(F'(c) - 2\kappa\Delta c)\} \quad \text{in } V, \tag{3.4}$$

with the boundary conditions

$$\frac{\partial c}{\partial n} = \frac{\partial \Delta c}{\partial n} = 0 \quad \text{on } \partial V. \tag{3.5}$$

The positive quantity M is related to the mobility of the two atomic species which comprise the alloy.

In the this thesis we consider the Cahn-Hilliard equation in the form

$$\begin{aligned}
u_t - \epsilon \Delta w \, \mathrm{d}t &= 0 \quad \text{in } \mathcal{D} \times [0,T], \\
w + \Delta u - f(u) &= 0 \quad \text{in } \mathcal{D} \times [0,T], \\
\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial\mathcal{D} \times [0,T], \\
u(0) &= u_0 \quad \text{in } \mathcal{D},
\end{aligned} \tag{3.6}$$

where $u_t = \partial u / \partial t$. The equation perturbed by noise is

$$\begin{aligned}
\mathrm{d}u - \epsilon \Delta w \, \mathrm{d}t &= \mathrm{d}W \quad \text{in } \mathcal{D} \times [0,T], \\
w + \Delta u - f(u) &= 0 \quad \text{in } \mathcal{D} \times [0,T], \\
\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial\mathcal{D} \times [0,T], \\
u(0) &= u_0 \quad \text{in } \mathcal{D},
\end{aligned} \tag{3.7}$$

where $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$ and $f(s) = s^3 - s$.

In the sequel we will write the equation (3.6) in operator form. By definition of $D(A)$ and H, equation (3.6) can be written as

$$\begin{aligned}
&u_t + A^2 u = -Af(u), \quad t > 0, \\
&u(0) = u_0,
\end{aligned} \tag{3.8}$$

which is equivalent to the fixed point equation

$$u(t) = \mathrm{e}^{-tA^2} u_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} A f(u(s)) \, \mathrm{d}s.$$

The generator $-A^2$ is the infinitesimal generator of an analytic semigroup e^{-tA^2} on H so that

$$\begin{aligned}
\mathrm{e}^{-tA^2} v &= \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j + \langle v, \varphi_0 \rangle \varphi_0 \\
&= \mathrm{e}^{-tA^2} P v + (I - P) v.
\end{aligned}$$

4 Stochastic analysis in Hilbert space

In this thesis we use the stochastic integrals and its properties frequently, so in this section we recall some definitions and theorems about stochastic integrals without proof. For more details one may refer to Prévôt and Röckner [20], Da Prato and Zabczyk [7], Klebaner [13] and Grigoriu [12].

4.1 Wiener process

Let Q be a selfadjoint, positive semidefinite, bounded linear operator on the Hilbert space U with $\mathrm{Tr}(Q) < \infty$. Let U and H be separable Hilbert spaces and assume that $\{W(t)\}_{t\in[0,T]}$ is a U-valued Q-Wiener process on a probability space $(\Omega, \mathcal{F}, P)$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, where $T > 0$ is fixed.

Definition 4.1. A U-valued stochastic process $\{W(t)\}_{t\geq 0}$ is called a Q-Wiener process if

- $W(0) = 0$,
- $\{W(t)\}_{t\geq 0}$ has continuous paths almost surely,
- $\{W(t)\}_{t\geq 0}$ has independent increments,
- The increments have a Gaussian law, that is,

$$P \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q), \quad 0 \leq s < t.$$

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal eigenbasis for Q with corresponding eigenvalues $\{\gamma_k\}_{k=1}^{\infty}$. Then we define

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{\frac{1}{2}} \beta_k(t) e_k,$$

where the β_k are real valued independent Brownian motions. The series converges in $L_2(\Omega, H)$.

4.2 Stochastic integral

Definition 4.2. Let $L(U, H)$ denote the space of bounded linear operators $U \to H$. An $L(U, H)$-valued process $\{\Phi(t)\}_{t\in[0,T]}$ is called elementary if there exist $0 = t_0 < t_1 < \cdots < t_N = T$, $N \in \mathbf{N}$, such that

$$\Phi(t) = \sum_{m=0}^{N-1} \Phi_m 1_{(t_m, t_{m+1}]}(t), \quad t \in [0, T],$$

where

- $\Phi_m \colon (\Omega, \mathcal{F}) \to L(U, H)$ is strongly $\mathcal{F}_{t_m}$ measurable,
- Φ_m takes only a finite number of values in $L(U, H)$.

We denote the (linear) space of elementary process by $\mathcal{E}$.

Definition 4.3 (Itô integral)**.** For $\Phi \in \mathcal{E}$, we define the stochastic integral by

$$\int_0^t \Phi \,\mathrm{d}W := \sum_{n=0}^{N-1} \Phi_n(\Delta W_n(t)), \quad t \in [0, T],$$

where

$$\Delta W_n(t) = W(t_{n+1} \wedge t) - W(t_n \wedge t) \quad t \wedge s = \min(t, s).$$

Definition 4.4 (Hilbert-Schmidt operators)**.** An operator $T \in L(U, H)$ is Hilbert-Schmidt if $\sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$ for an orthonormal basis $\{e_k\}_{k\in\mathbf{N}}$ in U. The Hilbert-Schmidt operators form a linear space denoted by $\mathcal{L}_2(U, H)$ which becomes a Hilbert space with scalar product and norm

$$\langle T, S\rangle_{\mathrm{HS}} = \sum_{k=1}^{\infty} \langle Te_k, Se_k\rangle_H, \ \|T\|_{\mathrm{HS}} = \Big(\sum_{k=1}^{\infty} \|Te_k\|_H^2\Big)^{\frac{1}{2}}.$$

We recall that the trace of a linear operator T is

$$\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k\rangle.$$

Consider the covariance operator $Q\colon U \to U$, selfadjoint, positive semidefinite, bounded and linear. Also assume that $W(t)$ is Q-Wiener process. If

$$\mathbf{E} \int_0^t \|T(s)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}s < \infty,$$

we can define the stochastic integral $\int_0^t T(s)\,\mathrm{d}W(s)$ as a limit in $L_2(\Omega, H)$ of integrals of elementary processes.

One important property the stochastic integral is the *isometry property*:

Proposition 4.5 (Isometry property)**.**

$$\mathbf{E}\Big\|\int_0^t T(s)\,\mathrm{d}W(s)\Big\|^2 = \mathbf{E}\int_0^t \|T(s)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}s. \tag{4.1}$$

4.3 Stochastic ordinary differential equation

Stochastic differential equations arise naturally in various engineering problems, where the effects of random *noise* perturbations to a system are being considered. For example in the problem of tracking a satellite, we know that it's motion will obey Newton's law to a very high degree of accuracy, so in theory we can integrate the trajectories from the initial points. However in practice there are rather random effects which perturb the motions.

For more details one can refer to Kuo [14], Klebaner [13] and Chung and Williams [6] The variety of SDE to be considered here describes a *diffusion* process and has the form

$$\mathrm{d}X_t = b(t, X_t) + \sigma(t, X_t)\,\mathrm{d}B_t, \tag{4.2}$$

where $b_i(x,t)$ and $\sigma_{ij}(t,x)$ for $1 \leq i \leq d$ and $1 \leq j \leq r$ are Borel measurable functions.

Definition 4.6 (Strong solution)**.** A strong solution of the SDE (4.2) on the given probability space $(\Omega, \mathcal{F}, P)$ with initial condition ξ is a process $\{X_t\}_{t\geq 0}$ which has continuous sample paths such that

- X_t is adapted to the augmented filtration generated by the Brownian motion B and initial condition ξ, which is denoted $\mathcal{F}_t$.
- $P(X_0 = \xi) = 1$.
- For every $0 \leq t < \infty$ and for each $1 \leq i \leq d$ and $1 \leq j \leq r$, then the following hold almost surely

$$\int_0^t |b_i(s, X_s)| + \sigma_{ij}^2(s, X_s)\mathrm{d}s < \infty.$$

- Almost surely the following holds

$$X_t = \xi + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s.$$

4.4 Stochastic partial differential equation

A stochastic partial differential equation (SPDE) is a partial differential equation containing a random (noise) term. The study of SPDEs is an exciting topic which brings together techniques from probability theory, functional analysis, and the theory of partial differential equations.

Stochastic partial differential equations appear in several different applications: study of random evolution of systems with a spatial extension (random interface growth, random evolution of surfaces, fluids subject to random forcing), study of stochastic models where the state variable is infinite dimensional (for example, a curve or surface), see Carmona [5], Musiela [16], Goldys et al. [11], Goldys and Maslowski [10], Peszat and Zabczyk [19], [18]. The solution to a stochastic partial differential equations may be viewed in several manners. One can view a solution as a random field (set of random variables indexed by a multidimensional parameter). In the case where the SPDE is an evolution equation, the infinite dimensional point of view consists in viewing the solution at a given time as a random element in a function space and thus view the SPDE as a stochastic evolution equation in an infinite dimensional space. In the pathwise point of view, one tries to give a meaning to the solution for (almost) every realization of the noise and then view the solution as a random variable on the set of (infinite dimensional) paths thus defined.

In this section we have a short introduction to the stochastic partial differential equations. For more details and proofs we refer to Frieler and Knoche [9], Da Prato and Zabczyk [7] and Prévôt and Röckner [20].

Definition 4.7. Let $\{W(t)\}_{t\in[0,T]}$ be a U-valued Q-Wiener process on the probability space $(\Omega, \mathcal{F}, P)$, adapted to a normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. The stochastic partial differential equation (SPDE) is of the form

$$\begin{aligned} &\mathrm{d}X(t) = (AX(t) + f(t))\,\mathrm{d}t + \mathrm{d}W(t), \quad 0 < t < T, \\ &X(0) = \xi, \end{aligned} \tag{4.3}$$

where the following assumptions hold:

1. A is a linear operator, generating a strongly continuous semigroup (C_0-semigroup) of bounded linear operators $\{E(t)\}_{t\geq 0}$,
2. $B \in L(U, H)$,
3. $\{f(t)\}_{t\in[0,T]}$ is a predictable H-valued process with Bochner integrable trajectories,
4. ξ is an $\mathcal{F}_0$-measurable H-valued random variable.

Definition 4.8 (Weak solution)**.** An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a weak solution of (4.3) if $\{X(t)\}_{t\in[0,T]}$ is H-predictable, $\{X(t)\}_{t\in[0,T]}$ has Bochner integrable trajectories P-almost surely and

$$\begin{aligned} \langle X(t), \eta\rangle = \langle \xi, \eta\rangle &+ \int_0^t (\langle X(s), A^*\eta\rangle + \langle f(s), \eta\rangle)\,\mathrm{d}s \\ &+ \int_0^t B\,\mathrm{d}W(s), \quad P\text{-a.s., } \forall \eta \in D(A),\ t \in [0,T]. \end{aligned}$$

Definition 4.9 (Mild solution)**.** An U-valued predictable process $X(t)$, $t \in [0, T]$, is called a *mild solution* of problem (4.3) if

$$X(t) = E(t)\xi + \int_0^t E(t-s)f(s)\,\mathrm{d}s + \int_0^t E(t-s)B(X(s))\,\mathrm{d}W(s)$$

P-a.s. for each $t \in [0, T]$. In particular, the appearing integrals have to be well defined.

Definition 4.10 (Strong solution)**.** An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a strong solution of (4.3) if $\{X(t)\}_{t\in[0,T]}$ is H-predictable, $X(t,\omega) \in \mathcal{D}(A)$ P_T-almost surely, $\int_0^T \|AX(t)\|\,\mathrm{d}t < \infty$ P-almost surely, and, for all $t \in [0,T]$,

$$X(t) = \xi + \int_0^t (AX(s) + f(s))\,\mathrm{d}s + \int_0^t B\,\mathrm{d}W(s), \quad P\text{-a.s.}$$

Recall that the integral $\int_0^t B\,\mathrm{d}W(s)$ is defined if and only if $\|B\|_{\mathrm{HS}}^2 = \mathrm{Tr}(BQB^*) < \infty$.

In a special case we have the stochastic Cahn-Hilliard equation as

$$\begin{aligned} &\mathrm{d}X(t) + A^2X(t)\,\mathrm{d}t + Af(X(t))\,\mathrm{d}t = \mathrm{d}W(t), \quad t > 0,\\ &X(0) = X_0, \end{aligned} \tag{4.4}$$

where $A = -\Delta$, P is the orthogonal projection of L_2 onto $\dot{H}$. By using the semigroup approach we can write the mild solution to the equation (4.4) as

$$X(t) = E(t)X_0 - \int_0^t E(t-s)Af(X(s))\,\mathrm{d}s + \int_0^t E(t-s)\,\mathrm{d}W(s), \tag{4.5}$$

where $\{E(t)\}_{t\geq 0} = \{\mathrm{e}^{-tA^2}\}_{t\geq 0}$ is the semigroup generated by $-A^2$. In this thesis we study the equation (4.4) in linear, $f \equiv 0$, and nonlinear cases.

4.5 Stochastic convolution

The last term in (4.5) is a stochastic convolution

$$\begin{aligned} W_A(t) &= \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s)\\ &= \int_0^t \mathrm{e}^{-(t-s)A^2}\,P\mathrm{d}W(s) + \int_0^t \langle \mathrm{d}W(s), \varphi_0\rangle\varphi_0\\ &= \int_0^t \mathrm{e}^{-(t-s)A^2}\,P\mathrm{d}W(s) + \langle W(t), \varphi_0\rangle\varphi_0.\\ &= \int_0^t \mathrm{e}^{-(t-s)A^2}\,P\mathrm{d}W(s) + (I-P)W(t). \end{aligned} \tag{4.6}$$

5 Finite element method

The finite element method (FEM) is a numerical technique for finding approximate solutions of partial differential equations (PDE). In solving PDEs, the primary challenge is to create an equation that approximates the equation to be studied, but is numerically stable, meaning that errors in the input data and intermediate calculations do not accumulate and cause the resulting output to be meaningless. There are many ways of doing this, all with advantages and disadvantages. The finite element method is a good choice for solving partial differential equations over complicated domains. For more details one can refer to Larsson and Thomée [15] and Thomée [21].

In this section we study the FEM for the Cahn-Hilliard equation in deterministic and stochastic cases.

Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of regular triangulations of $\mathcal{D}$ with maximal mesh size h. Let S_h the space of continuous functions on $\mathcal{D}$, which are piecewise polynomials of degree ≤ 1 with respect to $\mathcal{T}_h$. Hence $S_h \subset H^1$. We also define $\dot{S}_h = PS_h$, that is,

$$\dot{S}_h = \Big\{ v_h \in S_h : \int_{\mathcal{D}} v_h \, \mathrm{d}x = 0 \Big\}.$$

The space $\dot{S}_h$ is only used for the purpose of theory but not for computation. Now we define the "discrete Laplacian" $A_h \colon S_h \to \dot{S}_h$ by

$$\langle A_h v_h, w_h \rangle = \langle \nabla v_h, \nabla w_h \rangle, \quad \forall v_h \in S_h, \, w_h \in \dot{S}_h. \tag{5.1}$$

The operator A_h is selfadjoint, positive definite on $\dot{S}_h$ and A_h has an orthonormal eigenbasis $\{\varphi_{h,j}\}_{j=0}^{N_h}$ with corresponding eigenvalues $\{\lambda_{h,j}\}_{j=0}^{N_h}$. We have

$$0 = \lambda_{h,0} < \lambda_{h,1} < \cdots \leq \lambda_{h,j} \leq \lambda_{h,N_h},$$

and $\varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$. Moreover we define $\mathrm{e}^{-tA_h^2} \colon S_h \to S_h$ by

$$\mathrm{e}^{-tA_h^2} v_h = \sum_{j=0}^{N_h} \mathrm{e}^{-t\lambda_{h,j}^2} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j} = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}^2} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j} + \langle v_h, \varphi_0 \rangle \varphi_0,$$

and the orthogonal projector $P_h \colon H \to S_h$ by

$$\langle P_h v, w_h \rangle = \langle v, w_h \rangle \quad \forall v \in H, \, w_h \in S_h. \tag{5.2}$$

Clearly $P_h \colon \dot{H} \to \dot{S}_h$ and

$$\mathrm{e}^{-tA_h^2} P_h v = \mathrm{e}^{-tA_h^2} P_h P v + (I - P) v.$$

5.1 FEM for the deterministic Cahn-Hilliard equation

Consider the Cahn-Hilliard equation (3.6) with $\epsilon = 1$

$$\begin{aligned} u_t - \Delta w &= 0, && x \in \mathcal{D}, \; t > 0, \\ w + \Delta u - f(u) &= 0, && x \in \mathcal{D}, \; t > 0, \\ \frac{\partial u}{\partial n} = 0, \; \frac{\partial v}{\partial n} &= 0, && x \in \partial\mathcal{D}, \; t > 0, \\ u(x,0) &= u_0(x), && x \in \mathcal{D}. \end{aligned} \tag{5.3}$$

Multiply the first and the second equation of (5.3) by $\phi = \phi(x) \in H^1(\mathcal{D}) = H^1$ and integrate over $\mathcal{D}$. Using Green's formula gives

$$\begin{aligned} &\langle u_t, \phi\rangle + \langle \nabla w, \nabla \phi\rangle = 0 && \forall \phi \in H^1, \\ &\langle w, \phi\rangle = \langle \nabla u, \nabla \phi\rangle + \langle f(u), \phi\rangle && \forall \phi \in H^1. \end{aligned} \tag{5.4}$$

So the variational formulation is: Find $u(t), w(t) \in H^1$ such that (5.4) holds and such that $u(x,0) = u_0(x)$ for $x \in \mathcal{D}$.

Let $\mathcal{T}_h = \{K\}$ denote a triangulation of $\mathcal{D}$ and let S_h denote the continuous piecewise polynomial functions on $\mathcal{T}_h$. So the finite element problem is: Find $u_h(t), w_h(t) \in S_h$ such that

$$\begin{aligned} &\langle u_{h,t}, \chi\rangle + \langle \nabla w_h, \nabla \chi\rangle = 0 && \forall \chi \in S_h,\ t > 0, \\ &\langle w_h, \chi\rangle = \langle \nabla u_h, \nabla \chi\rangle + \langle f(u_h), \chi\rangle && \forall \chi \in S_h,\ t > 0, \\ &u_h(0) = u_{h,0}. \end{aligned} \tag{5.5}$$

Then we can write the equation (5.5) as

$$\begin{aligned} &u_{h,t} + A_h^2 u_h + A_h P_h f(u_h) = 0, \quad t > 0, \\ &u_h(0) = u_{0,h}, \end{aligned} \tag{5.6}$$

which is equivalent to the fixed point equation

$$u_h(t) = \mathrm{e}^{-tA_h^2} u_{0,h} - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(u_h(s))\,\mathrm{d}s,$$

where

$$\mathrm{e}^{-tA_h^2} v = \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_{h,j}^2} \langle v, \varphi_{h,j}\rangle \varphi_{h,j},$$

where $(\lambda_{h,j}, \varphi_{h,j})$ are the eigenpairs of A_h.

5.2 FEM for the stochastic Cahn-Hilliard equation

Consider the equation (4.4) and assume that $\{\mathcal{T}_h\}_{0<h<1}$ is a triangulation with mesh size h and $\{S_h\}_{0<h<1}$ is the set of continuous piecewise linear functions where $S_h \subset H^1(\mathcal{D})$. Also let A_h and P_h be the same as in (5.1) and (5.2). The finite element problem for (4.4) is:

Find $X_h(t) \in \dot{S}_h$ such that

$$\begin{aligned} &\mathrm{d}X_h(t) + A_h^2 X_h(t)\,\mathrm{d}t + A_h P_h f(X_h(s))\,\mathrm{d}t = P_h\,\mathrm{d}W(t), \\ &X_h(0) = P_h X_0, \end{aligned} \tag{5.7}$$

where $P_hW(t)$ is Q_h-Wiener process on S_h with $Q_h = P_hQP_h$. The mild solution is given by the equation

$$X_h(t) = E_h(t)P_hX_0 - \int_0^t E_h(t-s)A_hP_hf(X_h(s))\,\mathrm{d}s + \int_0^t E(t-s)P_h\,\mathrm{d}W(s),$$

where $E_h(t) = \mathrm{e}^{-tA_h^2}$. In the linear case, the finite element problem is

$$\begin{aligned} &\mathrm{d}X_h(t) + A_h^2X_h(t)\,\mathrm{d}t = P_h\,\mathrm{d}W(t),\\ &X_h(0) = P_hX_0, \end{aligned} \tag{5.8}$$

with mild solution

$$X_h(t) = E(t)P_hX_0 + \int_0^t E(t-s)P_h\,\mathrm{d}W(s).$$

Now define the stochastic convolution

$$\begin{aligned} W_{A_h}(t) &= \int_0^t \mathrm{e}^{-(t-s)A_h^2}\,P_h\mathrm{d}W(s)\\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2}\,P_hP\mathrm{d}W(s) + \langle W(t),\varphi_0\rangle\varphi_0\\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2}\,P_hP\mathrm{d}W(s) + (I-P)W(t). \end{aligned}$$

Hence, in view of (4.6),

$$W_{A_h}(t) - W_A(t) = \int_0^t \left(\mathrm{e}^{-(t-s)A_h^2}P_h - \mathrm{e}^{-(t-s)A^2}\right)P\,\mathrm{d}W(s),$$

so that the error can be analyzed in the spaces $\dot{H}$ and $\dot{S}_h$.

Let $k = \Delta t_n$, $t_n = nk$ and $\Delta W_n = W(t_n) - W(t_{n-1})$. Also consider $\Delta X_{h,n} = X_{h,n} - X_{h,n-1}$ and apply the backward Euler method to (5.8) to get

$$\begin{aligned} &X_{h,n} \in S_h,\\ &\Delta X_{h,n} + A_h^2X_{h,n}\Delta t_n = P_h\Delta W_n,\\ &X_{h,0} = P_hX_0. \end{aligned} \tag{5.9}$$

This implies

$$X_{h,n} - X_{h,n-1} + kA_h^2X_{h,n} = P_h\Delta W_n.$$

If we set $E_{k,h} = (I + kA_h^2)^{-1}$ we get

$$(I + kA_h^2)X_{h,n} = P_h\Delta W_n + X_{h,n-1}.$$

So

$$X_{h,n} = E_{k,h} P_h \Delta W_n + E_{k,h} X_{h,n-1}.$$

We repeat it to get

$$X_{h,n} = E_{k,h}^n P_h X_0 + \sum_{j=1}^{n} E_{k,h}^{n-j+1} P_h \Delta W_j. \tag{5.10}$$

6 A posteriori error estimate

In this section we recall some theorems and techniques for a posteriori error estimates for the Galerkin approximation of nonlinear variational problems. For more details and proofs, we refer to Bangerth and Rannacher [1] and Becker and Rannacher [2].

Let $A(u, \cdot)$ be a semi-linear form and $J(\cdot)$ an output functional, not necessarily linear, defined on some function space V. Consider the variational problem: Find $u \in V$ such that

$$A(u; \psi) = 0 \quad \forall \psi \in V, \tag{6.1}$$

and the corresponding finite element problem: Find $u_h \in V_h \subset V$ such that

$$A(u_h; \psi_h) = 0 \quad \forall \psi_h \in V_h. \tag{6.2}$$

Suppose that the directional derivatives of A and J up to order three exist and denoted by

$$A'(u; \varphi, \cdot),\ A''(u; \psi, \varphi, \cdot),\ A'''(u; \xi, \psi, \varphi, \cdot),$$

and

$$J'(u; \varphi),\ J''(u; \psi, \varphi),\ A''(u; \xi, \psi, \varphi),$$

respectively for increments φ, ψ, $\xi \in V$. We want to estimate $J(u) - J(u_h)$. Introduce dual variable $z \in V$ and define the Lagrangian functional

$$\mathcal{L}(u; z) := J(u) - J(u_h),$$

and seek for the stationary points $\{u, z\} \in V \times V$ of $\mathcal{L}(\cdot, \cdot)$. i.e. for all $\psi, \varphi \in V$

$$\mathcal{L}'(u; z, \varphi, \psi) = J'(u; \varphi) - A'(u; z, \varphi) - A(u; \psi) = 0.$$

We quote three lemmas from [1].

Lemma 6.1. *Let $L(\cdot)$ be a three times differentiable functional defined on a (real or complex) vector space X which has a stationary point $x \in X$, i.e.*

$$L'(x; y) = 0, \quad \forall y \in X,$$

Suppose that on a finite dimensional subspace $X_h \subset X$ the corresponding Galerkin approximation

$$L'(x_h; y_h) = 0 \quad \forall y_h \in X_h.$$

has a solution, $x_h \in X_h$. Then there holds the error representation

$$L(x) - L(x_h) = \frac{1}{2} L'(x_h; x - y_h) + \mathcal{R}_h \quad \forall y_h \in X_h,$$

with a remainder term $\mathcal{R}_h$, which is cubic in the error $e := x - x_h$,

$$\mathcal{R}_h := \frac{1}{2} \int_0^1 L'''(x_h + se; e, e, e) s(s-1) \, \mathrm{d}s.$$

From Lemma 6.1 we obtain the following result for the Galerkin approximation of the variational equation.

Lemma 6.2. *For any solutions of equations* (6.1) *and* (6.2) *we have the error representation*

$$J(u) - J(u_h) = \frac{1}{2} \rho(u_h; e_z) + \frac{1}{2} \rho^*(u_h; z_h, e_u) + \mathcal{R}_h^{(3)},$$

where

$$\begin{aligned} \rho(u_h; e_z) &= -A'(u_h; z_h, e_z), \\ \rho^*(u_h; z_h, e_u) &= J'(u_h; e_u) - A'(u_h; z_h, e_u), \end{aligned}$$

with $e_u = u - u_h$, $e_z = z - z_h$ and

$$\begin{aligned} \mathcal{R}_h^{(3)} = \frac{1}{2} \int_0^1 \Big(& J'''(u_h + se_u; e_u, e_u, e_u) - A'''(u_h + se_u; z_h + se_z, e_u, e_u, e_u) \\ & - 3A''(u_h + se_u; e_u, e_u, e_z) \Big) s(s-1) \, \mathrm{d}s \end{aligned}$$

The forms $\rho(\cdot, \cdot), \rho^*(\cdot; \cdot, \cdot)$ are the residuals of (6.1) and the linearized adjoint equation, respectively. The remainder $\mathcal{R}_h^{(3)}$ is cubic in the error. The following lemma shows that the residuals are equal up to a quadratic remainder.

Lemma 6.3. *With the notation from above, for any φ_h, $\psi_h \in V_h$ there holds*

$$\rho^*(u_h; z_h, u-\varphi_h) = \rho(u_h; z-\psi_h) + \Delta\rho \quad \forall \varphi_h, \psi_h \in V_h,$$

with

$$\Delta\rho = \int_0^1 \Big(A''(u_h + se_u; e_u, e_u, z_h + se_z) - J''(u_h + se_u; e_u, e_u) \Big)\, \mathrm{d}s.$$

Moreover, we we have the simplified error representation

$$J(u) - J(u_h) = \rho(u_h, z-\varphi_h) + \mathcal{R}_h^{(2)} \quad \forall \varphi_h \in V_h,$$

with quadratic remainder

$$\mathcal{R}_h^{(2)} = \int_0^1 \Big(A''(u_h + se_u, e_u, e_u, z) - J''(u_h + se_u; e_u, e_u) \Big)\, \mathrm{d}s.$$

In Paper III we apply this methodology to a space and time discretization of the deterministic Cahn-Hilliard equation.

7 Summary of appended papers

7.1 Paper I

In this paper we prove error bounds for the linear Cahn-Hilliard-Cook equation; that is, (3.7) with $f(u) = 0$. The main result is a mean square error estimate for the finite element approximation defined in (5.8):

$$\begin{aligned} \|X_h(t) - X(t)\|_{L_2(\Omega,H)} \\ \le C h^\beta (\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + |\log h| \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}). \end{aligned}$$

The proof is essentially based on applying the isometry (4.5) to

$$W_{A_h}(t) - W_A(t) = \int_0^t \Big(\mathrm{e}^{-(t-s)A_h^2} P_h - \mathrm{e}^{-(t-s)A^2} \Big)\, P\, \mathrm{d}W(s).$$

The proof is then reduced to proving bounds for the error operator $F_h(t) = E_h(t)P_hP - E(t)P$ for the corresponding linear deterministic problem. For this problem we show the following error bounds with optimal dependence on the regularity of the initial value v:

$$\begin{aligned} &\|F_h(t)v\| \le C h^\beta |v|_\beta, && v \in \dot{H}^\beta, \\ &\Big(\int_0^t \|F_h(\tau)v\|^2\, \mathrm{d}\tau \Big)^{\frac{1}{2}} \le C|\log h| h^\beta |v|_{\beta-2}, && v \in \dot{H}^{\beta-2}, \end{aligned}$$

for $1 \leq \beta \leq r$, where $r \geq 2$ is the order of the finite element method.

The same program is carried out for the the backward Euler method in (5.9). The result is the error bound

$$\begin{aligned}&\|X_{h,n}(t) - X(t_n)\|_{L_2(\Omega,H)} \\ &\qquad \leq \Big(C|\log h|h^{\beta} + C_{\beta,k}k^{\frac{\beta}{4}}\Big)\Big(\|X_0\|_{L_2(\Omega,\dot{H}^{\beta})} + \|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}\Big),\end{aligned}$$

where where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

7.2 Paper II

We study the nonlinear stochastic Cahn-Hilliard equation driven by additive colored noise (3.7). Using the framework of [7] we write this as an abstract evolution equation of the form

$$\mathrm{d}X + (A^2X + Af(X))\,\mathrm{d}t = \mathrm{d}W, \quad t > 0; \quad X(0) = X_0, \tag{7.1}$$

Our goal is to study the convergence properties of the spatially semidiscrete finite element approximation X_h of X, which is defined by an equation of the form

$$\mathrm{d}X_h + (A_h^2X + A_hP_hf(X))\,\mathrm{d}t = P_h\mathrm{d}W, \quad t > 0; \quad X(0) = P_hX_0.$$

In order to do so, we need to prove existence and regularity for solutions of (7.1).

Following the semigroup framework of [7] we write the equation (7.1) as the integral equation (mild solution)

$$X(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2}Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s).$$

This naturally splits the solution as $X = Y + W_A$, where $W_A(t)$ is the stochastic convolution that was studied in Paper I. The remaining part, Y, satisfies an evolution equation without noise, but with a random coefficient,

$$\dot{Y} + A^2Y + Af(X) = 0, \quad t > 0; \quad Y(0) = X_0.$$

The regularity and error analysis can now be performed on this equation.

An important step is to bound the functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x,$$

where $F(s)$ is a primitive function to $f(s)$. For the deterministic equation this is a Lyapunov functional, which means that it does not increase along solution paths. For the equation which is perturbed by noise we show that

$$\mathbf{E}[J(X(t))] \leq C(t),$$

where $C(t)$ grows quadratically in t. The same result holds for $X_h(t)$. By means of Chebyshev's inequality we may then show that for each $T > 0$ and $\epsilon \in (0, 1)$ there are K_T and $\Omega_\epsilon \subset \Omega$ with $\mathbf{P}(\Omega_\epsilon) \geq 1 - \epsilon$ and such that

$$\|X(t)\|_{H^1}^2 + \|X_h(t)\|_{H^1}^2 \leq \epsilon^{-1} K_T \quad \text{on } \Omega_\epsilon,\ t \in [0, T].$$

These bounds are then used to control the random term $f(X)$ and we show the necessary regularity and the error estimate

$$\|X_h(t) - X(t)\| \leq C(\epsilon^{-1} K_T, T) h^2 |\log(h)| \quad \text{on } \Omega_\epsilon,\ t \in [0, T].$$

We thus have optimal rate of convergence on sets of probability arbitrarily close 1, but the constant increases rapidly when $\epsilon \to 0$. Nevertheless, we show that this implies strong convergence but without known rate:

$$\max_{t \in [0,T]} \big(\mathbf{E}[\|X_h(t) - X(t)\|^2]\big)^{\frac{1}{2}} \to 0 \quad \text{as } h \to 0.$$

7.3 Paper III

In this paper we consider the deterministic Cahn-Hilliard equation (3.6) and we discretize it by a Galerkin finite element method, which is based on continuous piecewise linear functions with respect to x and discontinuous piecewise constant functions with respect to t. The numerical method is the same as the implicit Euler time stepping combined with spatial discretization by a standard finite element method.

We perform an a posteriori error analysis within the framework of dual weighted residuals as in section 6. If $J(u)$ is a given goal functional, this results in an error estimate essentially of the form

$$|J(u) - J(U)| \leq \sum_{n=1}^{N} \sum_{K \in \mathbf{T}_n} \Big\{ \rho_{u,K} \omega_{u,K} + \rho_{w,K} \omega_{w,K} \Big\} + \mathcal{R},$$

where U denotes the numerical solution and $\mathbf{T}_n$ is the spatial mesh at time level t_n. The terms $\rho_{u,K}, \rho_{w,K}$ are local residuals from the first and second equations in (3.6), respectively. The weights $\omega_{u,K}, \omega_{w,K}$ are derived from the solution of the linearized adjoint problem. The remainder $\mathcal{R}$ is quadratic in the error.

We also derive a variant of this, where the weights are replaced by stability constants, which are obtained by proving a priori estimates for the solution of the linearized adjoint problem.

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FINITE ELEMENT APPROXIMATION OF THE LINEARIZED CAHN-HILLIARD-COOK EQUATION

STIG LARSSON[1] AND ALI MESFORUSH

Abstract. The linearized Cahn-Hilliard-Cook equation is discretized in the spatial variables by a standard finite element method. Strong convergence estimates are proved under suitable assumptions on the covariance operator of the Wiener process, which is driving the equation. The backward Euler time stepping is also studied. The analysis is set in a framework based on analytic semigroups. The main part of the work consists of detailed error bounds for the corresponding deterministic equation.

1. Introduction

When the Cahn-Hilliard equation (cf. [2, 3]) is perturbed by noise, we obtain the so-called Cahn-Hilliard-Cook equation (cf. [1, 5])

$$
\begin{aligned}
&\mathrm{d}u - \Delta v \,\mathrm{d}t = \mathrm{d}W, && \text{for } x \in \mathcal{D},\ t > 0,\\
&v = -\Delta u + f(u), && \text{for } x \in \mathcal{D},\ t > 0,\\
&\frac{\partial u}{\partial n} = 0,\ \frac{\partial \Delta u}{\partial n} = 0, && \text{for } x \in \partial\mathcal{D},\ t > 0,\\
&u(\cdot, 0) = u_0,
\end{aligned}
\tag{1.1}
$$

where $u = u(x,t)$, $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$, and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\mathcal{D}$. We assume that $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d$ for $d \le 3$ with sufficiently smooth boundary. A typical f is $f(s) = s^3 - s$. The purpose of this work is to study numerical approximation by the finite element method of the linearized Cahn-Hilliard-Cook equation, where $f = 0$.

We use the semigroup framework of [12] in order to give (1.1) a rigorous meaning. Let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the usual norm and inner product in the

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Hilbert space $H = L_2(\mathcal{D})$ and let $H^s = H^s(\mathcal{D})$ be the usual Sobolev space with norm $\|\cdot\|_s$. We also let $\dot{H}$ be the subspace of H, which is orthogonal to the constants, that is, $\dot{H} = \{v \in H : (v,1) = 0\}$, and we let $P\colon H \to \dot{H}$ be the orthogonal projector.

We define the linear operator $A = -\Delta$ with domain of definition

$$D(A) = \Big\{ v \in H^2 : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D} \Big\}.$$

Then A is selfadjoint, positive definite, unbounded linear operator on $\dot{H}$ with compact inverse. When it is considered as an unbounded operator on H, it is positive semidefinite with an orthonormal eigenbasis $\{\varphi_j\}_{j=0}^{\infty}$ and corresponding eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ such that

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lambda_j \to \infty.$$

The first eigenfunction is constant, $\varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$. By spectral theory we also define $\dot{H}^s = D(A^{\frac{s}{2}})$ with norms

$$|v|_s = \|A^{\frac{s}{2}} v\| = \Big(\sum_{j=1} \lambda_j^s (v, \varphi_j)^2 \Big)^{1/2}, \quad s \in \mathbf{R}. \tag{1.2}$$

It is well known that, for integer $s \ge 0$, $\dot{H}^s$ is a subspace of $H^s \cap \dot{H}$ characterized by certain boundary conditions and that the norms $|\cdot|_s$ and $\|\cdot\|_s$ are equivalent on $\dot{H}^s$. In particular, we have $\dot{H}^1 = H^1 \cap \dot{H}$ and the norm $|v|_1 = \|A^{\frac{1}{2}} v\| = \|\nabla v\|$ is equivalent to $\|v\|_1$ on $\dot{H}^1$.

For $v \in H$ we define

$$\mathrm{e}^{-tA^2} v = \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2} (v, \varphi_j) \varphi_j.$$

Then $\{E(t)\}_{t\ge0} = \{\mathrm{e}^{-tA^2}\}_{t\ge0}$ is the analytic semigroup on H generated by $-A^2$. We note that

$$E(t)v = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2} (v, \varphi_j)\varphi_j + (v, \varphi_0)\varphi_0 = E(t)Pv + (I-P)v,$$

where $(I-P)v = |\mathcal{D}|^{-1} \int_{\mathcal{D}} v \,\mathrm{d}x$ is the average of v.

Let $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\ge0})$ be a filtered probability space, let Q be a selfadjoint, positive semidefinite, bounded linear operator on H, and let $\{W(t)\}_{t\ge0}$ be an H-valued Q-Wiener process adapted to the filtration $\{\mathcal{F}_t\}_{t\ge0}$.

Now the Cahn-Hilliard-Cook equation (1.1) may be written formally

$$\mathrm{d}X(t) + A^2 X(t)\,\mathrm{d}t + Af(X(t))\,\mathrm{d}t = \mathrm{d}W(t), \quad t>0; \quad X(0) = X_0. \tag{1.3}$$

The semigroup framework of [12] gives a rigorous meaning to this in terms of the mild solution, which satisfies the integral equation

$$X(t) = E(t)X_0 - \int_0^t E(t-s)Af(X(s))\,\mathrm{d}s + \int_0^t E(t-s)\,\mathrm{d}W(s),$$

where $\int_0^t \dots \mathrm{d}W(s)$ denotes the H-valued Itô integral. Existence and uniqueness of solutions is proved in [6]. This is based on the natural splitting of the solution as $X(t) = Y(t) + W_A(t)$, where

$$W_A(t) = \int_0^t E(t-s)\,\mathrm{d}W(s)$$

is a stochastic convolution, and where

$$Y(t) = E(t)X_0 - \int_0^t E(t-s)Af(X(s))\,\mathrm{d}s$$

satisfies the random evolution problem

$$\dot{Y}(t) + A^2Y(t) + Af\big(Y(t) + W_A(t)\big) = 0, \quad t > 0; \quad Y(0) = X_0.$$

The study of the stochastic convolution $W_A(t)$ is thus a first step towards the study of the nonlinear problem.

In this work we therefore study numerical approximation of the linearized Cahn-Hilliard-Cook equation

$$\mathrm{d}X + A^2X\,\mathrm{d}t = \mathrm{d}W, \quad t > 0; \quad X(0) = X_0, \tag{1.4}$$

with the mild solution

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s). \tag{1.5}$$

The nonlinear equation is studied in a forthcoming paper [11]. We remark that a linearized equation of the form (1.4), but with A^2 replaced by $A^2 + A$ is studied by numerical simulation in the physics literature [7, 9].

For the approximation of the Cahn-Hilliard equation we follow the framework of [8]. We assume that we have a family $\{S_h\}_{h>0}$ of finite-dimensional approximating subspaces of H^1. Let $P_h \colon H \to S_h$ denote the orthogonal projector. We then define $\dot{S}_h = \{\chi \in S_h : (\chi, 1) = 0\}$. The operator $A_h \colon S_h \to \dot{S}_h$ (the "discrete Laplacian") is defined by

$$(A_h\chi, \eta) = (\nabla\chi, \nabla\eta), \quad \forall \chi \in S_h,\ \eta \in \dot{S}_h,$$

The operator A_h is selfadjoint, positive definite on $\dot{S}_h$, positive semidefinite on S_h, and A_h has an orthonormal eigenbasis $\{\varphi_{h,j}\}_{j=0}^{N_h}$ with corresponding eigenvalues $\{\lambda_{h,j}\}_{j=0}^{N_h}$. We have

$$0 = \lambda_{h,0} < \lambda_{h,1} \le \dots \le \lambda_{h,j} \le \dots \le \lambda_{h,N_h},$$

and $\varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$. Moreover, we define $E_h(t)\colon S_h \to S_h$ by

$$\begin{aligned} E_h(t)v_h = \mathrm{e}^{-tA_h^2}v_h &= \sum_{j=0}^{N_h} \mathrm{e}^{-t\lambda_{h,j}}(v_h,\varphi_{h,j})\varphi_{h,j} \\ &= \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}}(v_h,\varphi_{h,j})\varphi_{h,j} + (v_h,\varphi_0)\varphi_0, \end{aligned}$$

Then $\{E_h(t)\}_{t\geq 0}$ is the semigroup generated by $-A_h^2$. Clearly, $P_h\colon \dot{H} \to \dot{S}_h$ and

$$E_h(t)P_h v = E_h(t)P_h P v + (I-P)v.$$

The finite element approximation of the linearized Cahn-Hilliard-Cook equation (1.4) is: Find $X_h(t) \in S_h$ such that,

$$\mathrm{d}X_h + A_h^2 X_h\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t>0; \quad X_h(0) = P_h X_0. \tag{1.6}$$

The mild solution of (1.6) is

$$X_h(t) = E_h(t)P_h X_0 + \int_0^t E_h(t-s)P_h\,\mathrm{d}W(s). \tag{1.7}$$

We note that

$$\int_0^t E(t-s)(I-P)\,\mathrm{d}W(s) = (I-P)\int_0^t \mathrm{d}W(s) = (I-P)W(t),$$

so that

$$\begin{aligned} X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s) &= E(t)PX_0 + (I-P)X_0 \\ &+ \int_0^t E(t-s)P\,\mathrm{d}W(s) + (I-P)W(t), \end{aligned} \tag{1.8}$$

and similarly

$$\begin{aligned} X_h(t) &= E_h(t)P_h P X_0 + (I-P)X_0 \\ &\quad + \int_0^t E_h(t-s)P_h P\,\mathrm{d}W(s) + (I-P)W(t). \end{aligned}$$

Therefore, the error analysis can be based on the formula

$$\begin{aligned} X_h(t) - X(t) &= \big(E_h(t)P_h - E(t)\big)PX_0 \\ &\quad + \int_0^t \big(E_h(t-s)P_h - E(t-s)\big)P\,\mathrm{d}W(s), \end{aligned} \tag{1.9}$$

and it is sufficient to work in the spaces $\dot{H}$ and $\dot{S}_h$. Note that the numerical computations are carried out in S_h and that $\dot{S}_h$ is only used in the analysis.

Let $k = \delta t$ be a timestep, $t_n = nk$, $\delta X_{h,n} = X_{h,n} - X_{h,n-1}$, $\delta W_n = W(t_n) - W(t_{n-1})$, and apply Euler's method to (1.6) to get

$$\delta X_{h,n} + A_h^2 X_{h,n}\,\delta t = P_h\,\delta W_n, \quad n \geq 1; \quad X_{h,0} = P_h X_0. \tag{1.10}$$

With $E_{kh} = (I + kA_h^2)^{-1}$ we obtain a discrete variant of the mild solution

$$X_{h,n} = E_{kh}^n P_h X_0 + \sum_{j=1}^{n} E_{kh}^{n-j+1} P_h\,\delta W_j.$$

In Section 2 we assume that $\{S_h\}_{h>0}$ admits an error estimate of order $\mathcal{O}(h^r)$ as the mesh parameter $h \to 0$ for some integer $r \geq 2$. Then we show error estimates for the semigroup $E_h(t)$ with minimal regularity requirement. More precisely, in Theorem 2.1 we show, for $\beta \in [1, r]$ and all $t \geq 0$,

$$\begin{aligned}
&\|F_h(t)v\| \leq Ch^\beta\,|v|_\beta, \quad v \in \dot{H}^\beta, \\
&\Big(\int_0^t \|F_h(\tau)v\|^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}} \leq C|\log h| h^\beta\,|v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2},
\end{aligned}$$

where $F_h(t) = E_h(t)P_h - E(t)$ is the error operator in (1.9).

Analogous estimates are obtained for the implicit Euler approximation in Theorem 2.2.

In Section 3 we follow the technique developed in [14, 13] and use these estimates to prove strong convergence estimates for approximation of the linear Cahn-Hilliard-Cook equation. Let $L_2(\Omega, H)$ be the space of square integrable H-valued random variables with norm

$$\|X\|_{L_2(\Omega,H)} = \Big(\mathbf{E}\big(\|X\|^2\big)\Big)^{\frac{1}{2}} = \Big(\int_\Omega \|X(\omega)\|^2\,\mathrm{d}\mathbf{P}(\omega)\Big)^{\frac{1}{2}},$$

and let $\|T\|_{\mathrm{HS}}$ denote the Hilbert-Schmidt norm of bounded linear operators on H, $\|T\|_{\mathrm{HS}}^2 = \sum_{j=1}^\infty \|T\phi_j\|^2$, where $\{\phi_j\}_{j=1}^\infty$ is an arbitrary orthonormal basis for H. In Theorem 3.1 we study the spatial regularity of the mild solution (1.5) and show

$$\|X(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big), \quad \text{for } \beta > 0.$$

Moreover, in Theorem 3.2 we show strong convergence for the mild solution X_h in (1.7):

$$\begin{aligned}
&\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \\
&\qquad \leq Ch^\beta\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + |\log h|\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big), \quad \beta \in [1, r].
\end{aligned}$$

In Theorem 3.3 for the fully discrete case we obtain similarly, for $\beta \in [1, \min(r, 4)]$,

$$\begin{aligned}
&\|X_{h,n}(t) - X(t_n)\|_{L_2(\Omega,H)} \\
&\qquad \leq \big(C|\log h|h^{\beta} + C_{k,\beta}k^{\frac{\beta}{4}}\big)\big(\|X_0\|_{L_2(\Omega,\dot H^{\beta})} + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\big),
\end{aligned}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

Note that these bounds are uniform with respect to $t \geq 0$.

Our results require that $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}} < \infty$. In order to see what this means we compute two special cases. For $Q = I$ (spatially uncorrelated noise, or space-time white noise), by using the asymptotics $\lambda_j \sim j^{\frac{2}{d}}$, we have

$$\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}^2 = \|A^{\frac{\beta-2}{2}}\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} \lambda_j^{\beta-2} \sim \sum_{j=1}^{\infty} j^{(\beta-2)\frac{2}{d}} < \infty,$$

if $\beta < 2 - \frac{d}{2}$. Hence, for example, $\beta < \frac12$ if $d = 3$. On the other hand, if Q is of trace class, $\mathrm{Tr}(Q) = \|Q^{\frac12}\|_{\mathrm{HS}}^2 < \infty$, then we may take $\beta = 2$.

There are few studies of numerical methods for the Cahn-Hilliard-Cook equation. We are only aware of [4] in which convergence in probability was proved for a difference scheme for the nonlinear equation in multiple dimensions, and [10] where strong convergence was proved for the finite element method for the linear equation in 1-D.

2. Error estimates for the deterministic Cahn-Hilliard equation

We start this section with some necessary inequalities. Let $\{E(t)\}_{t\geq 0} = \{\mathrm{e}^{-tA^2}\}_{t\geq 0}$ and $\{E_h(t)\}_{t\geq 0} = \{\mathrm{e}^{-tA_h^2}\}_{t\geq 0}$ be the semigroups generated by $-A^2$ and $-A_h^2$, respectively. By the smoothing property there exist positive constants c, C such that

$$\|A_h^{2\beta}E_h(t)P_hPv\| + \|A^{2\beta}E(t)Pv\| \leq Ct^{-\beta}\mathrm{e}^{-ct}\|v\|, \quad \beta \geq 0, \tag{2.1}$$

$$\int_0^t \|A_hE_h(s)P_hPv\|^2\,\mathrm{d}s + \int_0^t \|AE(s)Pv\|^2\,\mathrm{d}s \leq C\|v\|^2. \tag{2.2}$$

Let $R_h\colon \dot H^1 \to \dot S_h$ be the Ritz projector defined by

$$(\nabla R_hv, \nabla\chi) = (\nabla v, \nabla\chi), \quad \forall \chi \in \dot S_h.$$

It is clear that $R_h = A_h^{-1}P_hA$. We assume that for some integer $r \geq 2$, we have the error bound, with the norm defined in (1.2),

$$\|R_hv - v\| \leq Ch^{\beta}|v|_{\beta}, \quad v \in \dot H^{\beta},\ 1 \leq \beta \leq r. \tag{2.3}$$

This holds with $r = 2$ for the standard piecewise linear Lagrange finite element method in a bounded convex polygonal domain $\mathcal{D}$. For higher order elements the situation is more complicated and we refer to standard texts on the finite element method. In the next theorem we prove error estimates for the deterministic Cahn-Hilliard equation in the semidiscrete case.

Theorem 2.1. *Set $F_h(t) = E_h(t)P_h - E(t)$. Then there are h_0 and C, such that for $h \le h_0$, $1 \le \beta \le r$ and $t \ge 0$, we have*

$$\|F_h(t)v\| \le Ch^\beta\,|v|_\beta, \quad v \in \dot{H}^\beta, \tag{2.4}$$

$$\Big(\int_0^t \|F_h(\tau)v\|^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}} \le C|\log h|h^\beta\,|v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}. \tag{2.5}$$

Note that $F_h(t)v = F_h(t)Pv$ for $v \in H$, so that it is sufficient to take $v \in \dot{H}$. The reason why we assume $\beta \ge 1$ is that in (2.5) we need at least $v \in \dot{H}^{-1}$ for $E_h(t)P_hv$ to be defined.

Proof. Let $u(t) = E(t)v$, $u_h(t) = E_h(t)P_hv$, that is, u and u_h are solutions of

$$u_t + A^2u = 0, \quad t > 0; \quad u(0) = v, \tag{2.6}$$

$$u_{h,t} + A_h^2u_h = 0, \quad t > 0; \quad u_h(0) = P_hv. \tag{2.7}$$

Set $e(t) = u_h(t) - u(t)$. We want to prove that

$$\|e(t)\| \le Ch^\beta\,|v|_\beta, \quad v \in \dot{H}^\beta,$$

$$\Big(\int_0^t \|e(\tau)\|^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}} \le C|\log h|h^\beta\,|v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}.$$

Let $G = A^{-1}P$ and $G_h = A_h^{-1}P_hP$. Apply G to (2.6) to get $Gu_t + Au = 0$, and apply G_h^2 to (2.7) to get $G_h^2u_{h,t} + u_h = 0$. Hence

$$G_h^2e_t + e = -G_h^2u_t - u + G_h(Gu_t + Au) = (G_hA - I)u - G_h(G_hA - I)Gu_t,$$

that is,

$$G_h^2e_t + e = \rho + G_h\eta, \tag{2.8}$$

where $\rho = (R_h - I)u, \eta = -(R_h - I)Gu_t$. Take the inner product of (2.8) by e_t to get

$$\|G_he_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 = (\rho, e_t) + (\eta, G_he_t),$$

Since $(\eta, G_he_t) \le \|\eta\|\,\|G_he_t\| \le \frac{1}{2}\|\eta\|^2 + \frac{1}{2}\|G_he_t\|^2$, we obtain

$$\|G_he_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \le 2(\rho, e_t) + \|\eta\|^2.$$

Multiply this inequality by t to get $t\|G_h e_t\|^2 + t\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \le 2t(\rho, e_t) + t\|\eta\|^2$. Note that

$$t\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) - \|e\|^2, \quad t(\rho, e_t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho, e)\big) - (\rho, e) - t(\rho_t, e),$$

so that

$$t\|G_h e_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) \le 2\frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho, e)\big) + 2|(\rho, e)| + 2|t(\rho_t, e)| + t\|\eta\|^2 + \|e\|^2.$$

But

$$\begin{aligned}|(\rho, e)| &\le \|\rho\|\,\|e\| \le \frac{1}{2}\|\rho\|^2 + \frac{1}{2}\|e\|^2,\\ |t(\rho_t, e)| &\le t\|\rho_t\|\,\|e\| \le \frac{1}{2}t^2\|\rho_t\|^2 + \frac{1}{2}\|e\|^2.\end{aligned}$$

Hence

$$t\|G_h e_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) \le 2\frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho, e)\big) + \|\rho\|^2 + t^2\|\rho_t\|^2 + t\|\eta\|^2 + 3\|e\|^2.$$

Integrate over $[0, t]$ and use Young's inequality to get

$$\begin{aligned}\int_0^t \tau\|G_h e_t\|^2\,\mathrm{d}\tau + t\|e\|^2 \le 2t\|\rho\|^2 + \frac{1}{2}t\|e\|^2 + \int_0^t \|\rho\|^2\,\mathrm{d}\tau + \int_0^t \tau^2\|\rho_t\|^2\,\mathrm{d}\tau\\ + \int_0^t \tau\|\eta\|^2\,\mathrm{d}\tau + 3\int_0^t \|e\|^2\,\mathrm{d}\tau.\end{aligned}$$

Hence

$$t\|e\|^2 \le Ct\|\rho\|^2 + C\int_0^t \big(\|\rho\|^2 + \tau^2\|\rho_t\|^2 + \tau\|\eta\|^2 + \|e\|^2\big)\,\mathrm{d}\tau. \tag{2.9}$$

We must bound $\int_0^t \|e\|^2\,\mathrm{d}\tau$. Multiply (2.8) by e to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|G_h e\|^2 + \|e\|^2 \le \|\rho\|\,\|e\| + \|\eta\|\,\|G_h e\| \le \frac{1}{2}\|\rho\|^2 + \frac{1}{2}\|e\|^2 + \|\eta\| \max_{0\le\tau\le t}\|G_h e\|,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|G_h e\|^2 + \|e\|^2 \le \|\rho\|^2 + 2\|\eta\| \max_{0\le\tau\le t}\|G_h e\|. \tag{2.10}$$

Integrate (2.10), note that $G_h e(0) = A_h^{-1}P_h(P_h - I)v = 0$, to get

$$\|G_h e\|^2 + \int_0^t \|e\|^2\,\mathrm{d}\tau \le \int_0^t \|\rho\|^2\,\mathrm{d}\tau + \max_{0\le\tau\le t}\|G_h e\|^2 + \Big(\int_0^t \|\eta\|\,\mathrm{d}\tau\Big)^2.$$

Hence, since t is arbitrary,

$$\int_0^t \|e\|^2\,\mathrm{d}\tau \le \int_0^t \|\rho\|^2\,\mathrm{d}\tau + \Big(\int_0^t \|\eta\|\,\mathrm{d}\tau\Big)^2. \tag{2.11}$$

We insert (2.11) in (2.9) and conclude

$$
\begin{aligned}
t\|e\|^2 \le Ct\|\rho\|^2 + C\int_0^t \big(\|\rho\|^2 + \tau^2\|\rho_t\|^2 + \tau\|\eta\|^2\big)\,\mathrm{d}\tau \\
+ C\Big(\int_0^t \|\eta\|\,\mathrm{d}\tau\Big)^2.
\end{aligned} \tag{2.12}
$$

We compute the terms in the right hand side. With $v \in \dot{H}^\beta$, recalling $\rho = (R_h - I)u$ and using (2.3), we have

$$
\|\rho(t)\| \le Ch^\beta |u(t)|_\beta \le Ch^\beta \|E(t)A^{\frac{\beta}{2}}v\| \le Ch^\beta\|A^{\frac{\beta}{2}}v\| \le Ch^\beta|v|_\beta, \tag{2.13}
$$

so that,

$$
t\|\rho\|^2 \le Ch^{2\beta}t|v|_\beta^2, \quad \int_0^t \|\rho\|^2\,\mathrm{d}\tau \le Ch^{2\beta}t|v|_\beta^2.
$$

Similarly, by (2.1),

$$
\|\rho_t(t)\| \le Ch^\beta|u_t(t)|_\beta \le Ch^\beta\|A^2E(t)A^{\frac{\beta}{2}}v\| \le Ch^\beta t^{-1}|v|_\beta,
$$

so that

$$
\int_0^t \tau^2\|\rho_t\|^2\,\mathrm{d}\tau \le Ch^{2\beta}t|v|_\beta^2. \tag{2.14}
$$

Moreover, since $\eta = -(R_h - I)Gu_t$,

$$
\|\eta(t)\| \le Ch^\beta|Gu_t(t)|_\beta \le Ch^\beta\|AE(t)A^{\frac{\beta}{2}}v\| \le Ch^\beta t^{-\frac{1}{2}}|v|_\beta,
$$

so that

$$
\Big(\int_0^t \|\eta\|\,\mathrm{d}\tau\Big)^2 \le Ch^{2\beta}t|v|_\beta^2, \quad \int_0^t \tau\|\eta\|^2\,\mathrm{d}\tau \le Ch^{2\beta}t|v|_\beta^2.
$$

By inserting these in (2.12) we conclude

$$
t\|e\|^2 \le Ch^{2\beta}t|v|_\beta^2,
$$

which proves (2.4).

To prove (2.5) we recall (2.11) and let $v \in \dot{H}^{\beta-2}$. By using (2.3) and (2.2) we obtain

$$
\begin{aligned}
\int_0^t \|\rho\|^2\,\mathrm{d}\tau \le Ch^{2\beta}\int_0^t |u|_\beta^2\,\mathrm{d}\tau = Ch^{2\beta}\int_0^t \|AE(\tau)A^{\frac{\beta-2}{2}}v\|^2\,\mathrm{d}\tau \\
\le Ch^{2\beta}|v|_{\beta-2}^2.
\end{aligned} \tag{2.15}
$$

Now we compute $\int_0^t \|\eta\|\,\mathrm{d}\tau$. To this end we assume first $1 < \beta \le r$ and let $1 \le \gamma < \beta$. By using (2.1) and (2.3) we get

$$\int_0^t \|\eta\|\,\mathrm{d}\tau \le Ch^\gamma \int_0^t |Gu_t|_\gamma\,\mathrm{d}\tau = Ch^\gamma \int_0^t \|A^{2-\frac{\beta-\gamma}{2}} E(\tau) A^{\frac{\beta-2}{2}} v\|\,\mathrm{d}\tau$$
$$\le Ch^\gamma \int_0^t \tau^{-1+\frac{\beta-\gamma}{4}} \mathrm{e}^{-c\tau}\,\mathrm{d}\tau\,|v|_{\beta-2},$$

where, since $0 < \beta - \gamma \le r - 1$,

$$\int_0^t \tau^{-1+\frac{\beta-\gamma}{4}} \mathrm{e}^{-c\tau}\,\mathrm{d}\tau = \frac{4}{\beta-\gamma} \int_0^{t^{\frac{\beta-\gamma}{4}}} \mathrm{e}^{-cs^{\frac{4}{\beta-\gamma}}}\,\mathrm{d}s \le \frac{C}{\beta-\gamma} \int_0^\infty \mathrm{e}^{-cs^{\frac{4}{r-1}}}\,\mathrm{d}s.$$

Hence, with C independent of β,

$$\int_0^t \|\eta\|\,\mathrm{d}\tau \le \frac{Ch^\gamma}{\beta-\gamma} |v|_{\beta-2}. \tag{2.16}$$

Now let $\frac{1}{\beta-\gamma} = |\log h| = -\log h$, so $\gamma \to \beta$ as $h \to 0$, and

$$\gamma \log h = (\gamma - \beta + \beta) \log h = 1 + \beta \log h.$$

Therefore we have

$$\frac{h^\gamma}{\beta-\gamma} = |\log h| \mathrm{e}^{\gamma \log h} = |\log h| \mathrm{e}^{1+\beta \log h} \le C |\log h| h^\beta.$$

Put this in (2.16) to get, for $1 < \beta \le r$,

$$\int_0^t \|\eta\|\,\mathrm{d}\tau \le Ch^\beta |\log h| |v|_{\beta-2}, \tag{2.17}$$

and hence also for $1 \le \beta \le r$, because C is independent of β. Finally, we put (2.15) and (2.17) in (2.11) to get

$$\Big(\int_0^t \|e\|^2\,\mathrm{d}\tau \Big)^{\frac{1}{2}} \le C |\log h| h^\beta |v|_{\beta-2},$$

which is (2.5). □

Now we turn to the fully discrete case. The backward Euler method applied to

$$u_{h,t} + A_h^2 u_h = 0, \quad t > 0,$$
$$u_h(0) = P_h v,$$

defines $U_n \in S_h$ by

$$\begin{aligned} &\partial U_n + A_h^2 U_n = 0, \quad n \ge 1, \\ &U_0 = P_h v, \end{aligned} \tag{2.18}$$

where $\partial U_n = \frac{1}{k}(U_n - U_{n-1})$. Denoting $E_{kh}^n = (I + kA_h^2)^{-n}$, we have $U_n = E_{kh}^n v$. The next theorem provides error estimates in the L_2 norm for the deterministic Cahn-Hilliard equation in the fully discrete case.

Theorem 2.2. *Set $F_n = E_{kh}^n P_h - E(t_n)$. Then there are h_0, k_0 and C, such that for $h \le h_0$, $k \le k_0$, $1 \le \beta \le \min(r, 4)$, and $n \ge 1$, we have*

$$\|F_n v\| \le C(h^\beta + k^{\frac{\beta}{4}})|v|_\beta, \quad v \in \dot{H}^\beta, \tag{2.19}$$

$$\Big(k\sum_{j=1}^n \|F_j v\|^2\Big)^{\frac12} \le \big(C|\log h|h^\beta + C_{\beta,k}k^{\frac{\beta}{4}}\big)|v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}, \tag{2.20}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

Proof. Let G and G_h be as in the proof of Theorem 2.1. With $e_n = U_n - u_n = E_{kh}^n P_h v - E(t_n)v$, we get

$$G_h^2 \partial e_n + e_n = \rho_n + G_h \eta_n + G_h \delta_n, \tag{2.21}$$

where $u_n = u(t_n)$, $u_{t,n} = u_t(t_n)$ and

$$\rho_n = (R_h - I)u_n, \quad \eta_n = -(R_h - I)G\partial u_n, \quad \delta_n = -G(\partial u_n - u_{t,n}).$$

Multiply (2.21) by ∂e_n and note that

$$(\eta_n, G_h\partial e_n) \le \|\eta_n\|^2 + \frac14\|G_h\partial e_n\|^2, \ (\delta_n, G_h\partial e_n) \le \|\delta_n\|^2 + \frac14\|G_h\partial e_n\|^2,$$

to get

$$\|G_h\partial e_n\|^2 + 2(e_n, \partial e_n) \le 2(\rho_n, \partial e_n) + 2\|\eta_n\|^2 + 2\|\delta_n\|^2. \tag{2.22}$$

We have the following identities

$$\partial(a_n b_n) = (\partial a_n)b_n + a_{n-1}(\partial b_n) \tag{2.23}$$

$$= (\partial a_n)b_n + a_n(\partial b_n) - k(\partial a_n)(\partial b_n). \tag{2.24}$$

By using (2.24) we have

$$2(e_n, \partial e_n) = \partial\|e_n\|^2 + k\|\partial e_n\|^2,$$
$$(\rho_n, \partial e_n) = \partial(\rho_n, e_n) - (\partial\rho_n, e_n) + k(\partial\rho_n, \partial e_n).$$

Put these in (2.22) and cancel $k\|\partial e_n\|^2$ to get

$$\|G_h\partial e_n\|^2 + \partial\|e_n\|^2 \le 2\partial(\rho_n, e_n) - 2(\partial\rho_n, e_n) + k\|\partial\rho_n\|^2 + 2\|\eta_n\|^2 + 2\|\delta_n\|^2.$$

Multiply this by t_{n-1} and note that $k \le t_{n-1}$ for $n \ge 2$, so that we have for $n \ge 1$

$$\begin{aligned} t_{n-1}\|G_h\partial e_n\|^2 + t_{n-1}\partial\|e_n\|^2 &\le 2t_{n-1}\partial(\rho_n, e_n) - 2t_{n-1}(\partial\rho_n, e_n) + t_{n-1}^2\|\partial\rho_n\|^2 \\ &\quad + 2t_{n-1}\|\eta_n\|^2 + 2t_{n-1}\|\delta_n\|^2. \end{aligned} \tag{2.25}$$

By (2.23) we have

$$t_{n-1}\partial\|e_n\|^2 = \partial(t_n\|e_n\|^2) - \|e_n\|^2,$$
$$2t_{n-1}\partial(\rho_n, e_n) = 2\partial(t_n(\rho_n, e_n)) - 2(\rho_n, e_n).$$

Put these in (2.25) to get

$$\begin{aligned} t_{n-1}\|G_h\partial e_n\|^2 &+ \partial(t_n\|e_n\|^2) \\ &\le C\big(\partial(t_n(\rho_n, e_n)) + \|\rho_n\|^2 + t_{n-1}^2\|\partial\rho_n\|^2 + \|e_n\|^2\big) \\ &\quad + C\big(t_{n-1}\|\eta_n\|^2 + t_{n-1}\|\delta_n\|^2\big). \end{aligned} \tag{2.26}$$

Note that

$$k\sum_{j=1}^{n}\partial\big(t_j\|e_j\|^2\big) = t_n\|e_n\|^2, \quad k\sum_{j=1}^{n}\partial\big(t_j(\rho_j, e_j)\big) = t_n(\rho_n, e_n). \tag{2.27}$$

By summation in (2.26) and using (2.27) we get

$$\begin{aligned} k\sum_{j=1}^{n} t_{j-1}\|G_h\partial e_j\|^2 &+ t_n\|e_n\|^2 \le Ct_n\|\rho_n\|^2 \\ &+ Ck\sum_{j=1}^{n}\big(\|\rho_j\|^2 + t_{j-1}^2\|\partial\rho_j\|^2 + \|e_j\|^2\big) \\ &+ Ck\sum_{j=1}^{n}\big(t_{j-1}\|\eta_j\|^2 + t_{j-1}\|\delta_j\|^2\big). \end{aligned} \tag{2.28}$$

Now we estimate $k\sum_{j=1}^{n}\|e_j\|^2$. Take the inner product of (2.21) by e_n to get

$$2(G_h^2\partial e_n, e_n) + \|e_n\|^2 \le \|\rho_n\|^2 + 2\big(\|\eta_n\| + \|\delta_n\|\big)\|G_h e_n\|. \tag{2.29}$$

By (2.24) we have

$$2(G_h^2\partial e_n, e_n) = 2(\partial G_h e_n, G_h e_n) = \partial\|G_h e_n\|^2 + k\|\partial G_h e_n\|^2. \tag{2.30}$$

By summation in (2.29) and using $G_h e_0 = 0$, we get

$$\begin{aligned} \|G_h e_n\|^2 + k\sum_{j=1}^{n}\|e_j\|^2 &\le k\sum_{j=1}^{n}\|\rho_j\|^2 + \frac{1}{2}\max_{j\le n}\|G_h e_j\|^2 \\ &\quad + 2\Big(k\sum_{j=1}^{n}\big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2. \end{aligned}$$

Hence

$$k\sum_{j=1}^{n}\|e_j\|^2 \le k\sum_{j=1}^{n}\|\rho_j\|^2 + 2\Big(k\sum_{j=1}^{n}\big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2. \tag{2.31}$$

By putting (2.31) in (2.28) we get

$$
\begin{aligned}
t_n\|e_n\|^2 \le\, & Ct_n\|\rho_n\|^2 \\
& + Ck\sum_{j=1}^{n}\Big(\|\rho_j\|^2 + t_{j-1}^2\|\partial\rho_j\|^2 + t_{j-1}\|\eta_j\|^2 + t_{j-1}\|\delta_j\|^2\Big) \\
& + C\Big(k\sum_{j=1}^{n}\big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2.
\end{aligned} \tag{2.32}
$$

Now we compute the terms in the right hand side. With $v \in \dot{H}^\beta$ we have by (2.13),

$$
\|\rho_n\|^2 \le Ch^{2\beta}|v|_\beta^2, \quad k\sum_{j=1}^{n}\|\rho_j\|^2 \le Ch^{2\beta}t_n|v|_\beta^2. \tag{2.33}
$$

By using the Cauchy-Schwartz inequality we have

$$
\begin{aligned}
k\sum_{j=1}^{n} t_{j-1}^2\|\partial\rho_j\|^2 &= k\sum_{j=2}^{n} t_{j-1}^2\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}\rho_t\,\mathrm{d}\tau\Big\|^2 \\
&\le \sum_{j=2}^{n}\Big(t_{j-1}^2\frac{1}{k}\int_{t_{j-1}}^{t_j}\tau^{-2}\mathrm{d}\tau\int_{t_{j-1}}^{t_j}\tau^2\|\rho_t(\tau)\|^2\,\mathrm{d}\tau\Big), \\
&\le \int_0^{t_n}\tau^2\|\rho_t\|^2\mathrm{d}\tau.
\end{aligned}
$$

Hence, by (2.14),

$$
k\sum_{j=1}^{n} t_{j-1}^2\|\partial\rho_j\|^2 \le Ch^{2\beta}t_n|v|_\beta^2. \tag{2.34}
$$

By using (2.3) and (2.1) we have

$$
\begin{aligned}
\|\eta_j\| &\le Ch^\beta|G\partial u_j|_\beta \le \frac{Ch^\beta}{k}\Big\|\int_{t_{j-1}}^{t_j}AE(\tau)A^{\frac{\beta}{2}}v\,\mathrm{d}\tau\Big\| \\
&\le \frac{Ch^\beta}{k}\int_{t_{j-1}}^{t_j}\tau^{-\frac{1}{2}}\,\mathrm{d}\tau\|A^{\frac{\beta}{2}}v\| \le \frac{Ch^\beta}{k}(\sqrt{t_j}-\sqrt{t_{j-1}})|v|_\beta \le \frac{Ch^\beta}{\sqrt{t_j}}|v|_\beta.
\end{aligned}
$$

So

$$
k\sum_{j=1}^{n} t_{j-1}\|\eta_j\|^2 \le Ch^{2\beta}t_n|v|_\beta^2, \quad k\sum_{j=1}^{n}\|\eta_j\| \le Ch^\beta t_n^{\frac{1}{2}}|v|_\beta. \tag{2.35}
$$

By using (2.1) we have, for $j \geq 2$,

$$\begin{aligned}\|\delta_j\| &\leq \Big\| \frac{1}{k} \int_{t_{j-1}}^{t_j} (\tau - t_{j-1}) G u_{tt}(\tau) \mathrm{d}\tau \Big\| \leq \int_{t_{j-1}}^{t_j} \|A^{3-\frac{\beta}{2}} E(\tau) A^{\frac{\beta}{2}} v\| \, \mathrm{d}\tau \\ &\leq C \int_{t_{j-1}}^{t_j} \tau^{\frac{-6+\beta}{4}} \, \mathrm{d}\tau |v|_\beta,\end{aligned}$$

so that, by Hölder's inequality with $p = \frac{4}{\beta}$ and $q = \frac{4}{4-\beta}$, $1 \leq \beta < 4$,

$$\begin{aligned}\int_{t_{j-1}}^{t_j} \tau^{\frac{-6+\beta}{4}} \, \mathrm{d}\tau &\leq C k^{\frac{\beta}{4}} \Big(\int_{t_{j-1}}^{t_j} \big(\tau^{\frac{-6+\beta}{4}}\big)^{\frac{4}{4-\beta}} \, \mathrm{d}\tau \Big)^{\frac{4-\beta}{4}} \\ &\leq C k^{\frac{\beta}{4}} \Big(\frac{\beta - 4}{2} \big(t_{j-1}^{-\frac{2}{4-\beta}} - t_j^{-\frac{2}{4-\beta}} \big) \Big)^{\frac{4-\beta}{4}} \\ &\leq C k^{\frac{\beta}{4}} t_{j-1}^{-\frac{1}{2}}.\end{aligned}$$

The same result is obtained with $\beta = 4$. For $j = 1$ we have

$$\begin{aligned}\|\delta_1\| &\leq \Big\| \frac{1}{k} \int_0^k \tau G u_{tt}(\tau) \, \mathrm{d}\tau \Big\| \leq C \frac{1}{k} \int_0^k \tau^{\frac{-2+\beta}{4}} \, \mathrm{d}\tau |v|_\beta \\ &\leq C \frac{4}{2+\beta} k^{\frac{-2+\beta}{4}} |v|_\beta \leq C k^{\frac{\beta}{4}} t_1^{-\frac{1}{2}} |v|_\beta.\end{aligned}$$

So we have, for $j \geq 1$,

$$\|\delta_j\| \leq C k^{\frac{\beta}{4}} t_j^{-\frac{1}{2}} |v|_\beta.$$

Hence

$$k \sum_{j=1}^n \|\delta_j\| \leq c k^{\frac{\beta}{4}} t_n^{\frac{1}{2}} |v|_\beta, \quad k \sum_{j=1}^n t_{j-1} \|\delta_j\|^2 \leq C k^{\frac{\beta}{2}} t_n |v|_\beta^2. \tag{2.36}$$

Put (2.33), (2.34), (2.35), and (2.36) in (2.32), to get

$$\|e_n\| \leq C(h^\beta + k^{\frac{\beta}{4}}) |v|_\beta.$$

This completes the proof (2.19).

To prove (2.20) we recall (2.31) and let $v \in \dot{H}^{\beta-2}$. For the first term we write $k \sum_{j=1}^n \|\rho_j\|^2 = k \|\rho_1\|^2 + k \sum_{j=2}^n \|\rho_j\|^2$, where by (2.1)

$$k \|\rho_1\|^2 \leq k C h^{2\beta} \|A E(k) A^{\frac{\beta-2}{2}} v\|^2 \leq C h^{2\beta} |v|_{\beta-2},$$

and

$$\begin{aligned}
k\sum_{j=2}^{n}\|\rho_j\|^2 &= \sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\Big\|\rho(s)+\int_s^{t_j}\rho_t(\tau)\,\mathrm{d}\tau\Big\|^2\,\mathrm{d}s\\
&\le 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\|\rho(s)\|^2\,\mathrm{d}s + 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\Big\|\int_s^{t_j}\rho_t(\tau)\,\mathrm{d}\tau\Big\|^2\,\mathrm{d}s\\
&\le 2\int_{t_1}^{t_n}\|\rho(s)\|^2\,\mathrm{d}s + 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}(t_j-s)\int_{t_{j-1}}^{t_j}\|\rho_t(\tau)\|^2\,\mathrm{d}\tau\,\mathrm{d}s\\
&\le 2\int_0^{t_n}\|\rho\|^2\,\mathrm{d}\tau + 2k\int_{t_1}^{t_n}\tau\|\rho_t\|^2\,\mathrm{d}\tau,
\end{aligned}$$

since $t_j - s \le k \le \tau$ and where, by (2.15),

$$\int_0^{t_n}\|\rho\|^2\,\mathrm{d}\tau \le Ch^{2\beta}|v|_{\beta-2}^2,$$

and

$$\begin{aligned}
k\int_{t_1}^{t_n}\tau\|\rho_t\|^2\,\mathrm{d}\tau &\le Ch^{2\beta}k\int_{t_1}^{t_n}\tau\|A^3E(\tau)A^{\frac{\beta-2}{2}}v\|^2\,\mathrm{d}\tau\\
&\le Ch^{2\beta}k\int_k^{t_n}\tau^{-2}\,\mathrm{d}\tau\,|v|_{\beta-2}^2\\
&\le Ch^{2\beta}k(k^{-1}-t_n^{-1})|v|_{\beta-2}^2 \le Ch^{2\beta}|v|_{\beta-2}^2.
\end{aligned}$$

So

$$k\sum_{j=1}^{n}\|\rho_j\|^2 \le Ch^{2\beta}|v|_{\beta-2}^2. \tag{2.37}$$

Now we compute $k\sum_{j=1}^{n}\|\eta_j\|$. Recall that $\eta_j = -(R_h-I)G\partial u_j$ and $\eta = -(R_h-I)Gu_t$, so

$$\begin{aligned}
\|\eta_j\| = \Big\|(R_h-I)G\frac{1}{k}\int_{t_{j-1}}^{t_j}u_t\,\mathrm{d}\tau\Big\| &\le \frac{1}{k}\int_{t_{j-1}}^{t_j}\|(R_h-I)Gu_t\|\,\mathrm{d}\tau\\
&\le \frac{1}{k}\int_{t_{j-1}}^{t_j}\|\eta\|\,\mathrm{d}\tau,
\end{aligned}$$

and hence by (2.17) we have

$$k\sum_{j=1}^{n}\|\eta_j\| \le \int_0^{t_n}\|\eta\|\,\mathrm{d}\tau \le Ch^{\beta}|\log h||v|_{\beta-2}. \tag{2.38}$$

For computing $k\sum_{j=1}^n \|\delta_j\|$ we use (2.1) and obtain for $1\le\beta<4$,

$$\begin{aligned}\|\delta_j\| &\le \frac{1}{k}\int_{t_{j-1}}^{t_j}(\tau-t_{j-1})\|Gu_{tt}(\tau)\|\,\mathrm{d}\tau \le \int_{t_{j-1}}^{t_j}\|A^{4-\frac{\beta}{2}}E(\tau)A^{\frac{\beta-2}{2}}v\|\,\mathrm{d}\tau\\ &\le C\int_{t_{j-1}}^{t_j}\tau^{-2+\frac{\beta}{4}}\,\mathrm{d}\tau\,|v|_{\beta-2}.\end{aligned}$$

Hence

$$\begin{aligned}k\sum_{j=2}^n\|\delta_j\| &\le Ck\int_k^{t_n}\tau^{-2+\frac{\beta}{4}}\,\mathrm{d}\tau|v|_{\beta-2}\\ &\le Ck\frac{4}{4-\beta}\Big(k^{-1+\frac{\beta}{4}}-t_n^{-1+\frac{\beta}{4}}\Big)|v|_{\beta-2}\\ &\le \frac{C}{4-\beta}k^{\frac{\beta}{4}}|v|_{\beta-2}\end{aligned}$$

and

$$\begin{aligned}k\|\delta_1\| &\le \int_0^k\tau\|Gu_{tt}(\tau)\|\,\mathrm{d}\tau \le \int_0^k\tau\|A^{4-\frac{\beta}{2}}E(\tau)A^{\frac{\beta-2}{2}}v\|\,\mathrm{d}\tau\\ &\le C\int_0^k\tau^{\frac{\beta}{4}-1}\,\mathrm{d}\tau\,|v|_{\beta-2}\le\frac{C}{4-\beta}k^{\frac{\beta}{4}}|v|_{\beta-2}.\end{aligned}$$

Therefore, for $1\le\beta<4$,

$$k\sum_{j=1}^n\|\delta_j\|\le\frac{C}{4-\beta}k^{\frac{\beta}{4}}|v|_{\beta-2}.$$

If we put $\frac{1}{4-\beta}=|\log k|$, we also have

$$\begin{aligned}k\sum_{j=1}^n\|\delta_j\| &\le \frac{C}{4-\beta}k^{1-\frac{4-\beta}{4}}|v|_{\beta-2}=C|\log k|k\mathrm{e}^{-\frac{4-\beta}{4}\log k}|v|_{\beta-2}\\ &\le Ck|\log k||v|_{\beta-2}=C|\log k||v|_{\beta-2}.\end{aligned}$$

Therefore, for $1\le\beta\le4$, we have

$$k\sum_{j=1}^n\|\delta_j\|\le C_{\beta,k}k^{\frac{\beta}{4}}|v|_{\beta-2}. \tag{2.39}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$. Finally we put (2.37), (2.38) and (2.39) in (2.31), to get

$$\Big(k\sum_{j=1}^{n}\|e_j\|^2\Big)^{\frac12} \le \Big(Ch^{\beta}|\log h| + C_{\beta,k}k^{\frac{\beta}{4}}\Big)|v|_{\beta-2}.$$

□

3. Finite element method for the Cahn-Hilliard-Cook equation

Consider the linear Cahn-Hilliard-Cook equation (1.4) with mild solution

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s). \tag{3.1}$$

We recall the isometry of the Itô integral

$$\mathbf{E}\Big\|\int_0^t B(s)\,\mathrm{d}W(s)\Big\|^2 = \mathbf{E}\int_0^t \|B(s)Q^{\frac12}\|_{\mathrm{HS}}^2\,\mathrm{d}s, \tag{3.2}$$

where the Hilbert-Schmidt norm is defined by

$$\|T\|_{\mathrm{HS}}^2 = \sum_{l=1}^{\infty}\|T\phi_l\|^2,$$

where $\{\phi_l\}_{l=1}^{\infty}$ is an arbitrary orthonormal basis for H. In the next theorem we consider the regularity of the mild solution (3.1).

Theorem 3.1. *Let $X(t)$ be the mild solution* (3.1)*. If $X_0 \in L_2(\Omega,\dot H^{\beta})$ and $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ for some $\beta \ge 0$. If $\beta > 0$, then*

$$\|X(t)\|_{L_2(\Omega,\dot H^{\beta})} \le C\Big(\|X_0\|_{L_2(\Omega,\dot H^{\beta})} + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\Big), \quad t \ge 0.$$

If $\beta = 0$, then

$$\|X(t)\|_{L_2(\Omega,H)} \le C\Big(\|X_0\|_{L_2(\Omega,H)} + \|A^{-1}Q^{\frac12}\|_{\mathrm{HS}} + t^{\frac12}\Big), \quad t \ge 0.$$

Proof. By using the isometry (3.2), the definition of the Hilbert-Schmidt norm, and (2.1), (2.2) we get, for $\beta > 0$, see (1.8),

$$\begin{aligned}
\|X(t)\|^2_{L_2(\Omega,\dot H^\beta)} &= \mathbf{E}\Big|E(t)X_0 + \int_0^t E(t-s)P\,\mathrm{d}W(s)\Big|^2_\beta \\
&\le C\Big(\mathbf{E}\big|E(t)X_0\big|^2_\beta + \mathbf{E}\Big\|\int_0^t A^{\frac{\beta}{2}}E(t-s)P\,\mathrm{d}W(s)\Big\|^2\Big) \\
&\le C\Big(\|X_0\|^2_{L_2(\Omega,\dot H^\beta)} + \int_0^t \|A^{\frac{\beta}{2}}E(s)PQ^{\frac12}\|^2_{\mathrm{HS}}\,\mathrm{d}s\Big) \\
&\le C\Big(\|X_0\|^2_{L_2(\Omega,\dot H^\beta)} + \sum_{l=1}^\infty \|A^{\frac{\beta-2}{2}}Q^{\frac12}\phi_l\|^2\Big) \\
&\le C\Big(\|X_0\|^2_{L_2(\Omega,\dot H^\beta)} + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|^2_{\mathrm{HS}}\Big).
\end{aligned}$$

For $\beta = 0$, there is the additional term $\mathbf{E}\|(I-P)W(t)\|^2 = \mathbf{E}[(W(t),\varphi_0)^2] \le Ct$. □

The finite element problem for Cahn-Hilliard-Cook equation is: Find $X_h(t) \in S_h$ such that

$$\begin{aligned}
&\mathrm{d}X_h + A_h^2 X_h\,\mathrm{d}t = P_h\,\mathrm{d}W, \\
&X_h(0) = P_h X_0.
\end{aligned} \tag{3.3}$$

So the mild solution can be written as

$$X_h(t) = E_h(t)P_hX_0 + \int_0^t E_h(t-s)P_h\,\mathrm{d}W(s). \tag{3.4}$$

Theorem 3.2. *Let X_h and X be the mild solutions* (3.4) *and* (3.1) *with $X_0 \in L_2(\Omega,\dot H^\beta)$ and $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1,r]$. Then there are h_0 and C, such that, for $h \le h_0$ and $t \ge 0$,*

$$\begin{aligned}
\|X_h(t) - &X(t)\|_{L_2(\Omega,H)} \\
&\le Ch^\beta\big(\|X_0\|_{L_2(\Omega,\dot H^\beta)} + |\log h|\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\big).
\end{aligned}$$

Proof. Use (3.1) and (3.4) and set $F_h(t) = E_h(t)P_h - E(t)$ to get

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \le \|e_1(t)\|_{L_2(\Omega,H)} + \|e_2(t)\|_{L_2(\Omega,H)},$$

where

$$\begin{aligned}
e_1(t) &= F_h(t)X_0 = F_h(t)PX_0, \\
e_2(t) &= \int_0^t F_h(t-s)\,\mathrm{d}W(s) = \int_0^t F_h(t-s)P\,\mathrm{d}W(s).
\end{aligned}$$

By using Theorem 2.1 we get

$$\|e_1(t)\|_{L_2(\Omega,H)} = \big(\mathbf{E}\|F_h(t)X_0\|^2\big)^{\frac12} \le Ch^\beta\big(\mathbf{E}|X_0|_\beta^2\big)^{\frac12} = Ch^\beta\|X_0\|_{L_2(\Omega,\dot H^\beta)}.$$

For the second term we use the isometry (3.2), the definition of Hilbert-Schmidt norm and Theorem 2.1,

$$\begin{aligned}
\|e_2(t)\|^2_{L_2(\Omega,H)} &= \mathbf{E}\Big(\Big\|\int_0^t F_h(t-s)\,\mathrm{d}W(s)\Big\|^2\Big)\\
&= \int_0^t \|F_h(t-s)Q^{\frac12}\|^2_{\mathrm{HS}}\,\mathrm{d}s\\
&= \sum_{l=1}^\infty \int_0^t \|F_h(s)Q^{\frac12}\phi_l\|^2\,\mathrm{d}s\\
&\le C|\log h|^2h^{2\beta}\sum_{l=1}^\infty |Q^{\frac12}\phi_l|^2_{\beta-2}\\
&= C|\log h|^2h^{2\beta}\|A^{(\beta-2)/2}Q^{\frac12}\|^2_{\mathrm{HS}}.
\end{aligned}$$

□

Now we consider the fully discrete Cahn-Hilliard-Cook equation (1.10) with mild solution

$$X_{h,n} = E_{kh}^nP_hX_0 + \sum_{j=1}^n E_{kh}^{n-j+1}P_h\,\delta W_j, \tag{3.5}$$

where $E_{kh} = (I+kA_h^2)^{-1}$.

Theorem 3.3. *Let $X_{h,n}$ and X be given by* (3.5) *and* (3.1) *with $X_0 \in L_2(\Omega,\dot H^\beta)$ and $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1,\min(r,4)]$. Then there are h_0, k_0 and C, such that, for $h \le h_0$, $k \le k_0$, and $n \ge 1$,*

$$\begin{aligned}
&\|X_{h,n}(t) - X(t_n)\|_{L_2(\Omega,H)}\\
&\qquad \le \big(C|\log h|h^\beta + C_{\beta,k}k^{\frac{\beta}{4}}\big)\big(\|X_0\|_{L_2(\Omega,\dot H^\beta)} + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\big),
\end{aligned}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

Proof. By using (3.1) and (3.5) we get, with $F_n = E_{kh}^n P_h - E(t_n)$,

$$\begin{aligned} e_n &= F_n X_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} F_{n-j+1}\,\mathrm{d}W(s) \\ &\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(E(t_n - t_{j-1}) - E(t_n - s)\big)\,\mathrm{d}W(s) \\ &= e_{n,1} + e_{n,2} + e_{n,3}. \end{aligned}$$

By using Theorem 2.2 we have

$$\|e_{n,1}\|_{L_2(\Omega,H)} = \big(\mathbf{E}\|F_n X_0\|^2\big)^{\frac{1}{2}} \le C(h^\beta + k^{\frac{\beta}{4}})\|X_0\|_{L_2(\Omega,\dot{H}^\beta)}. \tag{3.6}$$

By using the isometry (3.2) and Theorem 2.2 we get

$$\begin{aligned} \|e_{n,2}\|^2_{L_2(\Omega,H)} &= \mathbf{E}\Big(\Big\|\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} F_{n-j+1}\,\mathrm{d}W(s)\Big\|^2\Big) \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|F_{n-j+1} Q^{\frac{1}{2}}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\ &= k \sum_{l=1}^{\infty} \sum_{j=1}^{n} \|F_{n-j+1} Q^{\frac{1}{2}} \phi_l\|^2 \\ &\le \sum_{l=1}^{\infty} \big(C|\log h| h^\beta + C_{\beta,k} k^{\frac{\beta}{4}}\big)^2 |Q^{\frac{1}{2}} \phi_l|^2_{\beta-2} \\ &= \big(C|\log h| h^\beta + C_{\beta,k} k^{\frac{\beta}{4}}\big)^2 \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|^2_{\mathrm{HS}}. \end{aligned}$$

By using the isometry property (3.2) again we have

$$\begin{aligned} &\|e_{n,3}\|^2_{L_2(\Omega,H)} \\ &\quad \le \mathbf{E}\Big(\Big\|\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (E(t_n - t_{j-1}) - E(t_n - s))\,\mathrm{d}W(s)\Big\|^2\Big) \\ &\quad = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|(E(t_n - t_{j-1}) - E(t_n - s)) Q^{\frac{1}{2}}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\ &\quad = \sum_{l=1}^{\infty} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|A^{-\frac{\beta}{2}} (E(s - t_{j-1}) - I) A E(t_n - s) A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}} \phi_l\|^2\,\mathrm{d}s. \end{aligned}$$

Using the well-known inequality

$$\|A^{\frac{-\beta}{2}}\big(E(t) - I\big)w\| \le C t^{\frac{\beta}{4}} \|w\|,$$

with $t = s - t_j, w = AE(t_n - s)A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\phi_l$, together with (2.2), we get

$$\begin{aligned}\|e_{n,3}\|^2_{L_2(\Omega,H)} &\leq Ck^{\frac{\beta}{2}}\sum_{l=1}^{\infty}\int_0^{t_n}\|AE(t_n-s)A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\phi_l\|^2\,\mathrm{d}s \\ &\leq Ck^{\frac{\beta}{2}}\sum_{l=1}^{\infty}\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\phi_l\|^2 = Ck^{\frac{\beta}{2}}\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|^2_{\mathrm{HS}}.\end{aligned}$$

Putting these together proves the desired result. □

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FINITE ELEMENT APPROXIMATION OF THE CAHN-HILLIARD-COOK EQUATION

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Abstract. We study the nonlinear stochastic Cahn-Hilliard equation driven by additive colored noise. We show almost sure existence and regularity of solutions. We introduce spatial approximation by a standard finite element method and prove error estimates of optimal order on sets of probability arbitrarily close to 1. We also prove strong convergence without known rate.

1. Introduction

We study the Cahn-Hilliard equation perturbed by noise, also known as the Cahn-Hilliard-Cook equation (cf. [1, 3]),

$$\begin{aligned}
&\mathrm{d}u - \Delta w \,\mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D} \times [0,T],\\
&w + \Delta u + f(u) = 0 && \text{in } \mathcal{D} \times [0,T],\\
&\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 && \text{on } \partial\mathcal{D} \times [0,T],\\
&u(0) = u_0. && \text{in } \mathcal{D}.
\end{aligned}$$

Here $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$, and $f(s) = s^3 - s$. Using the framework of [9] we write this as an abstract evolution equation of the form

$$\mathrm{d}X + (A^2 X + A f(X))\,\mathrm{d}t = \mathrm{d}W, \quad t > 0; \quad X(0) = X_0, \tag{1.1}$$

where A denotes the Neumann Laplacian considered as an unbounded operator in the Hilbert space $H = L_2(\mathcal{D})$ and W is a Q-Wiener process in H with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$. See Section 2 for details.

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Our goal is to study the convergence properties of the spatially semidiscrete finite element approximation X_h of X, which is defined by an equation of the form

$$\mathrm{d}X_h + (A_h^2 X_h + A_h P_h f(X_h))\,\mathrm{d}t = P_h \mathrm{d}W, \quad t > 0; \quad X_h(0) = P_h X_0.$$

In order to do so, we need to prove existence and regularity for solutions of (1.1). Such results were first proved in [4]. Under the assumption that the covariance operator $Q = I$ (space-time white noise, cylindrical noise) it was shown that there is a process which belongs to $C([0,T], H^{-1})$ almost surely (a.s.) and which is the unique solution of (1.1). Under the stronger assumption that A and Q commute and that $\mathrm{Tr}(A^{\delta-1}Q) < \infty$ for some $\delta > 0$ (colored noise) it was shown that the solution belongs to $C([0,T], H)$ a.s. Such regularity is insufficient for proving convergence of a numerical solution. Our first aim is therefore to prove existence of a solution in $C([0,T], H^\beta)$ a.s. for some $\beta > 0$.

Following the semigroup approach of [9] we write the equation (1.1) as the integral equation (mild solution)

$$\begin{aligned} X(t) &= \mathrm{e}^{-tA^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} A f(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s) \\ &= Y(t) + W_A(t), \end{aligned}$$

where e^{-tA^2} is the analytic semigroup generated by $-A^2$. This naturally splits the solution as $X = Y + W_A$, where $W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2} A\,\mathrm{d}W(s)$ is a stochastic convolution. This convolution, and its finite element approximation, was studied in [8]. In particular, it was shown there that if $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(A^{\beta-2}Q) < \infty$ for some $\beta \geq 0$, then we have regularity of order β in a mean square sense; that is,

$$\mathbf{E}[\|W_A(t)\|_{H^\beta}^2] \leq \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2, \quad t \geq 0. \tag{1.2}$$

The other part, Y, solves a differential equation with random coefficient,

$$\dot{Y} + A^2 Y + A f(Y + W_A) = 0, \quad t > 0; \quad Y(0) = X_0. \tag{1.3}$$

This can be solved once W_A is known. This approach was also used in [4], but while they used Galerkin's method and energy estimates to solve (1.3), we use a semigroup approach similar to that of [5]. However, published results for the deterministic Cahn-Hilliard equation do not apply directly due to the limited regularity in (1.3).

The nonlinear term is only locally Lipschitz and we need to control the Lipschitz constant. In the deterministic case studied in [5] this is achieved by the Lyapunov functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x, \quad u \in H^1, \quad F(s) = \tfrac{1}{4}s^4 - \tfrac{1}{2}s^2,$$

which is nonincreasing along paths, so that $\|X(t)\|_{H^1} \le C$ for $t \ge 0$. Due to the stochastic perturbation, this is not true for the stochastic equation (1.1). However, it is possible find a bound for the growth of the expected value of $J(X(t))$, and hence a bound

$$\mathbf{E}[\|X(t)\|_{H^1}^2] \le C(t), \quad t \ge 0. \tag{1.4}$$

This was shown in [4] under the assumption

$$\|A^{1/2}Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(AQ) < \infty, \tag{1.5}$$

which is consistent with $\beta = 3$ in (1.2). We repeat this in Theorem 3.1 with several improvements. First of all we reduce the growth of the bound from exponential to quadratic with respect to t. We also relax the assumptions: we do not assume that A and Q commute; that is, have a common eigenbasis, and we do not assume that the eigenbasis of Q consists of bounded functions. Moreover, we prove the same bound for the finite element solution X_h.

By means of Chebyshev's inequality we may then show that for each $T > 0$ and $\epsilon \in (0, 1)$ there are K_T and $\Omega_\epsilon \subset \Omega$ with $\mathbf{P}(\Omega_\epsilon) \ge 1 - \epsilon$ and such that

$$\|X(t)\|_{H^1}^2 + \|X_h(t)\|_{H^1}^2 \le \epsilon^{-1} K_T \quad \text{on } \Omega_\epsilon,\ t \in [0, T].$$

This bound controls the nonlinear term and we show that $X \in C([0, T], H^3)$ for $\omega \in \Omega_\epsilon$ under the assumption (1.5) (see Theorem 4.2). We also obtain an error estimate (see Theorem 5.3)

$$\|X_h(t) - X(t)\| \le C(\epsilon^{-1} K_T, T) h^2 |\log(h)| \quad \text{on } \Omega_\epsilon,\ t \in [0, T].$$

The constant grows rapidly with $\epsilon^{-1} K_T$, but nevertheless we may use this to show strong convergence (see Theorem 5.4),

$$\max_{t \in [0,T]} \mathbf{E}[\|X_h(t) - X(t)\|^2] \to 0 \quad \text{as } h \to 0. \tag{1.6}$$

To prove strong convergence with an estimate of the rate remains a challenge for future work. In this connection we note that even for numerical methods for stochastic ordinary differential equations with local Lipschitz nonlinearity there are few results on convergence rates (cf. [6]).

Numerical methods for the deterministic Cahn-Hilliard equation are well covered in literature. There are few studies of numerical methods for the Cahn-Hilliard-Cook equation. We are only aware of [2] in which convergence in probability was proved for a difference scheme for the nonlinear equation in multiple dimensions. For the linear equation there is [7], where strong convergence estimates were proved for the finite element method for the linear equation in 1-D, and the already mentioned work [8] on the finite element method for the stochastic convolution in multiple dimensions.

2. Preliminaries

2.1. **Norms.** Let $\mathcal{D} \subset \mathbf{R}^d$, $d = 1, 2, 3$, be a bounded convex domain with polygonal boundary $\partial\mathcal{D}$. Let $H = L_2(\mathcal{D})$ with standard inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and

$$\dot{H} = \Big\{ v \in H : \int_{\mathcal{D}} v \, \mathrm{d}x = 0 \Big\}.$$

We also denote by $H^k = H^k(\mathcal{D})$ the standard Sobolev space. We define $A = -\Delta$ with domain of definition

$$D(A) = \Big\{ v \in H^2 : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D} \Big\}.$$

Then A is a positive definite, selfadjoint, unbounded, linear operator on $\dot{H}$ with compact inverse. When extended to H it has an orthonormal eigenbasis $\{\varphi_j\}_{j=0}^{\infty}$ with corresponding eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ such that

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty.$$

The first eigenfunction is constant, $\varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$.

Let $P \colon H \to \dot{H}$ define the orthogonal projector. Then

$$(I - P)v = \langle v, \varphi_0 \rangle \varphi_0 = |\mathcal{D}|^{-1} \int_{\mathcal{D}} v \, \mathrm{d}x,$$

is the average of v. We define seminorms and norms

$$|v|_\alpha = \Big(\sum_{j=1}^{\infty} \lambda_j^\alpha |\langle v, \varphi_j \rangle|^2 \Big)^{\frac{1}{2}}, \quad \alpha \geq 0,$$

$$\|v\|_\alpha = \Big(\sum_{j=0}^{\infty} \lambda_j^\alpha |\langle v, \varphi_j \rangle|^2 \Big)^{\frac{1}{2}} = \big(|v|_\alpha^2 + |\langle v, \varphi_0 \rangle|^2 \big)^{\frac{1}{2}}, \quad \alpha \geq 0,$$

and corresponding spaces

$$\dot{H}^\alpha = D(A^{\frac{\alpha}{2}}) = \Big\{ v \in H : |v|_\alpha < \infty \Big\}, \quad H^\alpha = \Big\{ v \in H : \|v\|_\alpha < \infty \Big\}.$$

For integer order $\alpha = k$, H^k coincides with the standard Sobolev spaces with $\|\cdot\|_k$ equivalent to the standard norm $\|\cdot\|_{H^k}$. For example,

$$\|v\|_1^2 = |v|_1^2 + |\langle v, \varphi_0 \rangle|^2 = \|\nabla v\|^2 + |\langle v, \varphi_0 \rangle|^2 \tag{2.1}$$

is equivalent to $\|v\|_{H^1}^2$ by the Poincaré inequality.

2.2. **The semigroup.** The operator $-A^2$ is the infinitesimal generator of an analytic semigroup e^{-tA^2} on H,

$$\begin{aligned}\mathrm{e}^{-tA^2}v &= \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2}\langle v,\varphi_j\rangle\varphi_j = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2}\langle v,\varphi_j\rangle\varphi_j + \langle v,\varphi_0\rangle\varphi_0 \\ &= \mathrm{e}^{-tA^2}Pv + (I-P)v.\end{aligned}$$

The analyticity implies that

$$\|A^\alpha \mathrm{e}^{-tA^2}v\| \le Ct^{-\frac{\alpha}{2}}\mathrm{e}^{-ct}\|v\|, \quad \alpha > 0. \tag{2.2}$$

2.3. **The finite element method.** Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of regular triangulations of $\mathcal{D}$ with maximal mesh size h. Let S_h be the space of continuous functions on $\mathcal{D}$, which are piecewise polynomials of degree ≤ 1 with respect to $\mathcal{T}_h$. Hence, $S_h \subset H^1$. We also define $\dot{S}_h = PS_h$; that is,

$$\dot{S}_h = \Big\{ v_h \in S_h : \int_{\mathcal{D}} v_h \,\mathrm{d}x = 0 \Big\}.$$

The space $\dot{S}_h$ is introduced only for the purpose of theory but not for computation. Now we define the "discrete Laplacian" $A_h\colon S_h \to \dot{S}_h$ by

$$\langle A_h v_h, w_h\rangle = \langle \nabla v_h, \nabla w_h\rangle, \quad \forall v_h \in S_h,\ w_h \in \dot{S}_h.$$

We note that

$$|v_h|_1 = \|A^{\frac12}v_h\| = \|\nabla v_h\| = \|A_h^{\frac12}v_h\|, \quad v_h \in S_h. \tag{2.3}$$

The operator A_h is selfadjoint, positive definite on $\dot{S}_h$, positive semidefinite on S_h, and A_h has an orthonormal eigenbasis $\{\varphi_{h,j}\}_{j=0}^{N_h}$ with corresponding eigenvalues $\{\lambda_{h,j}\}_{j=0}^{N_h}$. We have

$$0 = \lambda_{h,0} < \lambda_{h,1} \le \cdots \le \lambda_{h,j} \le \cdots \le \lambda_{h,N_h},$$

and $\varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac12}$. Moreover, we define $\mathrm{e}^{-tA_h^2}\colon S_h \to S_h$ by

$$\mathrm{e}^{-tA_h^2}v_h = \sum_{j=0}^{N_h} \mathrm{e}^{-t\lambda_{h,j}}\langle v_h,\varphi_{h,j}\rangle\varphi_{h,j} = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}}\langle v_h,\varphi_{h,j}\rangle\varphi_{h,j} + \langle v_h,\varphi_0\rangle\varphi_0,$$

and the orthogonal projector $P_h\colon H \to S_h$ by

$$\langle P_h v, w_h\rangle = \langle v, w_h\rangle \quad \forall v \in H,\ w_h \in S_h. \tag{2.4}$$

Clearly, $P_h\colon \dot{H} \to \dot{S}_h$ and

$$\mathrm{e}^{-tA_h^2}P_h v = \mathrm{e}^{-tA_h^2}P_h P v + (I-P)v.$$

We have a discrete analog of (2.2),

$$\|A_h^\alpha \mathrm{e}^{-tA_h^2}v_h\| \le Ct^{-\frac{\alpha}{2}}\mathrm{e}^{-ct}\|v_h\|, \quad v_h \in S_h,\ \alpha > 0. \tag{2.5}$$

Finally, we define the Ritz projector $R_h\colon \dot H^1 \to \dot S_h$ by

$$\langle \nabla R_h v, \nabla w_h\rangle = \langle \nabla v, \nabla w_h\rangle, \quad \forall v \in \dot H^1,\, w_h \in \dot S_h.$$

We extend it to $R_h\colon H^1 \to S_h$ by

$$R_h v = R_h P v + (I-P)v, \quad v \in H^1. \tag{2.6}$$

We then have the error bound (cf. [10, Ch. 1])

$$\|R_h v - v\| \le C h^\beta |v|_\beta, \quad v \in H^\beta,\ \beta \in [1,2]. \tag{2.7}$$

In order to simplify the presentation, we assume that P_h is bounded with respect to the H^1 and L_4 norms, and that we have an inverse bound for A_h,

$$\begin{aligned} &\|P_h v\|_1 \le C\|v\|_1, && v \in H^1,\\ &\|P_h v\|_{L_4} \le C\|v\|_{L_4}, && v \in H^1,\\ &\|A_h v_h\| \le C h^{-2}\|v_h\|, && v_h \in S_h. \end{aligned} \tag{2.8}$$

This holds, for example, if the mesh family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform.

2.4. **The Wiener process.** We recall the definitions of the trace and the Hilbert-Schmidt norm of a linear operator T on H:

$$\mathrm{Tr}(T) = \sum_{k=1}^\infty \langle T f_k, f_k\rangle, \quad \|T\|_{\mathrm{HS}} = \Big(\sum_{k=1}^\infty \|T f_k\|^2\Big)^{\frac12},$$

where $\{f_k\}_{k=1}^\infty$ is an arbitrary orthonormal basis of H.

Let Q be a selfadjoint, positive semidefinite, bounded, linear operator on H with $\mathrm{Tr}(Q) < \infty$. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal eigenbasis for Q with eigenvalues $\{\gamma_k\}_{k=1}^\infty$. Then we define the Q-Wiener process

$$W(t) = \sum_{k=1}^\infty \gamma_k^{\frac12} \beta_k(t) e_k,$$

where the β_k are real-valued, independent Brownian motions. The series converges in $L_2(\Omega, H)$; that is, with respect to the norm $\|v\|_{L_2(\Omega,H)} = (\mathbf{E}[\|v\|^2])^{\frac12}$. The Q-Wiener process can be defined also when the covariance operator has infinite trace but this is not needed in the present work.

2.5. **The stochastic convolution.** We now define (cf. [9])

$$\begin{aligned} W_A(t) &= \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s) \\ &= \int_0^t \mathrm{e}^{-(t-s)A^2} P\,\mathrm{d}W(s) + \int_0^t \langle \mathrm{d}W(s), \varphi_0\rangle\varphi_0 \\ &= \int_0^t \mathrm{e}^{-(t-s)A^2} P\,\mathrm{d}W(s) + \langle W(t), \varphi_0\rangle\varphi_0 \\ &= \int_0^t \mathrm{e}^{-(t-s)A^2} P\,\mathrm{d}W(s) + (I-P)W(t). \end{aligned}$$

Similarly,

$$\begin{aligned} W_{A_h}(t) &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h\,\mathrm{d}W(s) \\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h P\,\mathrm{d}W(s) + \langle W(t), \varphi_0\rangle\varphi_0 \\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h P\,\mathrm{d}W(s) + (I-P)W(t). \end{aligned}$$

Hence, the constant eigenmodes cancel:

$$W_{A_h}(t) - W_A(t) = \int_0^t \Big(\mathrm{e}^{-(t-s)A_h^2} P_h - \mathrm{e}^{-(t-s)A^2}\Big) P\,\mathrm{d}W(s). \tag{2.9}$$

These convolutions were studied in [8]. We quote the following results from there. We use the norms

$$\|v\|_{L_2(\Omega,\dot H^\beta)} = (\mathbf{E}[\,|v|_\beta^2\,])^{\frac12}.$$

Theorem 2.1. *If* $\|A^{\frac{\beta-2}{2}} Q^{\frac12}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \ge 2$*, then*

$$\|W_A(t)\|_{L_2(\Omega,\dot H^\beta)} \le C\|A^{\frac{\beta-2}{2}} Q^{\frac12}\|_{\mathrm{HS}}, \quad t \ge 0.$$

Theorem 2.2. *If* $\|Q^{\frac12}\|_{\mathrm{HS}} < \infty$*, then*

$$\|W_{A_h}(t) - W_A(t)\|_{L_2(\Omega,H)} \le Ch^2|\log h|\|Q^{\frac12}\|_{\mathrm{HS}}, \quad t \ge 0.$$

Note that $\beta = 2$ in the latter theorem. In [8] these are stated with a slightly wider range of the order β, but this is not needed in the present work.

2.6. **Gronwall's lemma.** We need the following generalization of Gronwall's lemma. A proof can found in [5].

Lemma 2.3 (Generalized Gronwall lemma)**.** *Let the function* $\varphi(t) \geq 0$ *be continuous for* $0 \leq t \leq T$*. If*

$$\varphi(t) \leq At^{-1+\alpha} + B \int_0^t (t-s)^{-1+\beta} \varphi(s) \, \mathrm{d}s, \quad t \in (0, T],$$

for some constants $A, B \geq 0$ *and* $\alpha, \beta > 0$*, then there is a constant* $C = C(B, T, \alpha, \beta)$ *such that*

$$\varphi(t) \leq CAt^{-1+\alpha}, \quad t \in (0, T].$$

We also use the standard Gronwall lemma:

Lemma 2.4 (Gronwall's lemma)**.** *Let the function* $\varphi(t)$ *be continuous on* $[0, T]$*. If, for some* $A, C \geq 0$ *and* $B > 0$*,*

$$\varphi(t) \leq A + Ct + B \int_0^t \varphi(s) \, \mathrm{d}s, \quad t \in [0, T],$$

then

$$\varphi(t) \leq \Big(A + \frac{C}{B}\Big) \mathrm{e}^{Bt}, \quad t \in [0, T].$$

Proof. Set $\Phi(t) = A + Ct + B \int_0^t \varphi(s) \, \mathrm{d}s$. Then

$$\Phi'(t) = C + B\varphi(t) \leq C + B\Phi(t),$$

so that $\Phi'(t) - B\Phi(t) \leq C$, which gives $\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(t)\mathrm{e}^{-Bt}) \leq C\mathrm{e}^{-Bt}$. Hence

$$\Phi(t)\mathrm{e}^{-Bt} \leq \Phi(0) + C \int_0^t \mathrm{e}^{-Bs} \, \mathrm{d}s = \Big(A + \frac{C}{B}\Big) - \frac{C}{B} \mathrm{e}^{-Bt}.$$

Multiplying both sides by e^{Bt} gives

$$\Phi(t) \leq \Big(A + \frac{C}{B}\Big) \mathrm{e}^{Bt} - \frac{C}{B} \leq \Big(A + \frac{C}{B}\Big) \mathrm{e}^{Bt}.$$

But $\varphi(t) \leq \Phi(t)$, so the desired result follows. $\square$

2.7. **Bounds for the nonlinear term.**

Lemma 2.5. *For* $u, v \in H^3$ *and* $f(s) = s^3 - s$ *we have*

$$\|\Delta f(u)\| \leq C(1 + \|u\|_1^2)\|u\|_3, \tag{2.10}$$

$$\|A_h^{-\frac{1}{2}} P(f(u) - f(v))\| \leq C \left(1 + \|u\|_1^2 + \|v\|_1^2\right) \|u - v\|. \tag{2.11}$$

Proof. We have $f'(s) = 3s^2 - 2s$, $f''(s) = 6s^2$. Using Hölder's inequality, Sobolev's inequality $\|u\|_{L_6} \le C\|u\|_{H^1}$ (for $d \le 3$), and $\|u\|_{H^k} \le \|u\|_k$, we get

$$\begin{aligned}
\|\Delta f(u)\| &= \|f'(u)\Delta u + f''(u)|\nabla u|^2\| \\
&\le \|f'(u)\|_{L_3}\|\Delta u\|_{L_6} + \|f''(u)\|_{L_6}\|\nabla u\|_{L_6} \\
&\le C\left(1+\|u\|_{L_6}^2\right)\|\Delta u\|_{L_6} + C\|u\|_{L_6}\|\nabla u\|_{L_6}^2 \\
&\le C\left(1+\|u\|_{H^1}^2\right)\|u\|_{H^3} + C\|u\|_{H^1}\|\nabla u\|_{H^2}^2 \\
&\le C\left(1+\|u\|_1^2\right)\|u\|_3 + C\|u\|_1\|u\|_2^2 \\
&\le C\left(1+\|u\|_1^2\right)\|u\|_3,
\end{aligned}$$

where we used $\|u\|_2 \le C\|u\|_1^{\frac12}\|u\|_3^{\frac12}$ in the last step. This proves (2.10).

For (2.11) we apply (2.3) and Hölder and Sobolev's inequalities ($d \le 3$) to get

$$\begin{aligned}
\|A_h^{-\frac12}P\varphi\| &= \sup_{v_h\in S_h}\frac{\langle A_h^{-\frac12}P\varphi, v_h\rangle}{\|v_h\|} = \sup_{v_h\in S_h}\frac{\langle \varphi, A_h^{-\frac12}Pv_h\rangle}{\|v_h\|} \\
&= \sup_{w_h\in \dot S_h}\frac{\langle \varphi, w_h\rangle}{|w_h|_1} \le \sup_{w_h\in \dot S_h}\frac{\|\varphi\|_{L_{6/5}}\|w_h\|_{L_6}}{|w_h|_1} \le C\|\varphi\|_{L_{6/5}}.
\end{aligned}$$

We use this with $\varphi = f(u) - f(v) = \int_0^1 f'(su+(1-s)v)\,\mathrm{d}s\,(u-v) = \int_0^1 f'(u_s)\,\mathrm{d}s\,(u-v)$, where $u_s = su + (1-s)v$,

$$\begin{aligned}
\|A_h^{-\frac12}&P(f(u)-f(v))\| = \|A_h^{-\frac12}P\varphi\| \le C\|\varphi\|_{L_{6/5}} \\
&\le C\int_0^1 \|f'(u_s)\|_{L_3}\,\mathrm{d}s\,\|u-v\| \le C\int_0^1 (1+\|u_s\|_{L_6}^2)\,\mathrm{d}s\,\|u-v\| \\
&\le C\int_0^1 (1+\|u_s\|_1^2)\,\mathrm{d}s\,\|u-v\| \le C(1+\|u\|_1^2+\|v\|_1^2)\|u-v\|.
\end{aligned}$$

This is (2.11). □

3. The Cahn-Hilliard-Cook equation

3.1. **The continuous problem.**

The Cahn-Hilliard-Cook equation is

$$\begin{aligned}
&\mathrm{d}u - \Delta w\,\mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D}\times[0,T], \\
&w + \Delta u + f(u) = 0 && \text{in } \mathcal{D}\times[0,T], \\
&\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 && \text{on } \partial\mathcal{D}\times[0,T], \\
&u(0) = u_0. && \text{in } \mathcal{D}.
\end{aligned} \tag{3.1}$$

The finite element approximation is based on its weak form, which is

$$
\begin{aligned}
&\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t \langle w(s), \Delta v\rangle \,\mathrm{d}s + \int_0^t \langle \mathrm{d}W(s), v\rangle, \quad t>0,\\
&\langle w, v\rangle = \langle \nabla u, \nabla v\rangle + \langle f(u), v\rangle, \qquad\qquad\qquad\qquad t>0,\\
&u(0) = u_0,
\end{aligned}
\tag{3.2}
$$

for all $v \in H^2$ with $\frac{\partial v}{\partial n} = 0$ on $\partial\mathcal{D}$. With the operator A, defined in Section 2, we write (3.1) in the formal abstract form on $H = L_2(\mathcal{D})$:

$$
\mathrm{d}X + (A^2X + Af(X))\,\mathrm{d}t = \mathrm{d}W, \quad t>0; \quad X(0) = X_0. \tag{3.3}
$$

A weak solution of (3.3) satisfies

$$
\langle X(t), v\rangle - \langle X_0, v\rangle + \int_0^t \langle X, A^2 v\rangle \,\mathrm{d}s + \int_0^t \langle f(X(s)), Av\rangle \,\mathrm{d}s = \int_0^t \langle \mathrm{d}W(s), v\rangle,
$$

for all $v \in \dot{H}^4 = D(A^2)$. A mild solution of (3.3) is a solution of

$$
X(t) = \mathrm{e}^{-tA^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s). \tag{3.4}
$$

3.2. **The finite element problem.** Recalling (3.2), we define the finite element solution $u_h(t) \in S_h$ of (3.1) by

$$
\begin{aligned}
&\langle u_h(t), v_h\rangle = \langle u_0, v_h\rangle + \int_0^t \langle \nabla w_h(s), \nabla v_h\rangle \,\mathrm{d}s + \int_0^t \langle \mathrm{d}W(s), v_h\rangle,\\
&\langle w_h, v_h\rangle = \langle \nabla u_h, \nabla v_h\rangle + \langle f(u_h), v_h\rangle,\\
&u_h(0) = u_{h,0},
\end{aligned}
$$

for all $v_h \in S_h,\ t>0$. This may also be written in the abstract form in S_h:

$$
\mathrm{d}X_h + (A_h^2 X_h + A_h P_h f(X_h))\,\mathrm{d}t = P_h \mathrm{d}W, \quad t>0; \quad X_h(0) = P_h X_0, \tag{3.5}
$$

with mild solution

$$
\begin{aligned}
X_h(t) = \mathrm{e}^{-tA_h^2} X_0 &- \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(X(s))\,\mathrm{d}s\\
&+ \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h\,\mathrm{d}W(s).
\end{aligned}
\tag{3.6}
$$

3.3. **A Lyapunov functional.** Define the functional

$$
J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x, \quad u \in H^1, \tag{3.7}
$$

where $F(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2$ is a primitive of $f(s) = s^3 - s$. This is a Lyapunov functional for the deterministic Cahn-Hilliard equation, which means that in the deterministic case $J(X(t))$ does not increase along solution paths.

For the stochastic equation this is not true, but we have a bound for the expected value of $J(X(t))$.

Theorem 3.1. *Assume that* $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ *and* $X,\ X_h$ *are weak solutions of* (3.3) *and* (3.5) *with* $\mathbf{E}[J(X_0)] < \infty$ *and that* X_0 *is* $\mathcal{F}_0$*-measurable with values in* H^1*. Then, for all* $t > 0$*, we have*

$$\mathbf{E}[J(X(t))] \le C\Big(\mathbf{E}[J(X_0)] + 1 + tK_Q + t^2K_Q^2\Big), \tag{3.8}$$

and

$$\mathbf{E}[J(X_h(t))] \le C\Big(\mathbf{E}[J(P_hX_0)] + 1 + tK_Q + t^2K_Q^2\Big), \tag{3.9}$$

where $K_Q = \|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 + \langle Q\varphi_0, \varphi_0\rangle$*.*

Proof. We prove (3.9), the proof of (3.8) is essentially obtained by removing the subscript "h" everywhere (see also [4]).

We consider (3.5) as an Itô differential equation in S_h driven by P_hW, which is a $Q_h = P_hQP_h$-Wiener process in S_h. By assumption (2.8) it follows that $\mathbf{E}[J(P_hX_0)] < \infty$, if $\mathbf{E}[J(X_0)] < \infty$.

By applying Itô's formula ([9, Theorem 4.17]) to $J(X_h(t))$, we obtain

$$\begin{aligned} J(X_h(t)) &= J(X_h(0)) + \int_0^t \langle J'(X_h(s)), \mathrm{d}X_h(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s)Q_h)\,\mathrm{d}s \\ &= J(P_hX_0) + \int_0^t \langle J'(X_h(s)), -A_h^2X_h(s) - P_hA_hf(X_h(s))\rangle\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s)Q_h)\,\mathrm{d}s + \int_0^t \langle J'(X_h(s)), \mathrm{d}W(s)\rangle. \end{aligned}$$

But we have

$$\langle J'(u_h), v_h\rangle = \langle \nabla u_h, \nabla v_h\rangle + \langle f(u_h), v_h\rangle = \langle A_hu_h + P_hf(u_h), v_h\rangle,$$

and

$$\begin{aligned} \langle J''(u_h)v_h, w_h\rangle &= \langle \nabla v_h, \nabla w_h\rangle + \langle f'(u_h)v_h, w_h\rangle \\ &= \langle A_hv_h + P_h[f'(u_h)v_h], w_h\rangle, \end{aligned}$$

so that

$$J'(u_h) = A_hu_h + P_hf(u_h), \quad J''(u_h) = A_h + P_h[f'(u_h)\cdot].$$

Hence, by (2.3),

$$\begin{aligned} \mathbf{E}[J(X_h(t))] &= \mathbf{E}[J(P_hX_0)] - \mathbf{E}\left[\int_0^t |A_hX_h(s) + P_hf(X_h(s))|_1^2\,\mathrm{d}s\right] \\ &\quad + \frac{1}{2}\mathbf{E}\left[\int_0^t \Big(\mathrm{Tr}(A_hQ_h) + \mathrm{Tr}(P_h[f'(X_h(s))\cdot]Q_h)\Big)\,\mathrm{d}s\right]. \end{aligned}$$

We ignore the negative term on the right hand side to get

$$\begin{aligned}\mathbf{E}[J(X_h(t))] &\le \mathbf{E}[J(P_hX_0)] \\ &\quad + \frac{1}{2}\mathbf{E}\left[\int_0^t \Big(\mathrm{Tr}(A_hQ_h) + \mathrm{Tr}(P_h[f'(X_h(s))\cdot]Q_h)\Big)\,\mathrm{d}s\right].\end{aligned} \tag{3.10}$$

Now we compute $\mathrm{Tr}(A_hQ_h)$ and $\mathrm{Tr}(P_h[f'(X_h(s))\cdot]Q_h)$. To this end let $\{\varphi_{h,j}\}_{j=0}^{N_h}$ be an orthonormal basis of eigenvectors of A_h and $\{\lambda_{h,j}\}_{j=0}^{N_h}$ the corresponding eigenvalues. Then

$$\begin{aligned}&\mathrm{Tr}(A_hQ_h) = \mathrm{Tr}(Q_hA_h) = \sum_{j=1}^{N_h}\langle P_hQP_hA_h\varphi_{h,j}, \varphi_{h,j}\rangle = \sum_{j=1}^{N_h}\lambda_{h,j}\langle Q\varphi_{h,j}, \varphi_{h,j}\rangle \\ &= \sum_{j=1}^{N_h}\langle Q^{\frac12}A_h^{\frac12}\varphi_{h,j}, Q^{\frac12}A_h^{\frac12}\varphi_{h,j}\rangle = \sum_{j=1}^{N_h}\|Q^{\frac12}A_h^{\frac12}P_h\varphi_{h,j}\|^2 = \|Q^{\frac12}A_h^{\frac12}P_h\|_{\mathrm{HS}}^2 \\ &\le \|A_h^{\frac12}P_hQ^{\frac12}\|_{\mathrm{HS}}^2 = \|A_h^{\frac12}P_hA^{-\frac12}A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 \le \|A_h^{\frac12}P_hA^{-\frac12}\|_{B(\dot H)}^2\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 \\ &\le C\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2.\end{aligned}$$

Here we used (2.3) and (2.8) to get

$$\|A_h^{\frac12}P_hA^{-\frac12}v\| = |P_hA^{-\frac12}v|_1 \le C|A^{-\frac12}v|_1 = C\|v\|, \quad v \in \dot H,$$

so that $\|A_h^{\frac12}P_hA_h^{-\frac12}\|_{B(\dot H)} \le C$. Hence, with $K_Q = \|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 + \langle Q\varphi_0, \varphi_0\rangle$,

$$\|A_h^{\frac12}Q_h^{\frac12}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(A_hQ_h) \le C\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 \le CK_Q. \tag{3.11}$$

Let $\{e_{h,j}\}_{j=0}^{N_h}$ be an orthonormal eigenbasis of Q_h and $\{\gamma_{h,j}\}_{j=0}^{N_h}$ the corresponding eigenvalues. We get

$$\begin{aligned}\mathrm{Tr}\left(P_h[f'(X_h)\cdot]Q_h\right) &= \sum_{j=0}^{N_h}\langle P_h[f'(X_h)Q_he_{h,j}], e_{h,j}\rangle \\ &= \sum_{j=0}^{N_h}\gamma_{h,j}\langle f'(X_h)e_{h,j}, e_{h,j}\rangle \\ &= \sum_{j=0}^{N_h}\langle f'(X_h)Q_h^{\frac12}e_{h,j}, Q_h^{\frac12}e_{h,j}\rangle.\end{aligned} \tag{3.12}$$

By using the bound $|f'(s)| \le C(1+s^2)$, we get by Hölder's and Sobolev's inequalities,

$$|\langle f'(u)v, v\rangle| \le C(1+\|u\|_{L_4}^2)\|v\|_{L_4}^2 \le C(1+\|u\|_{L_4}^2)\|v\|_{H^1}^2 \le C(1+\|u\|_{L_4}^2)\|v\|_1^2.$$

By (2.1) and (2.3) we have, for $v_h \in S_h$,

$$\|v_h\|_1^2 = |v_h|_1^2 + \langle v_h, \varphi_0\rangle^2 = \|A_h^{\frac{1}{2}} v_h\|^2 + \langle v_h, \varphi_0\rangle^2,$$

so that, by (3.11),

$$\begin{aligned}\sum_{j=0}^{N_h} \|Q_h^{\frac{1}{2}} e_{h,j}\|_1^2 &= \sum_{j=0}^{N_h} \|A_h^{\frac{1}{2}} Q_h^{\frac{1}{2}} e_{h,j}\|^2 + \sum_{j=0}^{N_h} \langle Q_h^{\frac{1}{2}} e_{h,j}, \varphi_0\rangle^2 \\ &\le \|A_h^{\frac{1}{2}} Q_h^{\frac{1}{2}}\|_{\mathrm{HS}}^2 + \|Q_h^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \le \|A_h^{\frac{1}{2}} Q_h^{\frac{1}{2}}\|_{\mathrm{HS}}^2 + \|Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \\ &\le C\|A^{\frac{1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 + \langle Q\varphi_0, \varphi_0\rangle \le CK_Q.\end{aligned}$$

Here we used the boundedness of $A^{-\frac{1}{2}}$ to get

$$\begin{aligned}\|Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 &= \sum_{j=0}^{\infty} \|Q^{\frac{1}{2}} \varphi_j\|^2 = \sum_{j=1}^{\infty} \|A^{-\frac{1}{2}} A^{\frac{1}{2}} Q^{\frac{1}{2}} \varphi_j\|^2 + \|Q^{\frac{1}{2}} \varphi_0\|^2 \\ &\le C \sum_{j=1}^{\infty} \|A^{\frac{1}{2}} Q^{\frac{1}{2}} \varphi_j\|^2 + \langle Q\varphi_0, \varphi_0\rangle \\ &= C\|A^{\frac{1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} + \langle Q\varphi_0, \varphi_0\rangle \le CK_Q.\end{aligned} \tag{3.13}$$

Returning to (3.12), we now have

$$\mathrm{Tr}(P_h[f'(X_h)\cdot]Q_h) \le C(1 + \|X_h\|_{L_4}^2) \sum_{j=0}^{N_h} \|Q_h^{\frac{1}{2}} e_{h,j}\|_1^2 \le C(1 + \|X_h\|_{L_4}^2) K_Q, \tag{3.14}$$

Putting (3.11) and (3.14) in (3.10) gives

$$\mathbf{E}[J(X_h(t))] \le \mathbf{E}[J(P_h X_0)] + CK_Q\Big(t + \int_0^t \mathbf{E}[\|X_h(s)\|_{L_4}^2]\,\mathrm{d}s\Big). \tag{3.15}$$

It remains to bound $\int_0^t \mathbf{E}[\|X_h\|_{L_4}^2]\,\mathrm{d}s$. By definition of the Lyapunov functional (3.7) and noting that $F(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2 \ge c_1 s^4 - c_2$, we get

$$J(u) \ge \frac{1}{2}\|\nabla u\|^2 + C_1\|u\|_{L_4}^4 - C_2,$$

which implies

$$\|u\|_{L_4}^4 \le C_3(1 + J(u)).$$

Hence, by Hölder's inequality, we get, for $\epsilon > 0$,

$$\begin{aligned} CK_Q \int_0^t \mathbf{E}[\|X_h(s)\|_{L_4}^2] \,\mathrm{d}s &\le CK_Q \Big(\int_0^t \mathbf{E}[\|X_h(s)\|_{L_4}^2] \,\mathrm{d}s \Big)^{\frac12} t^{\frac12} \\ &\le \frac{\epsilon}{C_3} \int_0^t \mathbf{E}[\|X_h(s)\|_{L_4}^2] \,\mathrm{d}s + \frac{C_3}{4\epsilon} tCK_Q^2 \\ &\le \epsilon \int_0^t \mathbf{E}[1 + J(X_h(s)] \,\mathrm{d}s + C\epsilon^{-1} tK_Q^2 \\ &\le \epsilon \int_0^t \mathbf{E}[J(X_h(s))] \,\mathrm{d}s + \epsilon t + C\epsilon^{-1} tK_Q^2. \end{aligned}$$

Putting this in (3.15) gives

$$\mathbf{E}[J(X_h(t))] \le \mathbf{E}[J(P_h X_0)] + C\Big(\epsilon + K_Q + \epsilon^{-1} K_Q^2\Big) t + \epsilon \int_0^t \mathbf{E}[J(X_h(s))] \,\mathrm{d}s.$$

Now apply the Gronwall Lemma 2.4 to get, for $\epsilon > 0$,

$$\begin{aligned} \mathbf{E}[J(X_h(t))] &\le \mathrm{e}^{\epsilon t} \Big(\mathbf{E}[J(P_h X_0)] + C(1 + \epsilon^{-1} K_Q + \epsilon^{-2} K_Q^2) \Big) \\ &\le \mathrm{e} \Big(\mathbf{E}[J(P_h X_0)] + C(1 + tK_Q + t^2 K_Q^2) \Big), \end{aligned}$$

where for each fixed t we have chosen $\epsilon = t^{-1}$ to get an optimal bound. □

This theorem is adapted from [4]. We have improved it in several ways. Most importantly, the growth of the bound is reduced from exponential to quadratic with respect to t. Moreover, we have removed the assumption that A and Q have a common eigenbasis and that the eigenbasis of Q satisfies $\|e_j\|_{L_\infty} \le C$. It is also important that we obtain the same bound for X_h.

Note that the assumption $\|A^{\frac12} Q^{\frac12}\|_{\mathrm{HS}} < \infty$ is the same as the condition for regularity of order $\beta = 3$ for $W_A(t)$ in Theorem 2.1.

We now use the previous theorem to obtain norm bounds uniformly on subsets of Ω with probability arbitrarily close to 1.

Corollary 3.2. *Assume that* $\|A^{\frac12} Q^{\frac12}\|_{\mathrm{HS}} < \infty$ *and* X, X_h *are weak solutions of* (3.3) *and* (3.5) *with* X_0 $\mathcal{F}_0$*-measurable with values in* H^1 *and* $\|X_0\|_{L_2(\Omega,H^1)}^2 + \|X_0\|_{L_4(\Omega,L_4)}^4 \le \rho$. *Then, for every* $\epsilon \in (0,1)$, *there is* $\Omega_\epsilon \subset \Omega$ *with* $\mathbf{P}(\Omega_\epsilon) \ge 1 - \epsilon$ *and*

$$\|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 \le \epsilon^{-1} K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{3.16}$$

$$\|\nabla X_h(t)\|^2 + \|X_h(t)\|_{L_4}^4 \le \epsilon^{-1} K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{3.17}$$

$$\|X(t)\|_1^2 + \|X_h(t)\|_1^2 \le \epsilon^{-1} K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{3.18}$$

$$\|W_A(t)\|_3^2 \le \epsilon^{-1} K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{3.19}$$

where $K_T = C(1+\rho+K_QT+K_Q^2T^2)$.

Proof. Since $\mathbf{E}[J(X_0)] \le C(1+\rho)$, we obtain from Theorem 3.1,

$$\mathbf{E}[J(X(t))] \le C(1+\rho+K_QT+K_Q^2T^2) \le K_T \quad t \in [0,T].$$

We apply Chebyshev's inequality to get, for every $\alpha > 0$ and $t \in [0,T]$,

$$\begin{aligned}\mathbf{P}\Big(\{\omega \in \Omega : \|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 > \alpha\Big) &\le \frac{1}{\alpha}\mathbf{E}\left[\|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4\right] \\ \le \frac{1}{\alpha}C(1+\mathbf{E}\left[J(X(t))\right] &\le \frac{1}{\alpha}C(1+K_T) = \frac{K_T}{\alpha},\end{aligned}$$

where the C in K_T was adjusted. We choose $\alpha = \epsilon^{-1}K_T$ and set

$$\Omega_\epsilon = \{\omega \in \Omega : \|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 \le \epsilon^{-1}K_T\}.$$

So (3.16) holds and

$$\mathbf{P}(\Omega_\epsilon) = 1 - \mathbf{P}\Big(\{\omega \in \Omega : \|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 > \alpha\Big) \ge 1-\epsilon.$$

For (3.17) we replace X_h by X and note that we have $\mathbf{E}[J(P_hX_0)] \le C(1+\rho)$, by (2.8). For (3.18) we note that $\epsilon^{-1}K_T \ge 1$, and so

$$\|X(t)\|_1^2 \le \|\nabla X(t)\|^2 + \|X(t)\|^2 \le \|\nabla X(t)\|^2 + C\|X(t)\|_{L_4}^2 \le \epsilon^{-1}K_T$$

after an adjustment of the C in K_T. Finally, (3.19) follows in a similar way from Theorem 2.1 with $\beta = 3$ with a constant which can be absorbed in K_T. □

4. Regularity of the solution

We quote the following from [4].

Theorem 4.1. *Let* $T > 0$ *and assume that* $\mathrm{Tr}(A^{\delta-1}Q) < \infty$ *for some* $\delta > 0$ *and that* X_0 *is* $\mathcal{F}_0$*-measurable with values in* H. *Then there is a process* X*, which is in* $C([0,T],H)$ *a.s. and which is a mild solution of* (1.1).

We now show that, under the assumption $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$, the solution is actually in H^3. In order to do this we write $X(t) = Y(t) + W_A(t)$, where we already know that W_A is in H^3 from Theorem 2.1. The regularity of Y is studied in the next theorem. Since

$$Y(t) = X(t) - W_A(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2}Af(X(s))\,\mathrm{d}s,$$

it is a mild solution of

$$\dot{Y} + A^2Y + Af(X) = 0, \quad t > 0; \quad Y(0) = X_0. \tag{4.1}$$

Theorem 4.2. *Assume that* $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ *and that* X_0 *is* $\mathcal{F}_0$*-measurable with values in* H^3 *and* $\|X_0\|^2_{L_2(\Omega,H^1)} + \|X_0\|^4_{L_4(\Omega,L_4)} < \infty$*. Let* $T > 0$ *and* $\epsilon \in (0,1)$ *and let* Ω_ϵ *and* K_T *be as in Corollary 3.2. Let* X *be the solution from Theorem 4.1. Then, for each* $\omega \in \Omega_\epsilon$ *the mild solution* Y *of* (4.1) *belongs to* $C([0,T], H^3)$*. Moreover,*

$$\|Y(t)\|_3 \le C(\|X_0\|_3, \epsilon^{-1}K_T, T) \quad \text{on } \Omega_\epsilon,\ t \in [0,T],$$
$$\|X(t)\|_3 \le C(\|X_0\|_3, \epsilon^{-1}K_T, T) \quad \text{on } \Omega_\epsilon,\ t \in [0,T].$$

Proof. Let $T > 0$ and $\omega \in \Omega_\epsilon$. From Corollary 3.2 we have

$$\|X(t)\|_1^2 \le \epsilon^{-1}K_T, \quad \|W_A(t)\|_3 \le \epsilon^{-1}K_T. \tag{4.2}$$

We take norms in

$$Y(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2}Af(X(s))\,\mathrm{d}s, \tag{4.3}$$

to get

$$\begin{aligned}
|Y(t)|_3 &\le |\mathrm{e}^{-tA^2}X_0|_3 + \int_0^t |\mathrm{e}^{-(t-s)A^2}Af(X(s))|_3\,\mathrm{d}s \\
&= \|\mathrm{e}^{-tA^2}A^{\frac{3}{2}}X_0\| + \int_0^t \|A^{\frac{3}{2}}\mathrm{e}^{-(t-s)A^2}Af(X(s))\|\,\mathrm{d}s \\
&\le |X_0|_3 + C\int_0^t (t-s)^{-\frac{3}{4}}\|Af(X(s))\|\,\mathrm{d}s.
\end{aligned}$$

We apply (2.10) to $\|Af(X(s))\| = \|\Delta f(X(s))\|$ to get

$$\begin{aligned}
|Y(t)|_3 &\le |X_0|_3 + C\int_0^t (t-s)^{-\frac{3}{4}}(1 + \|X(s)\|_1^2)\|X(s)\|_3\,\mathrm{d}s \\
&\le |X_0|_3 + C\int_0^t (t-s)^{-\frac{3}{4}}(1 + \|X(s)\|_1^2)(\|Y(s)\|_3 + \|W_A(s)\|_3)\,\mathrm{d}s.
\end{aligned}$$

Since $(I-P)Y(t) = (I-P)X_0$ is constant, we get the same bound for the norm $\|Y(t)\|_3$. Using also (4.2) gives

$$\begin{aligned}
\|Y(t)\|_3 &\le \|X_0\|_3 + C\int_0^t (t-s)^{-\frac{3}{4}}(1 + \epsilon^{-1}K_T)(\|Y(s)\|_3 + \epsilon^{-1}K_T)\,\mathrm{d}s \\
&\le \|X_0\|_3 + C\epsilon^{-1}K_T(1 + \epsilon^{-1}K_T)T^{\frac{1}{4}} \\
&\quad + C(1 + \epsilon^{-1}K_T)\int_0^t (t-s)^{-\frac{3}{4}}\|Y(s)\|_3\,\mathrm{d}s.
\end{aligned}$$

Applying Gronwall's Lemma 2.3 with $\alpha = 1$, $\beta = \frac{1}{4}$ and

$$A = \|X_0\|_3 + C\epsilon^{-1}K_T\left(1 + \epsilon^{-1}K_T\right),\ B = C\left(1 + \epsilon^{-1}K_T\right), \tag{4.4}$$

gives

$$\|Y(t)\|_3 \le AC(B,T) = C(\|X_0\|_3, \epsilon^{-1}K_T, T), \quad t \in [0,T].$$

The bound for $\|X(t)\|_3$ then follows in view of (4.2). □

The constant $C(\|X_0\|_3, \epsilon^{-1}K_T, T)$ grows rapidly with $\epsilon^{-1}K_T$ and T. Hence, it is important that K_T grows only quadratically with T.

5. Error estimates

5.1. **Error estimate for deterministic Cahn-Hilliard equation.** Consider the linear Cahn-Hilliard equation

$$\begin{aligned} &\dot u + Av = 0, && t > 0,\\ &v - Au - f = 0, && t > 0\\ &u(0) = u_0, \end{aligned} \tag{5.1}$$

where f is a function of x, t, and the corresponding finite element problem

$$\begin{aligned} &\dot u_h + A_h v_h = 0, && t > 0,\\ &v_h - A_h u_h - P_h f = 0, && t > 0,\\ &u_h(0) = P_h u_0. \end{aligned} \tag{5.2}$$

We have the following error estimate. We will later use this for fixed $\omega \in \Omega_\epsilon$ with f replaced by $f(X)$ and u by the solution Y of (1.3).

Theorem 5.1. *Assume that u, v and u_h, v_h are the solutions of* (5.1) *and* (5.2)*, respectively. Then, for $t \ge 0$,*

$$\|u_h(t) - u(t)\| \le Ch^2 \bigg(|\log(h)| \max_{0\le s\le t} |u(s)|_2 + \Big(\int_0^t |v(s)|_2^2 \,\mathrm{d}s \Big)^{\frac12} \bigg). \tag{5.3}$$

Proof. The weak forms of (5.1) and (5.2) are

$$\begin{aligned} &\langle \dot u, \varphi_1\rangle + \langle \nabla v, \nabla\varphi_1\rangle = 0 && \forall \varphi_1 \in H^1,\\ &\langle v, \varphi_2\rangle - \langle \nabla u, \nabla\varphi_2\rangle - \langle f, \varphi_2\rangle = 0 && \forall \varphi_2 \in H^1,\\ &u(0) = u_0, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\langle \dot u_h, \varphi_{h,1}\rangle + \langle \nabla v_h, \nabla\varphi_{h,1}\rangle = 0 && \forall \varphi_{h,1} \in S_h,\\ &\langle v_h, \varphi_{h,2}\rangle - \langle \nabla u_h, \nabla\varphi_{h,2}\rangle - \langle f, \varphi_{h,2}\rangle = 0 && \forall \varphi_{h,2} \in S_h,\\ &u_h(0) = P_h u_0. \end{aligned} \tag{5.5}$$

Let P_h and R_h be as in (2.4) and (2.6) and set

$$e_u = u_h - u = (u_h - P_h u) + (P_h u - u) = \theta_u + \rho_u, \tag{5.6}$$

$$e_v = v_h - v = (v_h - R_h v) + (R_h v - v) = \theta_v + \rho_v. \tag{5.7}$$

We want to compute

$$\|e_u\| \le \|\theta_u\| + \|\rho_u\|. \tag{5.8}$$

In (5.4) choose $\varphi_1 = \varphi_{h,1}$ and $\varphi_2 = \varphi_{h,2}$ and subtract the first two equations of (5.4) from the corresponding equations in (5.5) to get

$$\begin{aligned}
\langle \dot e_u, \varphi_{h,1}\rangle + \langle \nabla e_v, \nabla \varphi_{h,1}\rangle &= 0 \quad \forall \varphi_{h,1} \in S_h,\\
\langle e_v, \varphi_{h,2}\rangle - \langle \nabla e_u, \nabla \varphi_{h,2}\rangle &= 0 \quad \forall \varphi_{h,2} \in S_h.
\end{aligned}$$

Hence, by (5.6) and (5.7),

$$\begin{aligned}
\langle \dot\theta_u, \varphi_{h,1}\rangle + \langle \nabla \theta_v, \nabla \varphi_{h,1}\rangle &= -\langle \dot\rho_u, \varphi_{h,1}\rangle - \langle \nabla \rho_v, \nabla \varphi_{h,1}\rangle \quad \forall \varphi_{h,1} \in S_h,\\
\langle \theta_v, \varphi_{h,2}\rangle - \langle \nabla \theta_u, \nabla \varphi_{h,2}\rangle &= -\langle \rho_v, \varphi_{h,2}\rangle + \langle \nabla \rho_u, \nabla \varphi_{h,2}\rangle \quad \forall \varphi_{h,2} \in S_h.
\end{aligned}$$

By the definitions of P_h and R_h we have

$$\begin{aligned}
\langle \dot\rho_u, \varphi_{h,1}\rangle &= \langle P_h \dot u - \dot u, \varphi_{h,1}\rangle = 0 \qquad \forall \varphi_{h,1} \in S_h,\\
\langle \nabla \rho_v, \nabla \varphi_{h,1}\rangle &= \langle \nabla R_h v - v, \nabla \varphi_{h,1}\rangle = 0 \quad \forall \varphi_{h,2} \in S_h,
\end{aligned}$$

so that

$$\begin{aligned}
\langle \dot\theta_u, \varphi_{h,1}\rangle + \langle \nabla \theta_v, \nabla \varphi_{h,1}\rangle &= 0 \qquad \forall \varphi_{h,1} \in S_h\\
\langle \theta_v, \varphi_{h,2}\rangle - \langle \nabla \theta_u, \nabla \varphi_{h,2}\rangle &= -\langle \rho_v, \varphi_{h,2}\rangle + \langle \nabla \rho_u, \nabla \varphi_{h,2}\rangle \quad \forall \varphi_{h,2} \in S_h.
\end{aligned}$$

In the second equation we set $\varphi_{h,2} = A_h \varphi_{h,1}$ to get

$$\langle \nabla \theta_v, \nabla \varphi_{h,1}\rangle = \langle A_h^2 \theta_u, \varphi_{h,1}\rangle - \langle A_h P_h \rho_v, \varphi_{h,1}\rangle + \langle A_h^2 R_h \rho_u, \varphi_{h,1}\rangle.$$

Inserting this into the first equation gives

$$\langle \dot\theta_u, \varphi_{h,1}\rangle + \langle A_h^2 \theta_u, \varphi_{h,1}\rangle = \langle A_h P_h \rho_v, \varphi_{h,1}\rangle - \langle A_h^2 R_h \rho_u, \varphi_{h,1}\rangle,$$

so the strong form is

$$\dot\theta_u + A_h^2 \theta_u = A_h P_h \rho_v - A_h^2 R_h \rho_u, \quad t > 0; \quad \theta_u(0) = 0,$$

with the mild solution

$$\theta_u(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h \rho_v(s)\,\mathrm{d}s - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h^2 R_h \rho_u(s)\,\mathrm{d}s.$$

Taking norms here gives

$$\begin{aligned}
\|\theta_u(t)\| \le &\Big\| \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h \rho_v(s)\,\mathrm{d}s \Big\| \\
&+ \Big\| \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h^2 R_h \rho_u(s)\,\mathrm{d}s \Big\| = I + II.
\end{aligned} \tag{5.9}$$

For I we define

$$w_h(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \rho_v(s)\,\mathrm{d}s,$$

which satisfies the equation

$$\dot{w}_h + A_h^2 w_h = P_h \rho_v, \quad t > 0; \quad w_h(0) = 0.$$

Multiply by $\dot{w}_h$ to get

$$\|\dot{w}_h\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_h w_h\|^2 = \langle P_h \rho_v, \dot{w}_h \rangle \le \|\rho_v\| \|\dot{w}_h\| \le \frac{1}{2}\|\rho_v\|^2 + \frac{1}{2}\|\dot{w}_h\|^2.$$

So we get

$$\|\dot{w}_h\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|A_h w_h\|^2 \le \|\rho_v\|^2.$$

Integrate and ignore $\int_0^t \|\dot{w}_h(s)\|^2 \,\mathrm{d}s$ to get

$$\Big\| A_h \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \rho_v(s) \,\mathrm{d}s \Big\| = \|A_h w_h(t)\| \le \Big(\int_0^t \|\rho_v(s)\|^2 \,\mathrm{d}s \Big)^{\frac{1}{2}},$$

where, from (2.7),

$$\|\rho_v\| = \|(R_h - I)v\| \le Ch^2 |v|_2.$$

So we get

$$\Big\| A_h \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \rho_v(s) \,\mathrm{d}s \Big\| \le Ch^2 \Big(\int_0^t |v(s)|_2^2 \,\mathrm{d}s \Big)^{\frac{1}{2}}. \tag{5.10}$$

For II we use

$$R_h \rho_u = R_h(P_h u - u) = P_h u - R_h u = P_h(u - R_h u).$$

Then

$$\begin{aligned} \Big\| \int_0^t A_h^2 \mathrm{e}^{-(t-s)A_h^2} R_h \rho_u(s) \,\mathrm{d}s \Big\| &\le \int_0^t \|A_h^2 \mathrm{e}^{-(t-s)A_h^2} P_h(u(s) - R_h u(s))\| \,\mathrm{d}s \\ &\le \int_0^t \|A_h^2 \mathrm{e}^{-(t-s)A_h^2} P_h\| \,\mathrm{d}s \max_{0 \le s \le t} \|u(s) - R_h u(s)\| \,\mathrm{d}s. \end{aligned}$$

Here we use $\|A_h\| \le Ch^{-2}$ from (2.8) and (2.5) to get

$$\begin{aligned} \int_0^t \|A_h^2 \mathrm{e}^{-(t-s)A_h^2} P_h\| \,\mathrm{d}s &= \int_0^{h^4} \|A_h\|^2 \|\mathrm{e}^{-sA_h^2}\| \,\mathrm{d}s + \int_{h^4}^t \|A_h^2 \mathrm{e}^{-sA_h^2}\| \,\mathrm{d}s \\ &\le Ch^{-4}h^4 + C\int_{h^4}^t s^{-1}\mathrm{e}^{-cs} \,\mathrm{d}s \le C(1 + \log(1/h)) \le C|\log(h)|. \end{aligned}$$

Hence, by (2.7), we have

$$\Big\| \int_0^t A_h^2 \mathrm{e}^{-(t-s)A_h^2} R_h \rho_u(s) \,\mathrm{d}s \Big\| \le Ch^2 |\log(h)| \max_{0 \le s \le t} |u(s)|_2. \tag{5.11}$$

Putting (5.10) and (5.11) in (5.9) gives

$$\|\theta_u(t)\| \le Ch^2\Big\{\Big(\int_0^t |v(s)|_2^2\,\mathrm{d}s\Big)^{\frac{1}{2}} + |\log(h)| \max_{0\le s\le t} |u(s)|_2\Big\}. \tag{5.12}$$

Finally, by the best approximation property of P_h,

$$\|\rho_u(t)\| = \|P_h u - u\| \le \|R_h u - u\| \le Ch^2|u(t)|_2. \tag{5.13}$$

Putting (5.12) and (5.13) in (5.8) gives the desired result (5.3). □

In the next lemma we prove a stabilty estimate for the deterministic Cahn-Hilliard equation (5.1).

Lemma 5.2. *Assume that u, v are the solutions of* (5.1)*. Then*

$$|u(t)|_2^2 + \int_0^t |v(s)|_2^2\,\mathrm{d}s \le |u_0|_2^2 + \int_0^t |f(s)|_2^2\,\mathrm{d}s.$$

Proof. Multiply the first equation in (5.1) by A^2u to get

$$\frac{1}{2}|u|_2^2 + \langle A^2 v, Au\rangle = 0.$$

The second equation of (5.1) gives $Au = v - f$, so we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + \langle A^2 v, v\rangle = \langle A^2 v, f\rangle \le |v|_2|f|_2 \le \frac{1}{2}|v|_2^2 + \frac{1}{2}|f|_2^2,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + |v|_2^2 \le |f|_2^2.$$

The proof is finished by integration. □

5.2. **Error estimate for the stochastic Cahn-Hilliard equation.** In the next theorem we prove an error estimate for the nonlinear Cahn-Hilliard-Cook equation.

Theorem 5.3. *Assume that $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ and X, X_h are the solutions of* (3.3) *and* (3.5) *with X_0 $\mathcal{F}_0$-measurable with values in H^3 and $\|X_0\|^2_{L_2(\Omega,H^1)} + \|X_0\|^4_{L_4(\Omega,L_4)} < \infty$. Let $T > 0$, $\epsilon \in (0,1)$, and let $\Omega_\epsilon \subset \Omega$ and K_T be as in Corollary 3.2. Then we have*

$$\|X_h(t) - X(t)\| \le C(\|X_0\|_3, \epsilon^{-1}K_T, T)h^2|\log(h)|, \quad \text{on } \Omega_\epsilon,\ t \in [0,T].$$

The constant $C(\|X_0\|_3, \epsilon^{-1}K_T, T)$ grows rapidly with $\epsilon^{-1}K_T$ and T due to the use of Gronwall's lemma in the proof.

Proof. Let $\omega \in \Omega_\epsilon$ be fixed. Set

$$X(t) = Y(t) + W_A(t), \tag{5.14}$$

where $W_A(t)$ is the stochastic convolution

$$W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s), \tag{5.15}$$

and $Y(t)$ is the mild solution (4.3) of (1.3). Also set

$$X_h(t) = Z_h(t) + W_{A_h}(t), \tag{5.16}$$

where $W_{A_h}(t)$ is the stochastic convolution

$$W_{A_h}(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h\,\mathrm{d}W(s), \tag{5.17}$$

and

$$Z_h(t) = \mathrm{e}^{-tA_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(X_h(s))\,\mathrm{d}s, \tag{5.18}$$

is the mild solution of

$$\dot{Z}_h + A_h^2 Z_h = -A_h P_h f(X_h), \quad t > 0; \quad Z_h(0) = P_h X_0. \tag{5.19}$$

Finally, let

$$Y_h(t) = \mathrm{e}^{-tA_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(X(s))\,\mathrm{d}s, \tag{5.20}$$

be the mild solution of

$$\dot{Y}_h + A_h^2 Y_h = -A_h P_h f(X), \quad t > 0; \quad Y_h(0) = P_h X_0. \tag{5.21}$$

We subtract (5.14) from (5.16),

$$\begin{aligned} X_h - X &= (Z_h + W_{A_h}) - (Y + W_A) \\ &= (W_{A_h} - W_A) + (Y_h - Y) + (Z_h - Y_h), \end{aligned}$$

and take norms,

$$\|X_h - X\| \le \|W_{A_h} - W_A\| + \|Y_h - Y\| + \|Z_h - Y_h\|. \tag{5.22}$$

We compute the three norms on the right hand side.

First we compute $\|W_{A_h}(t) - W_A(t)\|$. Since $\|A^{\frac{1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$, we have that $\|Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ and hence, by Theorem 2.2 and Chebyshev's inequality, we get

$$\begin{aligned} \|W_{A_h}(t) - W_A(t)\| &\le \epsilon^{-\frac{1}{2}} (\mathbf{E}[\|W_{A_h}(t) - W_A(t)\|^2])^{\frac{1}{2}} \\ &\le \epsilon^{-\frac{1}{2}} C h^2 |\log(h)| \|Q^{\frac{1}{2}}\|_{\mathrm{HS}} \le C(\epsilon^{-1} K_Q)^{\frac{1}{2}} h^2 |\log(h)|, \end{aligned}$$

see (3.13). Since $K_Q \le K_T$, we conclude

$$\|W_{A_h}(t) - W_A(t)\| \le C(\epsilon^{-1}K_T)^{\frac{1}{2}} h^2 |\log(h)|. \tag{5.23}$$

Now we consider $\|Y_h(t) - Y(t)\|$ and use Theorem (5.1) to get

$$\|Y_h(t) - Y(t)\| \le Ch^2 \Big\{ |\log(h)| \max_{0\le s\le t} |Y(s)|_2 + \Big(\int_0^t |V(s)|_2^2 \,\mathrm{d}s \Big)^{\frac{1}{2}} \Big\}, \tag{5.24}$$

where $Y(t)$ and $V(t)$ are the solutions of

$$\begin{aligned} &\dot{Y} + AV = 0, && t > 0, \\ &V = AY + f(X), && t > 0, \\ &Y(0) = X_0. \end{aligned} \tag{5.25}$$

By using Lemma 5.2, (2.10), and (3.19), we get

$$\begin{aligned} \int_0^t |V(s)|_2^2 \,\mathrm{d}s &\le |X_0|_2^2 + \int_0^t |f(X(s))|_2^2 \,\mathrm{d}s \\ &\le \|X_0\|_2^2 + C \int_0^t (1 + \|X(s)\|_1^2) \|X(s)\|_3 \,\mathrm{d}s \\ &\le \|X_0\|_3^2 + C \int_0^t (1 + \|X(s)\|_3^3) \,\mathrm{d}s \\ &\le \|X_0\|_3^2 + CT \Big(1 + (\epsilon^{-1}K_T)^{\frac{3}{2}} \Big). \end{aligned}$$

So

$$\int_0^t |V(s)|_2^2 \,\mathrm{d}s \le C(\|X_0\|_3, \epsilon^{-1}K_T, T). \tag{5.26}$$

Now we bound $|Y(t)|_2$. By Theorem 4.2 we have

$$|Y(t)|_2 \le \|Y(t)\|_3 \le C(\|X_0\|_3, \epsilon^{-1}K_T, T). \tag{5.27}$$

Using (5.26) and (5.27) in (5.24) gives

$$\|Y_h(t) - Y(t)\| \le C(\|X_0\|_3, \epsilon^{-1}K_T, T) h^2 |\log(h)|. \tag{5.28}$$

Finally we compute $\|e_h(t)\| = \|Z_h(t) - Y_h(t)\|$. By subtraction of (5.18) and (5.20), we obtain

$$\begin{aligned} \|e_h(t)\| &\le \int_0^t \|\mathrm{e}^{-(t-s)A_h^2} A_h P_h P(f(X_h(s)) - f(X(s))\| \,\mathrm{d}s \\ &= \int_0^t \|A_h^{\frac{3}{2}} \mathrm{e}^{-(t-s)A_h^2} A_h^{-\frac{1}{2}} P_h P(f(X_h(s)) - f(X(s))\| \,\mathrm{d}s \\ &\le \int_0^t \|A_h^{\frac{3}{2}} \mathrm{e}^{-(t-s)A_h^2} P_h\| \|A_h^{-\frac{1}{2}} P(f(X_h(s)) - f(X(s))\| \,\mathrm{d}s, \end{aligned}$$

since the constant eigenmodes cancel (cf. (2.9)). Using (2.11) and (2.5) gives

$$\|e_h(t)\| \le C \int_0^t (t-s)^{-\frac{3}{4}} (1 + \|X_h(s)\|_1^2 + \|X(s)\|_1^2) \|X_h(s) - X(s)\| \, \mathrm{d}s.$$

By Corollary (3.2) we have

$$\begin{aligned}
\|e_h(t)\| &\le C \int_0^t (t-s)^{-\frac{3}{4}} \Big(1 + \epsilon^{-1} K_T\Big) \Big(\|W_{A_h}(s) - W_A(s)\| \\
&\qquad + \|Y_h(s) - Y(s)\| + \|e_h(s)\| \Big) \, \mathrm{d}s \\
&\le C\Big(1 + \epsilon^{-1} K_T\Big) T^{\frac{1}{4}} \max_{0 \le s \le T} \Big(\|W_{A_h}(s) - W_A(s)\| + \|Y_h(s) - Y(s)\| \Big) \\
&\qquad + C\Big(1 + \epsilon^{-1} K_T\Big) \int_0^t (t-s)^{-\frac{3}{4}} \|e_h(s)\| \, \mathrm{d}s.
\end{aligned}$$

We apply Gronwall's Lemma 2.3 with $\alpha = 1$, $\beta = \frac{1}{4}$ and

$$\begin{aligned}
&A = C(1 + \epsilon^{-1} K_T) T^{\frac{1}{4}} \max_{0 \le s \le T} \Big(\|W_{A_h}(s) - W_A(s)\| + \|Y_h(s) - Y(s)\| \Big), \\
&B = C(1 + \epsilon^{-1} K_T),
\end{aligned}$$

to get

$$\|Z_h(t) - Y_h(t)\| = \|e_h(t)\| \le AC(B,T), \quad t \in [0,T]. \tag{5.29}$$

But we bounded $\|W_{A_h}(t) - W_A(t)\|$ and $\|Y_h(t) - Y(t)\|$ in (5.23) (5.28). By putting these values and (5.29) in (5.22) we get the desired result. □

We finally show that X_h converges strongly to X. More precisely, we show that $X_h(t) \to X(t)$ in $L_2(\Omega, H)$ uniformly on $[0,T]$ as $h \to 0$.

Theorem 5.4. *Assume that* $\|A^{\frac{1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ *and* X, X_h *are the solutions of* (3.3) *and* (3.5) *with* X_0 $\mathcal{F}_0$*-measurable with values in* H^3 *and* $\|X_0\|^2_{L_2(\Omega, H^1)} + \|X_0\|^4_{L_4(\Omega, L_4)} < \infty$. *Then*

$$\max_{t \in [0,T]} \big(\mathbf{E}[\|X_h(t) - X(t)\|^2]\big)^{\frac{1}{2}} \to 0 \quad \text{as } h \to 0.$$

Proof. From Theorem 3.1 it follows that

$$\mathbf{E}\left[\|X(t)\|_{L_4}^4\right] \le K_T, \quad \mathbf{E}\left[\|X_h(t)\|_{L_4}^4\right] \le K_T, \quad t \in [0,T],$$

with K_T as in Corollary 3.2. Let $\epsilon \in (0,1)$ and let Ω_ϵ be as in Corollary 3.2. Then

$$\begin{aligned}
\mathbf{E}\left[\|X_h(t) - X(t)\|^2\right] &\le \int_{\Omega_\epsilon} \|X_h(t) - X(t)\|^2 \, \mathrm{d}\mathbf{P} \\
&\quad + 2 \int_{\Omega_\epsilon^c} \left(\|X_h(t)\|^2 + \|X(t)\|^2 \right) \, \mathrm{d}\mathbf{P}.
\end{aligned}$$

Here, by Hölder's inequality, we have

$$\begin{aligned}\int_{\Omega_\epsilon^c} \|X(t)\|^2 \,\mathrm{d}\mathbf{P} &\le \Big(\int_{\Omega_\epsilon^c} 1^2 \,\mathrm{d}\mathbf{P}\Big)^{\frac12}\Big(\int_{\Omega_\epsilon^c} \|X(t)\|_{L_4}^4 \,\mathrm{d}\mathbf{P}\Big)^{\frac12}\\ &\le \epsilon^{\frac12}\big(\mathbf{E}\big[\|X(t)\|_{L_4}^4\big]\big)^{\frac12} \le \epsilon^{\frac12} K_T^{\frac12}.\end{aligned}$$

Therefore, by Theorem 5.3,

$$\max_{t\in[0,T]} \big(\mathbf{E}\big[\|X_h(t) - X(t)\|^2\big]\big)^{\frac12} \le C(\epsilon^{-1}K_T, T)h^2|\log(h)| + CK_T^{\frac14}\epsilon^{\frac14}.$$

Since $\frac{\epsilon^{\frac14}}{C(\epsilon^{-1}K_T,T)} \to 0$ monotonically as $\epsilon \to 0$, we may choose ϵ, depending on h, such that the two terms are equal. □

Since $C(\epsilon^{-1}K_T, T)$ grows rapidly with ϵ^{-1}, it is not possible to obtain a rate of convergence from this proof.

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A POSTERIORI ERROR ANALYSIS FOR THE CAHN-HILLIARD EQUATION

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Abstract. The Cahn-Hilliard equation is discretized by a Galerkin finite element method based on continuous piecewise linear functions in space and discontinuous piecewise constant functions in time. A posteriori error estimates are proved by using the methodology of dual weighted residuals.

1. Introduction

We consider the Cahn-Hilliard equation

$$\begin{aligned} u_t - \Delta w &= 0 \quad &&\text{in } \Omega \times [0,T], \\ w + \epsilon \Delta u - f(u) &= 0 \quad &&\text{in } \Omega \times [0,T], \\ \frac{\partial u}{\partial \nu} = 0,\ \frac{\partial w}{\partial \nu} &= 0 \quad &&\text{on } \partial\Omega \times [0,T], \\ u(\cdot,0) &= g_0 \quad &&\text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a polygonal domain in $\mathbf{R}^d, d = 1,2,3,\ u = u(x,t), w = w(x,t)$, $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2},\ u_t = \frac{\partial u}{\partial t}$, ν is the exterior unit normal to $\partial\Omega$, and $\epsilon > 0$ is a small parameter. The Cahn-Hilliard equation is a model for phase separation and spinodal decomposition [3]. The nonlinearity f is the derivative of a double-well potential. A typical example is $f(u) = u^3 - u$.

We discretize (1.1) by a Galerkin finite element method, which is based on continuous piecewise linear functions with respect to x and discontinuous piecewise constant functions with respect to t. This numerical method is the same as the implicit Euler time stepping combined with spatial discretization by a standard finite element method.

We perform an a posteriori error analysis within the framework of dual weighted residuals [2]. If $J(u)$ is a given goal functional, this results in an

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error estimate essentially of the form

$$|J(u) - J(U)| \leq \sum_{n=1}^{N} \sum_{K \in \mathbf{T}_n} \Big\{ \rho_{u,K}\omega_{u,K} + \rho_{w,K}\omega_{w,K} \Big\} + \mathcal{R},$$

where U denotes the numerical solution and $\mathbf{T}_n$ is the spatial mesh at time level t_n. The terms $\rho_{u,K}, \rho_{w,K}$ are local residuals from the first and second equations in (1.1), respectively. The weights $\omega_{u,K}, \omega_{w,K}$ are derived from the solution of the linearized adjoint problem. The remainder $\mathcal{R}$ is quadratic in the error.

There is an extensive literature on numerical methods for the Cahn-Hilliard equation; see, for example, [5] and [4] for a priori error estimates. Adaptive methods based on a posteriori estimates are presented in [1] and [6]. However, these estimates are restricted to spatial discretization. We are not aware of any completely discerete a posteriori error analysis.

2. Preliminaries

Here we present the methodology of dual weighted residuals [2] in an abstract form.

Let $A(\cdot;\cdot)$ be a semilinear form; that is, it is nonlinear in the first and linear in the second variable, and $J(\cdot)$ be an output functional, not necessarily linear, defined on some function space V. Consider the variational problem: Find $u \in V$ such that

$$A(u;\psi) = 0 \quad \forall \psi \in V, \tag{2.1}$$

and the corresponding finite element problem: Find $u_h \in V_h \subset V$ such that

$$A(u_h;\psi_h) = 0 \quad \forall \psi_h \in V_h. \tag{2.2}$$

We suppose that the derivatives of A and J with respect to the first variable u up to order three exist and are denoted by

$$A'(u;\varphi),\ A''(u;\psi,\varphi),\ A'''(u;\xi,\psi,\varphi),$$

and

$$J'(u;\varphi),\ J''(u;\psi,\varphi),\ J'''(u;\xi,\psi,\varphi),$$

respectively, for increments $\varphi,\ \psi,\ \xi \in V$. Here we use the convention that the forms are linear in the variables after the semicolon.

We want to estimate $J(u) - J(u_h)$. Introduce the dual variable $z \in V$ and define the Lagrange functional

$$\mathcal{L}(u;z) := J(u) - A(u;z)$$

and seek the stationary points $(u,z) \in V \times V$ of $\mathcal{L}(\cdot;\cdot)$; that is,

$$\mathcal{L}'(u;z,\varphi,\psi) = J'(u;\varphi) - A'(u;z,\varphi) - A(u;\psi) = 0 \quad \forall \varphi,\psi \in V. \tag{2.3}$$

By choosing $\varphi = 0$, we retrieve (2.1). By taking $\psi = 0$, we identify the linearized adjoint equation to find $z \in V$ such that

$$J'(u;\varphi) - A'(u;z,\varphi) = 0 \quad \forall \varphi \in V. \tag{2.4}$$

The corresponding finite element problem is: Find $(u_h, z_h) \in V_h \times V_h$ such that

$$\begin{aligned} \mathcal{L}'(u_h; z_h, \varphi_h, \psi_h) &= J'(u_h;\varphi_h) - A'(u_h; z_h, \varphi_h) - A(u_h;\psi_h) \\ &= 0 \quad \forall \varphi_h, \psi_h \in V_h. \end{aligned} \tag{2.5}$$

By choosing $\varphi_h = 0$, we retrieve (2.2). By taking $\psi_h = 0$, we identify the linearized adjoint equation to find $z_h \in V_h$ such that

$$J'(u_h;\varphi_h) - A'(u_h;z_h,\varphi_h) = 0 \quad \forall \varphi_h \in V_h. \tag{2.6}$$

We quote three propositions from [2, Ch. 6].

Proposition 2.1. *Let $L(\cdot)$ be a three times differentiable functional defined on a vector space X, which has a stationary point $x \in X$, that is,*

$$L'(x;y) = 0 \quad \forall y \in X.$$

Suppose that on a finite dimensional subspace $X_h \subset X$ the corresponding Galerkin approximation,

$$L'(x_h;y_h) = 0 \quad \forall y_h \in X_h,$$

has a solution, $x_h \in X_h$. Then there holds the error representation

$$L(x) - L(x_h) = \tfrac{1}{2} L'(x_h; x - y_h) + \mathcal{R} \quad \forall y_h \in X_h,$$

with a remainder term $\mathcal{R}$, which is cubic in the error $e := x - x_h$,

$$\mathcal{R} := \tfrac{1}{2} \int_0^1 L'''(x_h + se; e, e, e) s(s-1)\,\mathrm{d}s.$$

Since

$$\mathcal{L}(u;z) - \mathcal{L}(u_h;z_h) = J(u) - J(u_h),$$

at stationary points $(u,z), (u_h,z_h)$, Proposition 2.1 yields the following result for the Galerkin approximation (2.2) of the variational equation (2.1).

Proposition 2.2. *For any solutions u and u_h of equations* (2.1) *and* (2.2) *we have the error representation*

$$J(u) - J(u_h) = \tfrac{1}{2}\rho(u_h; z - \varphi_h) + \tfrac{1}{2}\rho^*(u_h; z_h, u - \psi_h) + \mathcal{R}^{(3)} \quad \forall \varphi_h, \psi_h \in V_h,$$

where z and z_h are solutions of the adjoint problems (2.4) *and* (2.6) *and*

$$\begin{aligned} \rho(u_h;\cdot) &= -A(u_h;\cdot), \\ \rho^*(u_h;z_h,\cdot) &= J'(u_h;\cdot) - A'(u_h;z_h,\cdot), \end{aligned}$$

and, with $e_u = u - u_h$, $e_z = z - z_h$, *the remainder is*

$$\begin{aligned}\mathcal{R}^{(3)} = \tfrac{1}{2}\int_0^1 \Big(&J'''(u_h + se_u; e_u, e_u, e_u) - A'''(u_h + se_u; z_h + se_z, e_u, e_u, e_u)\\ &- 3A''(u_h + se_u; e_u, e_u, e_z)\Big) s(s-1)\,\mathrm{d}s.\end{aligned}$$

The forms $\rho(\cdot;\cdot), \rho^*(\cdot;\cdot,\cdot)$ are the residuals of (2.1) and (2.4), respectively. The remainder $\mathcal{R}^{(3)}$ is cubic in the error. The following proposition shows that the residuals are equal up to a quadratic remainder.

Proposition 2.3. *With the notation from above, we have*

$$\rho^*(u_h; z_h, u - \psi_h) = \rho(u_h; z - \varphi_h) + \delta\rho \quad \forall \varphi_h, \psi_h \in V_h,$$

with

$$\delta\rho = \int_0^1 \Big(A''(u_h + se_u; z_h + se_z, e_u, e_u) - J''(u_h + se_u; e_u, e_u)\Big)\,\mathrm{d}s.$$

Moreover, we have the simplified error representation

$$J(u) - J(u_h) = \rho(u_h; z - \varphi_h) + \mathcal{R}^{(2)} \quad \forall \varphi_h \in V_h,$$

with quadratic remainder

$$\mathcal{R}^{(2)} = \int_0^1 \Big(A''(u_h + se_u; z, e_u, e_u) - J''(u_h + se_u; e_u, e_u)\Big)\,\mathrm{d}s.$$

3. Galerkin discretization and dual problem

In this section, we apply the dual weighted residuals methodology to the Cahn-Hilliard equation (1.1). We denote $I = [0, T]$, $Q = \Omega \times I$, and

$$\langle v, w\rangle_{\mathcal{D}} = \int_{\mathcal{D}} vw\,\mathrm{d}z, \quad \|v\|_{\mathcal{D}}^2 = \int_{\mathcal{D}} v^2\,\mathrm{d}z$$

for subsets $\mathcal{D}$ of Q or Ω with the relevant Lebesgue measure $\mathrm{d}z$. Let $V = H^1(\Omega)$ and $\mathcal{W} = C^1([0,T], V)$. By multiplying the first equation by $\psi_u \in V$ and the second equation by $\psi_w \in V$, integrating over Ω and using Green's formula, we obtain the weak formulation: Find $u, w \in \mathcal{W}$ such that $u(0) = g_0$ and

$$\begin{aligned}\langle u_t, \psi_u\rangle_\Omega + \langle \nabla w, \nabla \psi_u\rangle_\Omega = 0 \quad &\forall \psi_u \in V,\ t \in [0,T],\\ \langle w, \psi_w\rangle_\Omega - \epsilon\langle \nabla u, \nabla \psi_w\rangle_\Omega - \langle f(u), \psi_w\rangle_\Omega = 0 \quad &\forall \psi_w \in V,\ t \in [0,T].\end{aligned} \tag{3.1}$$

Split the interval $I = [0,T]$ into subintervals $I_n = [t_{n-1}, t_n)$ of lengths $k_n = t_n - t_{n-1}$,

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T.$$

For each time level $t_n, n \geq 1$, let $\mathcal{V}_n$ be the space of continuous piecewise linear functions with respect to regular spatial meshes $\mathbf{T}_n = \{K\}$, which may vary from time level to time level. By extending the spatial meshes $\mathbf{T}_n$ as constant in time to the time slab $\Omega \times I_n$, we obtain meshes $\mathcal{T}_k$ of the space-time domain $Q = \Omega \times I$, which consist of $(d+1)$-dimensional prisms $Q_K^n := K \times \bar{I}_n$. Define the finite element space

$$\mathcal{V} := \Big\{\varphi\colon \bar{Q} \to \mathbf{R} : \varphi(\cdot,t)|_{\bar{\Omega}} \in \mathcal{V}_n,\ t \in I_n,\ \varphi(x,\cdot)|_{I_n} \in \Pi_0,\ x \in \bar{\Omega}\Big\}.$$

Here Π_0 denotes the polynomials of degree 0. For functions from this space and their continuous analogues, we define

$$v_n^+ = \lim_{t \downarrow t_n} v(t), \quad v_n = v_n^- = \lim_{t \uparrow t_n} v(t), \quad [v]_n = v_n^+ - v_n^-.$$

For all $u, w, \psi_u, \psi_w \in \mathcal{V}$ or $\mathcal{W}$, consider the semilinear form

$$\begin{aligned} A(u,w;\psi_u,\psi_w) = \sum_{n=1}^{N} \int_{I_n} \Big\{ & \langle u_t, \psi_u\rangle_\Omega + \langle \nabla w, \nabla \psi_u\rangle_\Omega + \langle w, \psi_w\rangle_\Omega \\ & - \epsilon \langle \nabla u, \nabla \psi_w\rangle_\Omega - \langle f(u), \psi_w\rangle_\Omega \Big\}\,\mathrm{d}t \\ & + \sum_{n=2}^{N} \langle [u]_{n-1}, \psi_{u,n-1}^+\rangle_\Omega + \langle u_0^+ - g_0, \psi_{u,0}^+\rangle_\Omega. \end{aligned}$$

Solutions $u, w \in \mathcal{W}$ of (1.1) satisfy the variational problem

$$A(u,w;\psi_u,\psi_w) = 0 \quad \forall \psi_u, \psi_w \in \mathcal{W} \tag{3.2}$$

and the finite element problem can formulated: Find $U, W \in \mathcal{V}$ such that

$$A(U,W;\psi_u,\psi_w) = 0 \quad \forall \psi_u, \psi_w \in \mathcal{V}. \tag{3.3}$$

We now show that this is a standard time-stepping method. Since $U(t) = U_n = U_n^- = U_{n-1}^+$, $W(t) = W_n$ for $t \in I_n$, we have

$$\begin{aligned} A(U,W;\psi_u,\psi_w) = \sum_{n=1}^{N} \int_{I_n} \Big\{ & \langle \nabla W_n, \nabla \psi_u\rangle_\Omega + \langle W_n, \psi_w\rangle_\Omega \\ & - \epsilon \langle \nabla U_n, \nabla \psi_w\rangle_\Omega - \langle f(U_n), \psi_w\rangle_\Omega \Big\}\,\mathrm{d}t \\ & + \sum_{n=2}^{N} \langle U_n - U_{n-1}, \psi_{u,n-1}^+\rangle_\Omega + \langle U_1 - g_0, \psi_{u,0}^+\rangle_\Omega. \end{aligned} \tag{3.4}$$

By taking

$$\psi_u(t) = \begin{cases} \chi_u \in \mathcal{V}_n, & t \in I_n, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi_w(t) = \begin{cases} \chi_w \in \mathcal{V}_n, & t \in I_n, \\ 0, & \text{otherwise}, \end{cases}$$

we see that (3.3) amounts to the implicit Euler time-stepping,

$$\begin{aligned}\langle U_0 - g_0, \chi_u\rangle_\Omega &= 0 \quad \forall \chi_u \in \mathcal{V}_1,\\ k_n\langle \nabla W_n, \nabla \chi_u\rangle_\Omega + \langle U_n - U_{n-1}, \chi_u\rangle_\Omega &= 0 \quad \forall \chi_u \in \mathcal{V}_n, n \geq 1,\\ \langle W_n, \chi_w\rangle_\Omega - \epsilon\langle \nabla U_n, \nabla \chi_w\rangle_\Omega - \langle f(U_n), \chi_w\rangle_\Omega &= 0 \quad \forall \chi_w \in \mathcal{V}_n, n \geq 1.\end{aligned}$$

Now take a goal functional $J(u)$, which depends only on u, and set

$$\mathcal{L}(v; z) = J(u) - A(v; z),$$

where $v = (u, w), z = (z_u, z_w)$. With $\varphi = (\varphi_u, \varphi_w), \psi = (\psi_u, \psi_w)$, stationary points are given by

$$\mathcal{L}'(v; z, \varphi, \psi) = J'(u; \varphi_u) - A'(v; z, \varphi) - A(v; \psi) = 0 \quad \forall \varphi, \psi \in \mathcal{W} \times \mathcal{W}.$$

With $\psi = 0$ we obtain $A'(v; z, \varphi) = J'(u; \varphi_u)$, the adjoint problem. So we should compute $A'(u, w; z_u, z_w, \varphi_u, \varphi_w)$ and $J'(u; \varphi_u)$. To this end we write

$$\begin{aligned}A(u, w; \psi_u, \psi_w) = \langle u_t, \psi_u\rangle_Q + \langle \nabla w, \nabla \psi_u\rangle_Q + \langle w, \psi_w\rangle_Q - \epsilon\langle \nabla u, \nabla \psi_w\rangle_Q\\ - \langle f(u), \psi_w\rangle_Q + \langle u(0) - g_0, \psi_u(0)\rangle_\Omega.\end{aligned}$$

Hence,

$$\begin{aligned}A'(u, w; z_u, z_w, \varphi_u, \varphi_w) = \langle \varphi_{u,t}, z_u\rangle_Q + \langle \nabla \varphi_w, \nabla z_u\rangle_Q + \langle \varphi_w, z_w\rangle_Q\\ - \epsilon\langle \nabla \varphi_u, \nabla z_w\rangle_Q - \langle \varphi_u, z_w\rangle_Q + \langle \varphi_u(0), z_u(0)\rangle_\Omega.\end{aligned}$$

By integration by parts in t,

$$\langle \varphi_{u,t}, z_u\rangle_Q = -\langle \varphi_u, z_{u,t}\rangle_Q + \langle \varphi_u(T), z_u(T)\rangle_\Omega - \langle \varphi_u(0), z_u(0)\rangle_\Omega,$$

we obtain

$$\begin{aligned}A'(u, w; z_u, z_w, \varphi_u, \varphi_w) = &-\langle \varphi_u, z_{u,t}\rangle_Q + \langle \nabla \varphi_w, \nabla z_u\rangle_Q\\ &+ \langle \varphi_w, z_w\rangle_Q + \epsilon\langle \nabla \varphi_u, \nabla z_w\rangle_Q\\ &- \langle \varphi_u, f'(u) z_w\rangle_Q + \langle \varphi_u(T), z_u(T)\rangle_\Omega.\end{aligned}$$

The adjoint problem is thus to find $z_u, z_w \in \mathcal{W}$ such that

$$\begin{aligned}&\langle \varphi_u, -z_{u,t}\rangle_Q - \epsilon\langle \nabla \varphi_u, \nabla z_w\rangle_Q\\ &\qquad - \langle \varphi_u, f'(u) z_w\rangle_Q + \langle \varphi_u(T), z_u(T)\rangle_\Omega\\ &\qquad + \langle \nabla \varphi_w, \nabla z_w\rangle_Q + \langle \varphi_w, z_w\rangle_Q = J'(u; \varphi_u) \quad \forall \varphi_u, \varphi_w \in \mathcal{W}.\end{aligned} \tag{3.5}$$

We now specialize to the case of a linear goal functional of the form

$$J(u) = \langle u, g\rangle_Q + \langle u(T), g_T\rangle_\Omega,$$

for some $g \in L_2(Q),\ g_T \in L_2(\Omega)$. Then

$$J'(u; \varphi_u) = \langle \varphi_u, g\rangle_Q + \langle \varphi_u(T), g_T\rangle_\Omega. \tag{3.6}$$

The adjoint problem then becomes: Find $z_u, z_w \in \mathcal{W}$ such that

$$\begin{aligned} &\langle \varphi_u, -z_{u,t} - f'(u) z_w - g \rangle_Q - \epsilon \langle \nabla \varphi_u, \nabla z_w \rangle_Q \\ &\qquad + \langle \varphi_u(T), z_u(T) - g_T \rangle_\Omega = 0 \quad \forall \varphi_u \in \mathcal{W}, \\ &\langle \varphi_w, z_w \rangle_Q + \langle \nabla \varphi_w, \nabla z_u \rangle_Q = 0 \quad \forall \varphi_w \in \mathcal{W}. \end{aligned} \tag{3.7}$$

The strong form of this is

$$\begin{aligned} -\partial_t z_u + \epsilon \Delta z_w - f'(u) z_w &= g \quad && \text{in } Q, \\ z_w - \Delta z_u &= 0 && \text{in } Q, \\ \frac{\partial z_u}{\partial \nu} = 0, \ \frac{\partial z_w}{\partial \nu} &= 0 && \text{on } \partial\Omega \times I, \\ z_u(T) &= g_T && \text{in } \Omega. \end{aligned} \tag{3.8}$$

4. A posteriori error estimates

From Proposition 2.3 we have the error representation

$$J(u) - J(U) = -A(U, W; z_u - \pi z_u, z_w - \pi z_w) + \mathcal{R}^{(2)}, \tag{4.1}$$

where $z = (z_u, z_w)$ is the solution of the adjoint problem (3.5) and $\pi z_u, \pi z_w \in \mathcal{V}$ are appropriate approximations to be defined below. The remainder is quadratic in the error.

In order to write this as a sum of local contributions we must rewrite $A(U, W; \psi_u, \psi_w)$ in (3.4). First we compute $\int_{I_n} \langle \nabla W, \nabla \psi_u \rangle_\Omega \, \mathrm{d}t$. By using Green's formula elementwise, we have

$$\begin{aligned} \int_{I_n} \langle \nabla W, \nabla \psi_u \rangle_\Omega \, \mathrm{d}t &= \int_{I_n} \sum_{K \in \mathbf{T}_n} \langle \nabla W, \nabla \psi_u \rangle_K \, \mathrm{d}t \\ &= \int_{I_n} \sum_{K \in \mathbf{T}_n} -\langle \Delta W, \psi_u \rangle_K \, \mathrm{d}t + \int_{I_n} \sum_{K \in \mathbf{T}_n} \langle \partial_\nu W, \psi_u \rangle_{\partial K} \, \mathrm{d}t, \end{aligned}$$

where $\partial_\nu W = \nu \cdot \nabla W$. We divide the boundary $\partial K \in \mathbf{T}_n$ into two parts: internal edges, denoted by $\mathcal{E}_I^n$, and edges on the boundary $\partial\Omega$, denoted by $\mathcal{E}_{\partial\Omega}^n$. So we get, with $[\ ]$ denoting the jump across the edge,

$$\begin{aligned} &\int_{I_n} \sum_{K \in \mathbf{T}_n} \langle \partial_\nu W, \psi_u \rangle_{\partial K} \, \mathrm{d}t \\ &\quad = \int_{I_n} \sum_{E \in \mathcal{E}_I^n} \langle \partial_\nu W, \psi_u \rangle_E \, \mathrm{d}t + \int_{I_n} \sum_{E \in \mathcal{E}_{\partial\Omega}^n} \langle \partial_\nu W, \psi_u \rangle_E \, \mathrm{d}t \\ &\quad = \int_{I_n} \sum_{K \in \mathbf{T}_n} -\tfrac{1}{2} \langle [\partial_\nu W], \psi_u \rangle_{\partial K \setminus \partial\Omega} \, \mathrm{d}t + \int_{I_n} \sum_{K \in \mathbf{T}_n} \langle \partial_\nu W, \psi_u \rangle_{\partial K \cap \partial\Omega} \, \mathrm{d}t. \end{aligned}$$

Let ∂_x denote the spatial boundary and define $\partial_x Q = \partial\Omega \times I$ and $\partial_x Q_K^n = \partial K \times I_n$. Hence,

$$\begin{aligned}\int_{I_n} \langle \nabla W, \nabla \psi_u \rangle_\Omega \, \mathrm{d}t = \sum_{K\in\mathbf{T}_n} \Big\{ &- \langle \Delta W, \psi_u \rangle_{Q_K^n} - \tfrac{1}{2} \langle [\partial_\nu W], \psi_u \rangle_{\partial_x Q_K^n \setminus \partial_x Q} \\ &+ \langle \partial_\nu W, \psi_u \rangle_{\partial_x Q_K^n \cap \partial_x Q} \Big\},\end{aligned}$$

and in the same way

$$\begin{aligned}\epsilon \int_{I_n} \langle \nabla U, \nabla \psi_w \rangle_\Omega \, \mathrm{d}t = \sum_{K\in\mathbf{T}_n} \Big\{ &- \epsilon \langle \Delta U, \psi_w \rangle_{Q_K^n} - \tfrac{1}{2} \epsilon \langle [\partial_\nu U], \psi_w \rangle_{\partial_x Q_K^n \setminus \partial_x Q} \\ &+ \epsilon \langle \partial_\nu U, \psi_w \rangle_{\partial_x Q_K^n \cap \partial_x Q} \Big\}.\end{aligned}$$

Note that $\Delta W = \Delta U = 0$ on Q_K^n for piecewise linear functions, but we find it instructive to keep these terms. Inserting this into (3.4) and noting that

$$\int_{I_n} \langle W, \psi_w \rangle_\Omega \, \mathrm{d}t = \sum_{K\in\mathbf{T}_n} \langle W, \psi_w \rangle_{Q_K^n},$$

and

$$\int_{I_n} \langle f(U), \psi_w \rangle_\Omega \, \mathrm{d}t = \sum_{K\in\mathbf{T}_n} \langle f(U), \psi_w \rangle_{Q_K^n},$$

gives

$$\begin{aligned}A(U, W; \psi_u, \psi_w) = \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ &- \langle \Delta W, \psi_u \rangle_{Q_K^n} \\ &+ \langle \epsilon\Delta U + W - f(U), \psi_w \rangle_{Q_K^n} - \tfrac{1}{2} \langle [\partial_\nu W], \psi_u \rangle_{\partial_x Q_K^n \setminus \partial_x Q} \\ &+ \tfrac{1}{2} \epsilon \langle [\partial_\nu U], \psi_w \rangle_{\partial_x Q_K^n \setminus \partial_x Q} + \langle \partial_\nu W, \psi_u \rangle_{\partial_x Q_K^n \cap \partial_x Q} \\ &- \epsilon \langle \partial_\nu U, \psi_w \rangle_{\partial_x Q_K^n \cap \partial_x Q} + \langle [U]_{n-1}, \psi_{u,n-1}^+ \rangle_K \Big\},\end{aligned}$$

where we have set $U_0^- = g_0$ for simplicity. Hence (4.1) becomes

$$\begin{aligned}J(u) - J(U) = \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ &\langle R_u, z_u - \pi z_u \rangle_{Q_K^n} + \langle R_w, z_w - \pi z_w \rangle_{Q_K^n} \\ &+ \langle r_u, z_u - \pi z_u \rangle_{\partial_x Q_K^n} + \langle r_w, z_w - \pi z_w \rangle_{\partial_x Q_K^n} \\ &- \langle [U]_{n-1}, (z_u - \pi z_u)_{n-1}^+ \rangle_K \Big\} + \mathcal{R}^{(2)},\end{aligned} \tag{4.2}$$

with the interior residuals

$$R_u = \Delta W, \quad R_w = -\epsilon\Delta U - W + f(U),$$

the edge residuals

$$r_w|_\Gamma = \begin{cases} -\frac{1}{2}\epsilon[\partial_\nu U], & \Gamma \subset \partial_x Q_K^n \setminus \partial_x Q, \\ 0, & \text{otherwise,} \end{cases}$$

$$r_u|_\Gamma = \begin{cases} \frac{1}{2}[\partial_\nu W], & \Gamma \subset \partial_x Q_K^n \setminus \partial_x Q, \\ 0, & \text{otherwise,} \end{cases}$$

and the boundary residuals

$$r_w|_\Gamma = \begin{cases} \epsilon\partial_\nu U, & \Gamma \subset \partial_x Q_K^n \cap \partial_x Q, \\ 0, & \text{otherwise,} \end{cases}$$

$$r_u|_\Gamma = \begin{cases} -\partial_\nu W, & \Gamma \subset \partial_x Q_K^n \cap \partial_x Q, \\ 0, & \text{otherwise.} \end{cases}$$

Here the subscript u refers to residuals from the first equation in (3.1) and the subscript w to residuals from the second equation.

We now define $\pi z_u, \pi z_w \in \mathcal{V}$. Let

$$(P_n v)(t) = \frac{1}{k_n}\int_{I_n} v(s)\,\mathrm{d}s$$

be the orthogonal projector onto constants. Let $\pi_n \colon C(\bar\Omega) \to \mathcal{V}_n$ be the nodal interpolator; that is, it is defined by

$$(\pi_n v)(a) = v(a),$$

for all nodal points a in $\mathbf{T}_n$. Then we define $\pi \colon C(\bar Q) \to \mathcal{V}$ by $\pi v|_{I_n} = P_n \pi_n v$. Since R_u, R_w, r_u, and r_w are piecewise constant in t, we have

$$\begin{aligned} &J(u) - J(U) \\ &\quad = \sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\Big\{\langle R_u, P_n(z_u - \pi_n z_u)\rangle_{Q_K^n} + \langle R_w, P_n(z_w - \pi_n z_w)\rangle_{Q_K^n} \\ &\qquad + \langle r_u, P_n(z_u - \pi_n z_u)\rangle_{\partial_x Q_K^n} + \langle r_w, P_n(z_w - \pi_n z_w)\rangle_{\partial_x Q_K^n} \\ &\qquad - \langle [U]_{n-1}, (z_u - \pi z_u)_{n-1}^+\rangle_K\Big\} + \mathcal{R}^{(2)}. \end{aligned} \tag{4.3}$$

Applying the Cauchy-Schwartz inequality to each term gives

$$
\begin{aligned}
|J(u)-J(U)| \le \sum_{n=1}^{N}\sum_{K\in \mathbf{T}_n} \Big\{ & \|R_u\|_{Q_K^n}\|P_n(z_u-\pi_n z_u)\|_{Q_K^n} \\
& + h_K^{-\frac12}\|r_u\|_{\partial_x Q_K^n} h_K^{\frac12}\|P_n(z_u-\pi_n z_u)\|_{\partial_x Q_K^n} \\
& + \|R_w\|_{Q_K^n}\|P_n(z_w-\pi_n z_w)\|_{Q_K^n} \\
& + h_K^{-\frac12}\|r_w\|_{\partial_x Q_K^n} h_K^{\frac12}\|P_n(z_w-\pi_n z_w)\|_{\partial_x Q_K^n} \\
& + k_n^{-\frac12}\|[U]_{n-1}\|_K k_n^{\frac12}\|(z_u-\pi z_u)_{n-1}^{+}\|_K \Big\} + |\mathcal{R}^{(2)}|.
\end{aligned}
$$

Here $h_K = \mathrm{diam}(K)$. For $a, b, c, d \ge 0$ we have

$$(ab+cd) \le (a^2+c^2)^{\frac12}(b^2+d^2)^{\frac12}.$$

We use this inequality for each term in the previous inequality and set

$$
\begin{aligned}
\rho_{u,K} &= \Big(\|R_u\|_{Q_K^n}^2 + h_K^{-1}\|r_u\|_{\partial_x Q_K^n}^2\Big)^{\frac12}, \\
\omega_{u,K} &= \Big(\|P_n(z_u-\pi_n z_u)\|_{Q_K^n}^2 + h_K\|P_n(z_u-\pi_n z_u)\|_{\partial_x Q_K^n}^2\Big)^{\frac12}, \\
\rho_{w,K} &= \Big(\|R_w\|_{Q_K^n}^2 + h_K^{-1}\|r_w\|_{\partial_x Q_K^n}^2\Big)^{\frac12}, \\
\omega_{w,K} &= \Big(\|P_n(z_w-\pi_n z_w)\|_{Q_K^n}^2 + h_K\|P_n(z_w-\pi_n z_w)\|_{\partial_x Q_K^n}^2\Big)^{\frac12}, \\
\rho_K &= \Big(k_n^{-1}\|[U]^{n-1}\|_K^2\Big)^{\frac12}, \\
\omega_K &= \Big(k_n\|(z_u-\pi z_u)_{n-1}^{+}\|_K^2\Big)^{\frac12}.
\end{aligned}
$$

Note that, since $R_u = \Delta W = 0$ for piecewise linear functions, the first term in $\rho_{u,K}$ and $\omega_{u,K}$ can actually be removed. So we have

$$|J(u)-J(U)| \le \sum_{n=1}^{N}\sum_{K\in \mathbf{T}_n} \Big\{\rho_{u,K}\omega_{u,K} + \rho_{w,K}\omega_{w,K} + \rho_K\omega_K\Big\} + |\mathcal{R}^{(2)}|.$$

We have proved the following theorem:

Theorem 4.1. *We have the a posteriori error estimate*

$$|J(u)-J(U)| \le \sum_{n=1}^{N}\sum_{K\in \mathbf{T}_n} \Big\{\rho_{u,K}\omega_{u,K} + \rho_{w,K}\omega_{w,K} + \rho_K\omega_K\Big\} + |\mathcal{R}^{(2)}|. \tag{4.4}$$

Note that on each space-time cell Q_K^n, the terms $\rho_{u,K}\omega_{u,K}$ and $\rho_{w,K}\omega_{w,K}$ can be used to control the spatial mesh and the term $\rho_K\omega_K$ to control the time step k_n in an adaptive algorithm; see [2]. We do not pursue this here.

In the following we want to obtain a weight free a posteriori error estimate where the weights in (4.4) are replaced by a global stability constant. We need the following interpolation error estimate, see [2, Lemma 9.4].

Lemma 4.2. *With π and π_n as defined as before, there holds*

$$\|P_n(z-\pi_n z)\|_{Q_K^n} + h_K^{\frac{1}{2}}\|P_n(z-\pi_n z)\|_{\partial_x Q_K^n} \le C h_K^2 \|\mathrm{D}^2 z\|_{Q_K^n}, \tag{4.5}$$

$$\|z(t_{n-1}) - P_n z\|_K \le C k_n^{\frac{1}{2}} \|\partial_t z\|_{Q_K^n}. \tag{4.6}$$

Here $\|\mathrm{D}^2 z\|_{Q_K^n}$ denotes the seminorm $\Big(\sum_{|\alpha|=2}\|D^\alpha z\|_{Q_K^n}^2\Big)^{\frac{1}{2}}$.

In the following we assume that $J(\cdot)$ is a linear functional given by (3.6) and Ω is such that we have the elliptic regularity estimate

$$\|\mathrm{D}^2 v\|_\Omega \le C\|\Delta v\|_\Omega \quad \forall v \in H^2(\Omega) \text{ with } \frac{\partial v}{\partial \nu}\Big|_\Gamma = 0. \tag{4.7}$$

We also assume a global bound for $f'(u)$, which is reasonable since it is known that $\|u\|_{L_\infty(Q)} \le C$ (c.f. [5]).

In particular, with

$$g = (u-U)/\|u-U\|_Q \text{ and } g_T = (u_N - U_N)/\|u_N - U_N\|_\Omega$$

the following theorem provides bounds for the norms of the error, $\|u-U\|_Q$ and $\|u_N - U_N\|_\Omega$.

Theorem 4.3. *Assume that $\|f'(u)\|_{L_\infty} \le \beta$ and that (4.7) holds. Let z_u, z_w be the solutions of (3.8). Then there is $C = C(\beta)$ such that the following a posteriori error estimates hold.*

(i) Let $g \in L_2(Q)$ with $\|g\|_Q = 1$ and $g_T = 0$. Then

$$\begin{aligned}&|\langle u-U, g\rangle_Q| \\ &\le C C_S \sum_{n=1}^N \sum_{K\in\mathbf{T}_n} \Big\{h_K^4(\rho_{u,K}^2 + \rho_{w,K}^2) + (h_K^4 + k_n^2)\rho_K^2\Big\}^{\frac{1}{2}} + |\mathcal{R}^{(2)}|,\end{aligned} \tag{4.8}$$

where

$$C_S = \sup_{g\in L_2(Q)} \frac{\Big(\|\mathrm{D}^2 z_u\|_Q^2 + \|\partial_t z_u\|_Q^2 + \|\mathrm{D}^2 z_w\|_Q^2\Big)^{\frac{1}{2}}}{\|g\|_Q}.$$

(ii) Let $g_T \in L_2(\Omega)$ with $\|g_T\|_\Omega = 1$ and $g = 0$. Then

$$
\begin{aligned}
&|\langle u - U, g_T\rangle_\Omega| \\
&\le C C_S \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ h_K^4(\rho_{u,K}^2 + \sigma_n^{-1}\rho_{w,K}^2 + \sigma_n^{-1}\rho_K^2) + k_n^2\sigma^{-1}\rho_K^2 \Big\}^{\frac12} \\
&\quad + |\mathcal{R}^{(2)}|,
\end{aligned}
\tag{4.9}
$$

where $\sigma(t) = T - t$,

$$
\sigma_n = \begin{cases} \sigma(t_n) = T - t_n, & n = 1, \cdots, N-1, \\ k_N, & n = N, \end{cases}
$$

and

$$
\begin{aligned}
C_S = \sup_{g_T\in L_2(\Omega)} \Big(&\epsilon^{-1} \max_I \|z_u\|_\Omega^2 + \epsilon^{-1}\|z_w\|_Q^2 \\
&+ \|\mathrm{D}^2 z_u\|_Q^2 + \|\sigma^{\frac12}\partial_t z_u\|_Q^2 + \epsilon^2\|\sigma^{\frac12}\mathrm{D}^2 z_w\|_Q^2 \Big)^{\frac12} / \|g_T\|_\Omega.
\end{aligned}
$$

Proof. Part (i). From Theorem 4.2 we have

$$
\begin{aligned}
\omega_{u,K} &= \Big(\|P_n(z_u - \pi_n z_u)\|_{Q_K^n}^2 + h_K \|P_n(z_u - \pi_n z_u)\|_{\partial_x Q_K^n}^2 \Big)^{\frac12} \\
&\le C h_K^2 \|\mathrm{D}^2 z_u\|_{Q_K^n},
\end{aligned}
$$

$$
\begin{aligned}
\omega_{w,K} &= \Big(\|P_n(z_w - \pi_n z_w)\|_{Q_K^n}^2 + h_K \|P_n(z_w - \pi_n z_w)\|_{\partial_x Q_K^n}^2 \Big)^{\frac12} \\
&\le C h_K^2 \|\mathrm{D}^2 z_w\|_{Q_K^n},
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_K &= k_n^{\frac12} \|(z_u - \pi_n z_u)_{n-1}^+\|_K \\
&\le k_n^{\frac12} \|P_n(z_u - \pi_n z_u)\|_K + k_n^{\frac12}\|z_u(t_{n-1}) - P_n z_u\|_K \\
&\le C h_K^2 \|\mathrm{D}^2 z_u\|_{Q_K^n} + C k_n \|\partial_t z_u\|_{Q_K^n} + |\mathcal{R}^{(2)}|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\langle u - U, g\rangle_Q| &\le \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ \rho_{u,K}\omega_{u,K} + \rho_{w,K}\omega_{w,K} + \rho_K\omega_K \Big\} \\
&\le \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{ C h_K^2 \rho_{u,K} \|\mathrm{D}^2 z_u\|_{Q_K^n} + C h_K^2 \rho_{w,K}\|\mathrm{D}^2 z_w\|_{Q_K^n} \\
&\quad + \rho_K (C h_K^2 \|\mathrm{D}^2 z_u\|_{Q_K^n} + C k_n \|\partial_t z_u\|_{Q_K^n}) \Big\}
\end{aligned}
$$

and the desired estimate (4.8) follows by the Cauchy-Schwartz inequality

$$
\begin{aligned}
&\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} h_K^2 \rho_{u,K} \|\mathrm{D}^2 z_u\|_{Q_K^n} \\
&\qquad \le \Big(\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} h_K^4 \rho_{u,K}^2 \Big)^{\frac12} \Big(\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} h_K^2 \rho_{u,K} \|\mathrm{D}^2 z_u\|_{Q_K^n}^2 \Big)^{\frac12} \\
&\qquad = \Big(\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} h_K^4 \rho_{u,K}^2 \Big)^{\frac12} \|\mathrm{D}^2 z_u\|_Q \le C_S \Big(\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} h_K^4 \rho_{u,K}^2 \Big)^{\frac12} \|g\|_Q,
\end{aligned}
$$

and similarly for the other terms.

Part (ii). The previous bound for $\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \rho_{u,K}\omega_{u,K}$ applies here also. Consider then

$$
\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \rho_{w,K}\omega_{w,K} \le \sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \rho_{w,K} C h_K^2 \|\mathrm{D}^2 z_w\|_{Q_K^n} + \sum_{K\in\mathbf{T}_N} \rho_{w,K}\omega_{w,K}.
$$

Here,

$$
\begin{aligned}
&\sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \rho_{w,K} C h_K^2 \|\mathrm{D}^2 z_w\|_{Q_K^n} \\
&\qquad = \sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \rho_{w,K} C h_K^2 \|\sigma^{-\frac12}\sigma^{\frac12}\mathrm{D}^2 z_w\|_{Q_K^n} \\
&\qquad \le C \sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \rho_{w,K} \sigma_n^{-\frac12} h_K^2 \|\sigma^{\frac12}\mathrm{D}^2 z_w\|_{Q_K^n} \\
&\qquad \le C \Big(\sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \sigma_n^{-1} h_K^4 \rho_{w,K}^2 \Big)^{\frac12} \Big(\sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \|\sigma^{\frac12}\mathrm{D}^2 z_w\|_{Q_K^n}^2 \Big)^{\frac12} \\
&\qquad \le C \Big(\sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \sigma_n^{-1} h_K^4 \rho_{w,K}^2 \Big)^{\frac12} \|\sigma_n^{\frac12}\mathrm{D}^2 z_w\|_Q \\
&\qquad \le C_S C \Big(\sum_{n=1}^{N-1} \sum_{K\in\mathbf{T}_n} \sigma_n^{-1} h_K^4 \rho_{w,K}^2 \Big)^{\frac12} \|g_T\|_\Omega.
\end{aligned}
$$

The term with $n = N$ is special. We go back to (4.3) and replace it by

$$\begin{aligned}\sum_{K\in\mathbf{T}_N} \langle R_w, z_w - \pi_N z_w\rangle_{Q_K^N} &= \sum_{K\in\mathbf{T}_N} \Big\langle R_w, (I-\pi_N)\int_{I_N} z_w \,\mathrm{d}t\Big\rangle_K \\ &\le \sum_{K\in\mathbf{T}_N} \|R_w\|_K C h_K^2 \Big\| \mathrm{D}^2 \int_{I_N} z_w \,\mathrm{d}t\Big\|_K.\end{aligned}$$

Here, by the regularity estimate (4.7), $\epsilon\Delta z_w = \partial_t z_u + f'(u) z_w$ from the first equation in (3.8), and $\|f'(u)\|_{L_\infty} \le \beta$, we have

$$\begin{aligned}\Big\|\mathrm{D}^2 \int_{I_N} z_w \,\mathrm{d}t\Big\|_K &\le C \Big\| \int_{I_N} \Delta z_w \,\mathrm{d}t\Big\|_K \\ &= C\epsilon^{-1} \Big\| \int_{I_N} (\partial_t z_u + f'(u) z_w)\,\mathrm{d}t\Big\|_K \\ &\le C\epsilon^{-1}\Big(\|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + \beta k_N^{\frac12} \|z_w\|_{Q_K^N}\Big).\end{aligned}$$

Hence, since $\rho_{w,K} = \|R_w\|_{Q_K^N} = k_N^{\frac12}\|R_w\|_K$, we have

$$\begin{aligned}&\sum_{K\in\mathbf{T}_N} \langle R_w, z_w - \pi_N z_w\rangle_{Q_K^N} \\ &\quad\le \sum_{K\in\mathbf{T}_N} \|R_w\|_K C h_K^2 \epsilon^{-1}\Big(\|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + k_N^{\frac12}\|z_w\|_{Q_K^N}\Big) \\ &\quad= C\epsilon^{-1} \sum_{K\in\mathbf{T}_N} k_N^{-\frac12} h_K^2 \rho_{w,K}\Big(\|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + k_N^{\frac12}\|z_w\|_{Q_K^N}\Big) \\ &\quad\le C\epsilon^{-1}\Big(\sum_{K\in\mathbf{T}_N} k_N^{-1} h_K^4 \rho_{w,K}^2\Big)^{\frac12} \Big(\|z_u(t_N)\|_\Omega + \|z_u(t_{N-1})\|_\Omega + k_N^{\frac12}\|z_w\|_Q\Big) \\ &\quad\le C\epsilon^{-1} C_S \|g_T\|_\Omega \Big(\sum_{K\in\mathbf{T}_N} \sigma_N^{-1} h_K^4 \rho_{w,K}^2\Big)^{\frac12},\end{aligned}$$

where we have used $\sigma_N = k_N$. So we have

$$\sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \rho_{w,K}\omega_{w,K} \le C C_S \|g_T\|_\Omega \Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n} \sigma_n^{-1} h_K^4 \rho_{w,K}^2\Big)^{\frac12}. \tag{4.10}$$

Now we compute $\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\rho_K\omega_K$. For $K\in\mathbf{T}_N$ we use

$$\begin{aligned}\omega_K &= k_N^{\frac{1}{2}}\|(z_u-\pi z_u)_{N-1}^+\|_K\\ &\le k_N^{\frac{1}{2}}\|P_N(z_u-\pi_N z_u)\|_K + k_N^{\frac{1}{2}}\|z_u(t_{N-1})-P_N z_u\|_K\\ &= \|P_N(z_u-\pi_N z_u)\|_{Q_K^N} + k_N^{\frac{1}{2}}\|z_u(t_{N-1})-P_N z_u\|_K\\ &\le Ch_K^2\|\mathrm{D}^2 z_u\|_{Q_K^N} + k_N^{\frac{1}{2}}\|z_u(t_{N-1})-P_N z_u\|_K.\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\rho_K\omega_K\\ &\quad= C\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\rho_K h_K^2\|\mathrm{D}^2 z_u\|_{Q_K^n} + C\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\rho_K k_n\sigma_n^{-\frac{1}{2}}\|\sigma^{\frac{1}{2}}\partial_t z_u\|_{Q_K^n}\\ &\qquad+\sum_{K\in\mathbf{T}_N}\rho_K k_N^{\frac{1}{2}}\|z_u(t_{N-1})-P_N z_u\|_K\\ &\quad\le C\Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}h_K^4\rho_K^2\Big)^{\frac{1}{2}}\|\mathrm{D}^2 z_u\|_Q + C\Big(\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\rho_K^2 k_n^2\sigma_n^{-1}\Big)^{\frac{1}{2}}\|\sigma^{\frac{1}{2}}\partial_t z_u\|_Q\\ &\qquad+C\Big(\sum_{K\in\mathbf{T}_N}k_N\rho_K^2\Big)^{\frac{1}{2}}\|z_u(t_{N-1})-P_N z_u\|_\Omega.\end{aligned}$$

Using $\sigma_N = k_N$ and

$$\|z_u(t_{N-1})-P_N z_u\|_\Omega \le 2\max_I\|z_u\|_\Omega \le 2C_S\|g_T\|_\Omega,$$

gives

$$\begin{aligned}\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\rho_K\omega_K &\le C\Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}h_K^4\rho_K^2\Big)^{\frac{1}{2}}C_S\|g_T\|_\Omega\\ &\quad+C\Big(\sum_{n=1}^{N-1}\sum_{K\in\mathbf{T}_n}\rho_K^2 k_n^2\sigma_n^{-1}\Big)^{\frac{1}{2}}C_S\|g_T\|_\Omega + C\Big(\sum_{K\in\mathbf{T}_N}k_N\rho_K^2\Big)^{\frac{1}{2}}C_S\|g_T\|_\Omega\\ &= CC_S\Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}h_K^4\rho_K^2\Big)^{\frac{1}{2}}\|g_T\|_\Omega + CC_S\Big(\sum_{n=1}^{N}\sum_{K\in\mathbf{T}_n}\rho_K^2 k_n^2\sigma_n^{-1}\Big)^{\frac{1}{2}}\|g_T\|_\Omega.\end{aligned}$$

This completes the proof.

□

Finally, we prove a priori bounds for the stability constants C_S.

Theorem 4.4. *Assume that* $\|f'(u)\|_{L_\infty(Q)} \le \beta$ *and* $\epsilon \in (0,1]$ *and that* (4.7) *holds. Then the solution of* (3.8) *admits the following a priori bounds, where* $C = C(\beta)$. *If* $g_T = 0$, *then*

$$\|\mathrm{D}^2 z_u\|_Q^2 + \|\partial_t z_u\|_Q^2 + \epsilon^2 \|\mathrm{D}^2 z_w\|_Q^2 \le C\|g\|_Q^2 \mathrm{e}^{C\epsilon^{-1}T}. \tag{4.11}$$

If $g = 0$, *then, with* $\sigma(t) = T - t$,

$$\begin{aligned} &\epsilon^{-1}\max_I \|z_u\|_\Omega^2 + \|z_w\|_Q^2 + \|\mathrm{D}^2 z_u\|_Q^2 + \|\sigma^{\frac12}\partial_t z_u\|_Q^2 + \epsilon^2\|\sigma^{\frac12}\mathrm{D}^2 z_w\|_Q^2 \\ &\qquad \le C\epsilon^{-1}\|g_T\|_\Omega^2 \mathrm{e}^{C\epsilon^{-1}T}. \end{aligned} \tag{4.12}$$

Proof. We first estimate $\|z_w\|_Q^2$. To this end we use $\Delta z_u = z_w$ from the second equation of (3.8) to get

$$\langle \Delta z_w, z_u\rangle_\Omega = \langle z_w, \Delta z_u\rangle_\Omega = \|z_w\|_\Omega^2.$$

Then we multiply the first equation of (3.8) by z_u, and integrate over $[t,T]$,

$$\int_t^T \langle -\partial_t z_u, z_u\rangle_\Omega \,\mathrm{d}s + \epsilon\int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s - \int_t^T \langle f'(u) z_w, z_u\rangle_\Omega \,\mathrm{d}s = \int_t^T \langle g, z_u\rangle_\Omega \,\mathrm{d}s.$$

By assumption we know that $\|f'(u)\|_{L_\infty(Q)} \le \beta$, so we have

$$\begin{aligned} &\tfrac12\|z_u(t)\|_\Omega^2 - \tfrac12\|z_u(T)\|_\Omega^2 + \epsilon\int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s \\ &\quad\le \int_t^T \|f'(u)\|_{L_\infty(Q)}\|z_w\|_\Omega\|z_u\|_\Omega \,\mathrm{d}s + \int_t^T \|g\|_\Omega\|z_u\|_\Omega \,\mathrm{d}s \\ &\quad\le \int_t^T \big(\tfrac{\beta^2}{2\epsilon}\|z_u\|_\Omega^2 + \tfrac{\epsilon}{2}\|z_w\|_\Omega^2\big)\,\mathrm{d}s + \int_t^T \big(\tfrac{c}{2}\|g\|_\Omega^2 + \tfrac{1}{2c}\|z_u\|_\Omega^2\big)\,\mathrm{d}s \\ &\quad\le \tfrac{\beta^2}{\epsilon}\int_t^T \|z_u\|_\Omega^2 \,\mathrm{d}s + \tfrac{\epsilon}{2}\int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s + \int_t^T \big(\tfrac{c}{2}\|g\|_\Omega^2 + \tfrac{1}{2c}\|z_u\|_\Omega^2\big)\,\mathrm{d}s. \end{aligned}$$

Hence, with $z_u(T) = g_T$ and $c = \frac{\epsilon}{\beta^2}$,

$$\begin{aligned} &\|z_u(t)\|_\Omega^2 + \epsilon\int_t^T \|z_w\|_\Omega^2 \,\mathrm{d}s \\ &\qquad\le \frac{\epsilon}{\beta^2}\|g\|_Q^2 + \|g_T\|_\Omega^2 + 2\beta^2\epsilon^{-1}\int_t^T \|z_u\|_\Omega^2 \,\mathrm{d}s \\ &\qquad\le \tfrac{C}{\epsilon}\|g\|_Q^2 + \|g_T\|_\Omega^2 + C\epsilon^{-1}\int_t^T \|z_u\|_\Omega^2 \,\mathrm{d}s. \end{aligned}$$

Define

$$\Phi(t) = \|z_u(t)\|_\Omega^2 + \epsilon\int_t^T \|z_w(s)\|_\Omega^2 \,\mathrm{d}s.$$

Obviously we have $\|z_u(s)\|_\Omega^2 \le \Phi(s)$, so that

$$\Phi(t) \le C\epsilon\|g\|_Q^2 + \|g_T\|_\Omega^2 + C\epsilon^{-1}\int_t^T \Phi(s)\,\mathrm{d}s.$$

We apply Gronwall's lemma to get

$$\Phi(t) \le C(\epsilon\|g\|_Q^2 + \|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}(T-t)}.$$

This means

$$\|z_u(t)\|_\Omega^2 + \epsilon\int_t^T \|z_w\|_\Omega^2\,\mathrm{d}s \le C(\epsilon\|g\|_Q^2 + \|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}(T-t)}.$$

We conclude

$$\max_I \|z_u\|_\Omega^2 \le C(\epsilon\|g\|_Q^2 + \|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}T}.$$

$$\|z_w\|_Q^2 \le C(\|g\|_Q^2 + \epsilon^{-1}\|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}T}. \tag{4.13}$$

From the second equation we know $z_w = \Delta z_u$. So, by (4.7) and (4.13),

$$\|\mathrm{D}^2 z_u\|_Q^2 \le C\|\Delta z_u\|_Q^2 = C\|z_w\|_Q^2 \le C(\|g\|_Q^2 + \epsilon^{-1}\|g_T\|_\Omega^2)\mathrm{e}^{C\epsilon^{-1}T}. \tag{4.14}$$

This takes care of the first terms in (4.11) and (4.12).

Now assume that $g_T = 0$. Consider the dual problem (3.8) and multiply the first equation by $-\partial_t z_u$ and integrate over Q to get

$$\langle \partial_t z_u, \partial_t z_u\rangle_Q - \epsilon\langle \Delta z_w, \partial_t z_u\rangle_Q - \langle f'(u) z_w, \partial_t z_u\rangle_Q = -\langle g, \partial_t z_u\rangle_Q. \tag{4.15}$$

So, by using $z_w = \Delta z_u$ from the second equation, we get

$$\langle \Delta z_w, \partial_t z_u\rangle_Q = \langle z_w, \partial_t \Delta z_u\rangle_Q = \langle \Delta z_u, \partial_t \Delta z_u\rangle_Q = \tfrac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\|\Delta z_u\|_\Omega^2\,\mathrm{d}t.$$

By putting this in (4.15) and using that $\|f'(u)\|_{L_\infty(Q)} \le \beta$, we have

$$\begin{aligned}
\|\partial_t z_u\|_Q^2 &- \tfrac{\epsilon}{2}\|\Delta z_u(T)\|_\Omega^2 + \tfrac{\epsilon}{2}\|\Delta z_u(0)\|_\Omega^2 \\
&\le \|f'(u)\|_{L_\infty(Q)}\|z_w\|_Q\|\partial_t z_u\|_Q + \|g\|_Q\|\partial_t z_u\|_Q \\
&\le \tfrac{c\beta^2}{2}\|z_w\|_Q^2 + \tfrac{1}{2c}\|\partial_t z_u\|_Q^2 + \tfrac{c}{2}\|g\|_Q^2 + \tfrac{1}{2c}\|\partial_t z_u\|_Q^2.
\end{aligned}$$

Put $c = 2$ and kick back $\|\partial_t z_u\|_Q^2$ to get, with $z_u(T) = g_T = 0$,

$$\tfrac{1}{2}\|\partial_t z_u\|_Q^2 + \tfrac{\epsilon}{2}\|\Delta z_u(0)\|_\Omega^2 \le \beta^2\|z_w\|_Q^2 + \|g\|_Q^2.$$

Hence, by (4.13) with $C = C(\beta)$,

$$\|\partial_t z_u\|_Q^2 \le C\|z_w\|_Q^2 + C\|g\|_Q^2 \le C\|g\|_Q^2\mathrm{e}^{-C\epsilon^{-1}T}. \tag{4.16}$$

It remains to bound $\|\mathrm{D}^2 z_w\|_Q^2$. From the first equation of (3.8) we get $\epsilon\Delta z_w = g + \partial_t z_u + f'(u) z_w$. Taking norms and using (4.7), (4.13), and (4.16) gives

$$\begin{aligned}\epsilon^2\|\mathrm{D}^2 z_w\|_Q^2 &\le \epsilon^2 C\|\Delta z_w\|_Q^2 = C\|g + \partial_t z_u + f'(u) z_w\|_Q^2\\ &\le C\Big(\|g\|_Q^2 + \|\partial_t z_u\|_Q^2 + \|f'(u)\|_{L_\infty(Q)}^2\|z_w\|_Q^2\Big)\\ &\le C\|g\|_Q^2 \mathrm{e}^{C\epsilon^{-1}T}.\end{aligned}$$

This completes the proof of (4.11)

Now let $g = 0$ and set $\sigma(t) = T - t$. Multiply the first equation of (3.8) by $-\sigma\partial_t z_u$ to get

$$\langle\partial_t z_u, \sigma\partial_t z_u\rangle_Q - \epsilon\langle\Delta z_w, \sigma\partial_t z_u\rangle_Q - \langle f'(u) z_w, \sigma\partial_t z_u\rangle_Q = 0.$$

Here, since $z_w = \Delta z_u$ and $\sigma'(t) = -1$,

$$\begin{aligned}\langle\Delta z_w, \sigma\partial_t z_u\rangle_Q &= \langle z_w, \sigma\Delta\partial_t z_u\rangle_Q\\ &= \langle\Delta z_u, \sigma\Delta\partial_t z_u\rangle_Q\\ &= \tfrac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(\sigma\|\Delta z_u\|_\Omega^2)\,\mathrm{d}t - \tfrac{1}{2}\int_0^T \sigma'\|\Delta z_u\|_\Omega^2\,\mathrm{d}t\\ &= \tfrac{1}{2}\sigma(T)\|\Delta z_u(T)\|_\Omega^2 - \tfrac{1}{2}\sigma(0)\|\Delta z_u(0)\|_\Omega^2 + \tfrac{1}{2}\int_0^T \|z_w\|_\Omega^2\,\mathrm{d}t\\ &= -\tfrac{1}{2}T\|\Delta z_u(0)\|_\Omega^2 + \tfrac{1}{2}\|z_w\|_Q^2.\end{aligned}$$

Hence,

$$\begin{aligned}\|\sigma^{\frac{1}{2}}\partial_t z_w\|_Q^2 + \|\Delta z_u(0)\|_\Omega^2 &\le \tfrac{\epsilon}{2}\|z_w\|_Q^2 + \|f'(u)\|_{L_\infty}\|\sigma^{\frac{1}{2}} z_w\|_Q\|\sigma^{\frac{1}{2}}\partial_t z_u\|_Q\\ &\le \tfrac{1}{2}(\epsilon + \beta^2 T)\|z_w\|_Q^2 + \tfrac{1}{2}\|\sigma^{\frac{1}{2}}\partial_t z_u\|_Q^2.\end{aligned}$$

So by (4.13) we have

$$\|\sigma^{\frac{1}{2}}\partial_t z_u\|_Q \le (\epsilon + \beta^2 T)\|z_w\|_Q^2 C\epsilon^{-1}\|g_T\|_\Omega^2 \mathrm{e}^{C\epsilon^{-1}T}.$$

Finally, from (4.7) and $\epsilon\Delta z_w = \partial_t z_u + f'(u) z_w$ we get

$$\begin{aligned}\epsilon^2\|\sigma^{\frac{1}{2}}\mathrm{D}^2 z_w\|_Q^2 &\le \epsilon^2 C\|\sigma^{\frac{1}{2}}\Delta z_w\|_Q^2 = C\|\sigma^{\frac{1}{2}}(\partial_t z_u + f'(u) z_w)\|_Q^2\\ &\le C\Big(\|\sigma^{\frac{1}{2}}\partial_t z_u\|_Q^2 + T\|z_w\|_Q^2\Big)\\ &\le C\epsilon^{-1}\|g_T\|_\Omega^2 \mathrm{e}^{C\epsilon^{-1}T}.\end{aligned}$$

This completes the proof of (4.12). □

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