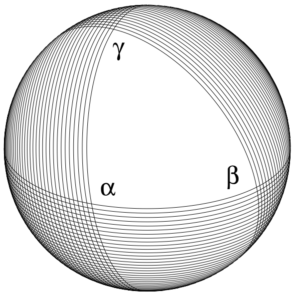
From Angular excess to Curvature, Euler characteristics and Gauss-Bonnet

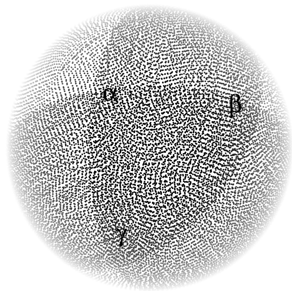
Let us consider the problem of computing the area of a spherical triangle. There are general ways of computing areas of various figures, depending on how they are presented. In the case of a spherical triangle one would naturally start out with a parametrization of a sphere, one traditional one being spherical coordinates (as we are familiar with from the geographical notions of longitudes and latitudes) and then we describe the area on the parameter base corresponding to the triangle and then we set up an integral and try and compute it. In a way it is all straightforward but in practice quite complicated and rather daunting and anyone who has had to evaluate integrals knows, exact expressions may be inaccessible. There is however another way that is surprisingly simple and ends up in giving a very beautiful expression relating the area of the triangle to its angles, and what is more, opening up a new vista of related problems and concepts. I do not claim that this is the way things historically evolved, but that is beside the point, it shows how one set of problems and concepts naturally leads to other problems and concepts, which is the way mathematics evolves. So let us start in an elementary way.

Given a set X let us denote by $\mu(X)$ the number of elements in X then for any two sets A, B we get that $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ because the elements of the intersection $A \cap B$ will be counted twice. The formula also holds for any μ which gives the 'size' of a set, provided it is additive, meaning $\mu(A \cup B) = \mu(A) + \mu(B)$ for any A, B disjoint. In particular this holds for areas. Note that this is from a logical point essentially tautologous, as the formula above is just a reformulation of additivity ($A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$ a union of three disjoint sets). The formula is easily generalized to three (or any number of sets). In fact

$$\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) - (\mu(A \cap B) + \mu(B \cap C) + \mu(C \cap A)) + \mu(A \cap B \cap C)$$

Now we consider a sphere of radius 1 (for the more general case of radius R we need only scale areas by R^2) its surface area is given by 4π which was shown already by Archimedes (a result which nowadays any student of elementary calculus can easily derive). Now as A, B, C we take hemispheres all obviously of area 2π. If we intersect two hemispheres we get a segment, bordered by two great circles (which could be thought of as meridians) which are an angle α apart and whose area obviously is 2α as the most natural measure of angle is in radians. Now the intersection of three hemispheres is a triangle T with angles α, β, γ and the union of three hemispheres is the whole sphere minus the antipodal triangle T which is congruent with T and has the same area.





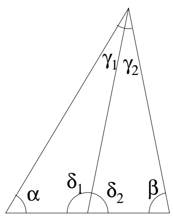
So putting everything into the equation we get

$$3 \cdot 2\pi - 2(\alpha + \beta + \gamma) + \mu(T) = 4\pi - \mu(T)$$

which simplifies to

$$\mu(T) = (\alpha + \beta + \gamma) - \pi$$

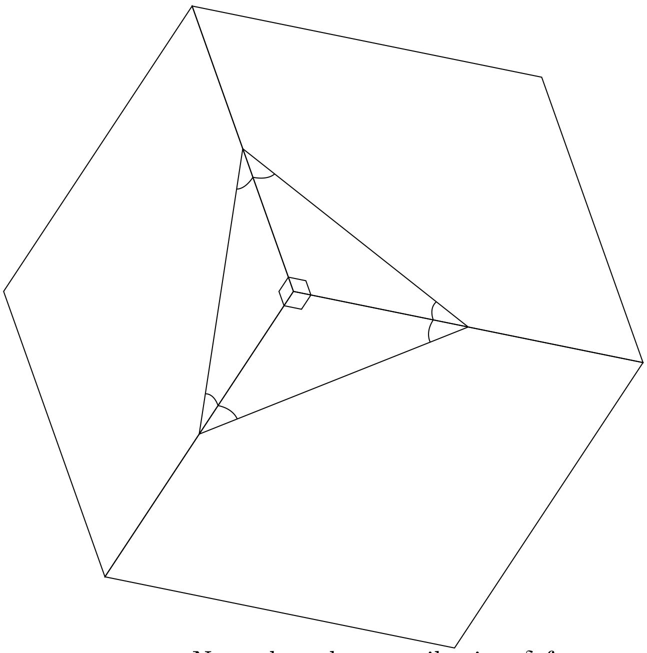
which is a striking and beautiful formula and also unexpected, the derivation of which is very simple yet touched with black magic. What does it all mean? The quantity $(\alpha+\beta+\gamma)-\pi$ is referred to angular excess, and on a sphere every triangle has an angular excess, and the bigger the triangle the larger the excess, in fact we have just shown that the angular excess is additive, as it is given by the area, but this can be shown directly.

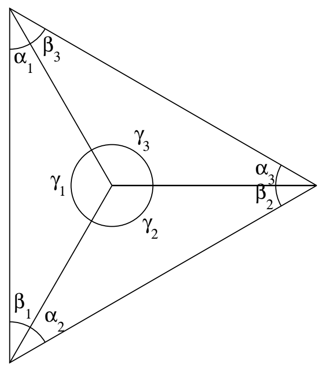


The two excesses are given by $\alpha + \delta_1 + \gamma_1 - \pi$ and $\beta + \delta_2 + \gamma_2 - \pi$ respectively. Adding them up we get $\alpha+\beta+(\gamma_1+\gamma_2)+(\delta_1+\delta_2-\pi)-\pi$ which simplifies by setting $\gamma = \gamma_1+\gamma_2$ to the angular excess $\alpha+\beta+\gamma-\pi$ of the big triangle.

But this fact reveals what may be really going on. Areas and angular excesses of spherical triangles seem to run on parallel tracks, maybe on the same? If every spherical triangle could be made up by special ones for which the two measures coincide we would be done. Such building blocks exist, namely the hemispheres (take three arbitrary points on it and we have a triangle where all the sides make up a straight line, and whose angular excess is $3 \cdot \pi - \pi = 2\pi$ which is also its area). However, a triangle cannot be made up as a union of such, but on the other hand as a complement of a union, and this was exactly what we exploited above, without really understanding that this was what we were doing. What was initially seen as black magic does turn out, after some exploration, as the most natural thing possible. The proof taken together with the after-sight (and in a presentation we could have started with the insight) does much more than verifies, it makes us understand why something is true. Without such, at least occasional flashes of understanding mathematics becomes a tedious game.

Now this idea of a measure can (and ought to) be further pursued. Look at a cube and triangles on it, they will all have angular excess of zero, unless they contain a corner, then the angular excess jumps to $\pi/2$. This is easy to verify. If the triangle would contain two corners, it would be twice that. In fact the angular excess is measured by the number of corners contained in it. This can be generalized to any polyhedron, and each corner will contribute a certain amount easily computed, and the angular excess is simply computed by adding up all the contributions from corners. This makes the additive structure very direct.



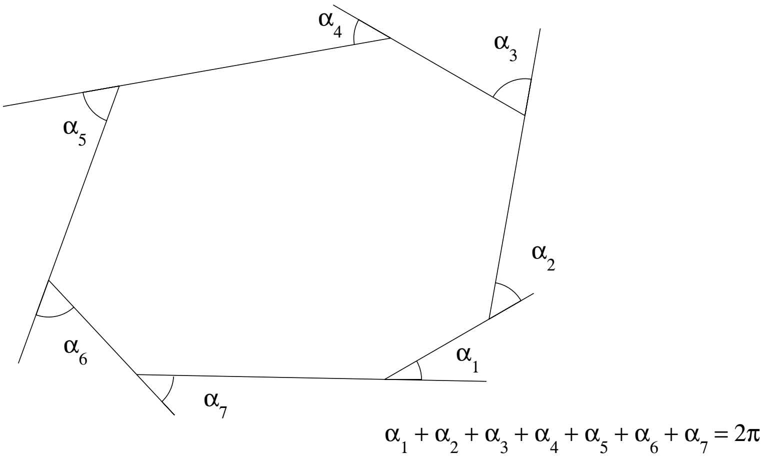


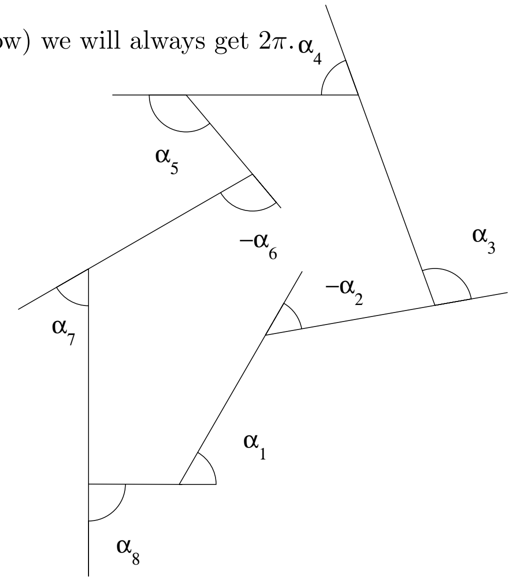
$$\alpha_\iota + \beta_\iota + \gamma_\iota = \pi$$

$$\sum_\iota \alpha_\iota + \beta_\iota = 3\pi - \sum_\iota \gamma_\iota = \pi + (2\pi - \textstyle\sum_\iota \gamma_\iota) = \pi + \delta$$

Note that the contribution δ from each corner is given by the defect $\delta = 2\pi - \sum_i \gamma_i$. Note that if C is a circle of radius r centered at the corner its circumference will be given by $(2\pi - \delta)r$ If $\delta = 0$ the corner is flat. If $\delta > 0$ we can call it an elliptic corner and the circumference is shorter than it 'should' be (i.e. in comparison with the flat case) and if $\delta < 0$ we can call it a hyperbolic one, and then the circumference is larger. Note that each corner will contribute the negative of its defect to the angular excess (which surrounding an hyperbolic corner will be negative).

We can go further. Consider a general polyhedron made out of triangles. Let us add all the angles of the triangles. This will obviously add up to $N\pi$ where N is the number of triangles, where we group them three by three, each triplet associated to a triangle. But we can also add them up corner by corner, and in this case we will get $2p\pi - \sum_i \delta_i$ where p is the number of corners and the δ_i refer to their respective contributions. Now for a polyhedron made up of triangles, we always have $2p - N = 4$ and thus the sum Δ of the contributions for any triangular polyhedron will be given by $\Delta = 4\pi$ a rather striking formula. There is an analogy in two-dimensions. Consider a polygon that does not intersect itself. It will then constitute a so called piece-wise linear approximation of the circle. If we then add up the exterior angles (with appropriate signs, plus at convex corners and negative at concave cf. figure below) we will always get 2π.

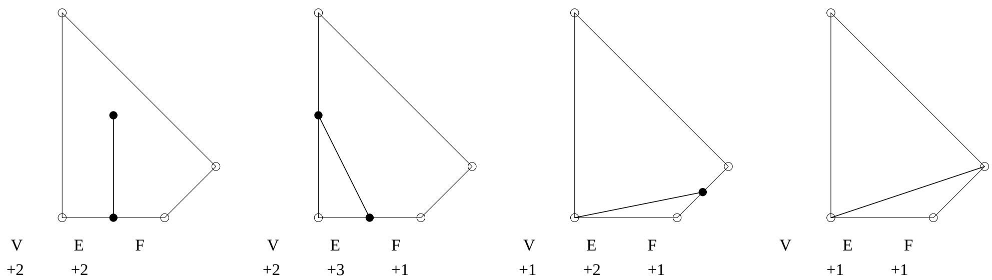




This is very intuitive as we have transversed the polygon we have turned around once. The directions at which the sides point will change at each corner with the amount of the corresponding external angle (taking into account the signs). However, if the polygon would be on a sphere, this would no longer be the case. The sum of the external angles will not add up to 2π but to $2\pi - \delta$ where δ is the angular excess of the interior of the polygon, in other words its area (as always we tacitly assume that our sphere has radius one). This may appear counter-intuitive. It has to do with the notion of parallel directions (vectors), in Euclidean space this is no problem as we just take the corresponding lines to be parallel, but there are no parallel lines on a sphere (two great circles always intersect in two antipodal points), instead we have the notion of parallel transport. Given two points p, q we can take the arc $L_{p,q}$ from p to q and if v is a vector at p, of if you prefer a short line segment, you choose a segment w at q making the same angle with L_{pq} at q as v at p and call that the parallel to v. The intuitive idea is that we have moved v to q without turning it (it points to the same direction). Note we could have made the same definition in the Euclidean space using a line joining the two points, but as noted there is no need for it. Formally the definition is based on choosing the great circle joining the two points, but could there be another definition? In Euclidean geometry two lines being parallel is an equivalence relation, which allows us to speak about directions, in particular it is transitive. Our definition of parallelity on the sphere is obviously reflexive and symmetric but it is not transitive. If we have three points p, q, r we can transport a vector v at p to a vector w at q, but we could also move v to a parallel vector u at r and that to a vector t at q but then t will not be the same as w. Equivalently we could move the vector v along the perimeter of the spherical triangle defined by p, q, r and find out that the direction of v has changed with the amount given by the angular excess[1].

The reader may ask how do we know that for a polyhedron with triangular sides, or as we mathematicians say, faces. The number p of vertices (corners) and number N of faces satisfy $2p - N = 4$? This can readily be verified on the tetrahedron, octahedra and icosahedra, but that is hardly a proof. Now in our special case of triangular polyhedra we notice that the number of edges (L) is going to be given by $3N/2$ (every face has three sides (edges) and every edge will be counted twice in that way). Then we can rewrite $2p-N = 2(p-E+N)$ and this innocent looking expression $(p-E+N)$ turns out to be amazingly powerful. In fact given any polyhedron, no matter the shape of the faces, we will have $p - E + N = 2$ as the reader can verify to his hearts content by considering all kinds of polyhedra he can imagine. But why is that true?

For simplicity assume that we can fit a small sphere inside the polyhedron such that all rays from its center intersect the polyhedron in just one point. In this way the vertices and edges on the polyhedron will correspond to vertices and arcs of great circles joining them, forming polygons in the sphere corresponding to the faces of the polyhedron. We get a combinatorial structure on the sphere, which is a spherical graph (denote it by Γ_1), meaning a graph on the sphere consisting of edges and vertices (the faces we get for free). If we have another polyhedron we will likewise get another graph Γ_2 and the union $\Gamma_1 \cup \Gamma_2$ will be a refinement of both. A refinement of a graph means adding new edges and vertices, creating as well new faces. This can be achieved by a succession of elementary steps as indicated below



Each step changing the number of vertices, edges and faces, but not their alternating sum. Hence this alternating sum will be the same for all polyhedra. This combinatorial invariant is called the Euler characteristics, although it goes further back then Euler, and is a very fundamental concept in mathematics. Polyhedra may look very different, but one thing they have in common is that they can be deformed to a sphere without tearing (think of them as being made out of very pliable rubber). One says that they are topologically equivalent, or using a more technical term homeomorphic, which requires a rather technical definition to be amenable to a precise mathematical treatment. We can now turn things on its head and define the Euler characteristic $e(X)$ of a topological space (which we have not defined but think of spheres, tori etc) to be a topological

[1] In fact it is impossible to find a notion of parallel vectors without this phenomenon, as it turns out that you cannot comb a sphere without hair sticking up at some point.

invariant (i.e. topologically equivalent spaces should have the same invariant) that satisfies the following axioms.

- $e(p) = 1$ where p is a point
- $e(P \cup Q) = e(P) + e(Q) - e(P \cap Q)$ (finite additivity)
- $e(P \times Q) = e(P) \cdot e(Q)$

Does such an invariant exist? In other words there are no contradictions? (Meaning that by dividing the space up in two different ways we get different results). Leave this question wisely aside! Let us first derive some consequences[2]. We have a $2:1$ map from the circle C onto itself thus $e(C) = 2e(C)$ and hence $e(C) = 0$[3]. Now take away a point from a circle and we have an open segment I with $e(I) = -1$, thus an open square $X = I \times I$ we get $e(X) = 1$ and hence also for a disc (with or without its boundary) or more generally for any polygon.

Now the Euler characteristics should be thought of a very versatile counting function. and the axioms indicates how it should be computed from a decomposition into simpler parts, thus to compute the number necessarily requires as well as forces an understanding of how a space is built up. Take away the poles of the spherical earth (remember we are thinking topologically so we do not have to worry about metric irregularities) what we have left can be thought of as the equator (a circle C) times an open interval I hence Euler characteristics zero. Thus the sphere has Euler characteristic two. A torus can be thought of as the product of two circles hence Euler characteristics zero. Take away a disc from a torus and you get a so called handle, which will have Euler-characteristics -1. Take away N disjoint discs from a sphere and along each of the circular borders of the ensuing holes, attach a handle in the obvious way. We then get a surface of eulercharacteristic $(2-N)+(-N) = 2-2N$ where N is the number of 'holes' in the surface.

We have now very elementary building blocks consisting of points, lines, and discs, and if we are interested in surfaces which can be built up by a finite number of such blocks, it will mean that the surface comes with a graph, consisting of the edges along which the discs (polygons), are glued to each other. The Euler characteristics of such a surface must by necessity be given by $V - E + F$ corresponding to the Euler characteristics of vertices, (open) edges, and faces given by $1, -1, 1$ respectively. Thus there is a deeper reason that the combinatorial invariant works and is the right one.

Now we can tie the ends together. Consider some surface, it could be some deformed sphere or a cup with a handle. We might now want to make sense

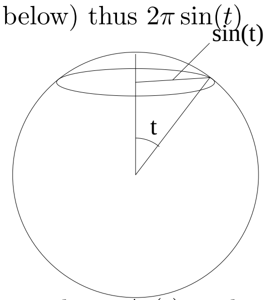
[2] Incidentally the fist axiom can be weakened to $e(p) \neq 0$ as $p \times p = p$ and hence $e(p)^2 = e(p \times p) = e(p)$ hence $e(p) = 1$. This is pedantry, but it points out that the third axiom forces a normalization

[3] If the circle is represented by the points $C(\theta) = (\cos\theta, \sin\theta)$ look at the map $C(\theta) \mapsto C(2\theta)$ which identifies antipodal points $C(\theta), C(\theta+\pi)$. If we let $\mathbb{Z}_2$ denote a space of two points (by additivity we have $e(\mathbb{Z}_2) = 2$) it is, however, not true that $C = C \times \mathbb{Z}_2$ as the right side denotes two disjoint circles, so we cannot use the third axiom directly. But think of a Moebius strip whose boundary is actually a connected circle. Cut it and it splits into two disjoint segments, thus $e(C) - 2 = 2(e(C) - 1)$ from which follows $e(C) = 2e(C)$.

of what is meant by a straight line (a so called geodesic) on a surface. If it is a plane it is obvious, and for a sphere we are talking about great circles. With an obvious caveat those will provide the shortest distances between two points, which of course begs the question of what will be distances in general on a surface. On a surface which is made up flat polygons, a geodesic is easy to describe, it will be made up by straight lines on the faces which when they meet at an edge have to satisfy a certain simple condition. Could we approximate a given surface with surfaces made up by flat pieces and go to the limit? This is a time-honored approach of mathematicians since the time of Archimedes. But let us assume that we have solved this problem. Could we define something like curvature at each point so that when we integrate this curvature over a triangle (assumed to have geodesic edges) we get the angular excess (or defect). This is true for spheres, but can we make it more general? One cheap way of achieving this would be to define the curvature at a given point to be the limit of the quotient of the angular excess and the area of small triangles converging to the point. If so we would end up, basically tautologically that the integral of the curvature of a surface X would be $2\pi e(X)$. For a sphere of radius 1 would give us that its area is indeed 4π. The point is though that no matter how the curvature κ it is defined we would have

$$\iint_X \kappa = 2\pi e(X)$$

which is a beautiful formula, known as Gauss-Bonnet. It really becomes interesting if we have an independent definition of curvature, which is what Gauss came up with in the early 19th century. Recall that at corner points of a polyhedron, the angles around the point did not add up to the customary 2π which could also be expressed as the length of a circumference of a circle of radius r is not the expected $2\pi r$. If we look at a circle of radius t on the unit sphere sphere it will be the same as that of the circle of radius $\sin(t)$ in Euclidean space (see below) thus $2\pi \sin(t)$



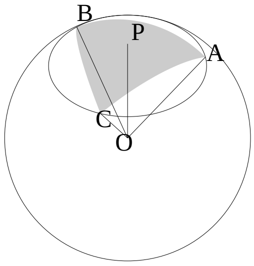
Now $\sin(t) = t - \frac{1}{3}t^3 + \frac{1}{120}t^4 \dots$ so we see that the smaller t is the better does the circle approximate that of the plane [4]. One may also compute the area of the circle which will be given by $2\pi(1 - \cos(t) = \pi t^2 - \pi \frac{t^4}{12}$. This give the clue to a definition of curvature as the limit

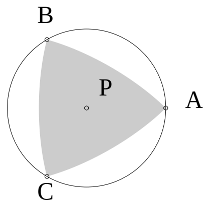
$$\lim_{t \to 0} \frac{A_0(t) - A(t)}{A_0 t^2}$$

where $A_0(t)$ is the area of a circle of radius t in Euclidean space, and $A(t)$ the area of a circle of radius t on the surface. Applied to the unit sphere we get 1 as desired.

[4]There will always be a 'bulge' but it will rapidly flatten out. On a human scale, an individual say 2 meters tall standing on level ground would see the horizon 5km away. That he cannot see the ground beyond that modest distance is due to the curvature of the Earth. If we would slice off that part of the bulge, he would only sink 3 cm in comparison with inevitable irregularities, this discrepancy would be dwarfed, no matter how level we would try to make the ground.

We will now make two local calculations. They would not have been part of the tool-kit of the Ancient Greeks but is nowadays expected of anyone with a modicum of mathematical education (say that of the first year of modern university studies). Consider on the unit sphere three points (A, B, C) equidistantly positioned on a small circle of radius t centered around a point O.





Now we can give 3-dimensional Cartesian coordinates for them namely
$P = (0, 0, 1)$
$A = (\sin(t), 0, \cos(t))$
$B = (-\frac{1}{2}\sin(t), \frac{\sqrt{3}}{2}\sin(t), \cos(t))$
$C = (-\frac{1}{2}\sin(t), -\frac{\sqrt{3}}{2}\sin(t), \cos(t))$

The great arc joining A and B is given by the intersection of the plane that goes through A, B and the origin. Similarly for the arc given from A to C. We would now like to compute the angle with which those two planes intersect, which is the same as the angle made by the two normals. A normal to a plane is a vector orthogonal to all the vectors in the plane. Given two vectors V, W we can form its vector product $V \times W$ (something which was part of the standard curriculum in most high-schools around the world) which will be orthogonal both to V and W and can hence serve as a normal to the plane spanned by V, W. Now if we have to vectors X, Y we can express the angle θ between them by $< X \cdot Y >= |X||Y|\cos(\theta)$ where $< X \cdot Y >$ denotes the inner product of X, Y and $|X|, |Y|$ are the lengths of X, Y respectively, in particular $< X \cdot X >= |X|^2$ [5]. We can now write down normals to the planes AB and AC to be given by $(-\frac{\sqrt{3}}{2}\cos(t), -\frac{3}{2}\cos(t), \frac{\sqrt{3}}{2}\sin(t)$ and $(\frac{\sqrt{3}}{2}\cos(t), -\frac{3}{2}\cos(t), -\frac{\sqrt{3}}{2}\sin(t)$ (I have factored out $\sin(t)$ from the formula given by the previous footnote, for simplicity). We now take their inner-product and divide by the products of their lengths and end up with

$$\cos(\theta(t)) = \frac{\frac{9}{4}\cos^2(t) - \frac{3}{4}}{\frac{9}{4}\cos^2(t) + \frac{3}{4}}$$

through some simple trigonometric simplification due to $\cos^2(t) + \sin^2(t) = 1$.

[5] The inner product is a so called bi-linear form so $|X+Y|^2 =< (X+Y) \cdot (X+Y) >=< X \cdot X > +2 < X \cdot Y > + < Y \cdot Y >= |X|^2 + 2\cos(\theta)|X||Y| + |Y|^2$ which is just Pythagoras theorem when $\theta = \frac{\pi}{2}$ and thus provides the well-known generalization of it. The point is that $< X \cdot Y >$ is easily computed when X, Y are given in coordinate forms. In fact if $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3)$ then $< X \cdot Y >= x_1y_1 + x_2y_2 + x_3y_3$. Furthermore we can also write down a formula for the vector product $X \times Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$. It is easy to verify that $< (X \times Y) \cdot X >=< (X \times Y) \cdot Y >= 0$ (incidentally $|X \times Y| = \sin(\theta)|X||Y|$).

Now we are only interested in very small values of t so we can replace $\cos^2(t)$ by $1-t^2+..$ plus higher powers of t which can be ignored. This essentially goes back to Newton. We then get $\frac{6-9t^2}{12-9t^2}$ after some straightforward algebraic manipulation. Further development yields

$$\frac{6-9t^2}{12-9t^2} = \frac{1}{2}\frac{1-\frac{3}{2}t^2}{1-\frac{3}{4}t^2} = \frac{1}{2}(1-\frac{3}{2}t^2)(1+\frac{3}{4}t^2) = \frac{1}{2}(1-\frac{3}{4}t^2)$$

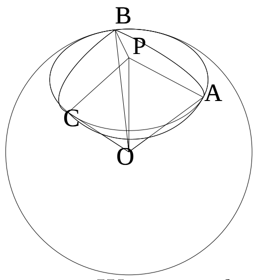
where the next to last step may warrant some explanation. We have in fact that $\frac{1}{1-t} = 1+t+..$ plus higher order terms (cf the sum of an infinite geometric series $\frac{1}{1-t} = 1+t+t^2+t^3+\dots$) so we just substitute $\frac{3}{4}t^2$ for t.

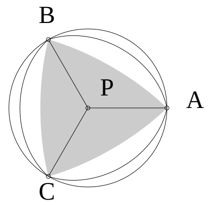
Now by the addition theorem for cosine (which all school children are expected to know[6]) we have that

$$\cos[\frac{\pi}{3}+u) = \cos(\frac{\pi}{3})\cos(u) - \sin(\frac{\pi}{3})\sin(u) = \frac{1}{2}(1-u^2) - \frac{\sqrt{3}}{2}u + ..$$

comparing with $\cos(\theta(t)) = \frac{1}{2} - \frac{3}{8}t^2$ we conclude that $\cos(\theta(t)) = \frac{\pi}{3}+u$ where $u = \frac{1}{4\sqrt{3}}$. The angular excess $3(\frac{\pi}{3}+u)-\pi = 3u = \frac{\sqrt{3}}{4}t^2$ If we then compare with the area of the triangle ABC we get $\frac{\sqrt{3}}{2}t\frac{3}{2}t = \frac{\sqrt{3}}{4}t^2$ and everything tallies. In fact had we not known the area A of a sphere, our initial argument would have yielded the formula $\frac{A}{4\pi}(\alpha+\beta+\gamma-\pi)$ and we would have obtained $A=4\pi$ without integration!

Finally we can compute the angles $APB = BPC = CPA$ denoting it by θ





We proceed as before and get
$v = (-\sin(t), 0, 1-\cos(t)), w = (\frac{1}{2}\sin(t), \frac{\sqrt{3}}{2}\sin(t), 1-\cos(t))$

$$\cos\theta = \frac{< v\cdot w >}{|v||w|} = \frac{-\frac{1}{2}\sin^2(t) + (1-\cos(t))^2}{2(1-\cos(t))}$$

Using well-known identities such as $1-\cos(t) = 2\sin^2(\frac{t}{2})$ and $\sin(t) = 2\sin(\frac{t}{2})\cos(\frac{t}{2})$ this can be simplified to $-\frac{1}{2}+\frac{3}{8}t^2+\dots$. Comparing with $\cos(\frac{2\pi}{3}+u) = \cos(\frac{2\pi}{3})\cos(u) - \sin(\frac{2\pi}{3})\sin(u) = -\frac{1}{2} - \frac{\sqrt{3}}{2}u + \dots$ we get $u = -\frac{1}{4\sqrt{3}}t^2+\dots$ Adding up the three angles we get $2\pi - \frac{\sqrt{3}}{4}t^2$ and hence on the polyhedron a circle with radius r is given by $(2\pi - \frac{\sqrt{3}}{4}t^2)\frac{r^2}{2}$. As $\sin(t) = r$ we can express

[6] $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$t = r + \frac{r^3}{6} + \dots$ and then get the area to $\pi r^2 - \frac{\sqrt{3}}{8} r^4 + \dots$ which does not tally with $\pi r^2 - \frac{\pi}{12} r^4 + \dots$ for the unit sphere. This shows that our naive idea of approximating with polyhedra does not work, the approximation is too crude, and other more powerful ideas are needed to show that the definition above is the one we want for curvature.

Now how should we interpret the preceeding discussion? A naive impression of mathematics is that one learns how to solve problems systematically as in school. For this and that problem one proceeds as follows etc. When it comes to compute the area of some region say on a sphere or more generally on any 'curvy' surface calculus students learn how to set up a specific integral; however to do this in practice is quite another matter and as everyone knows who has had a fleeting acquaintance with integration the answer seldom has a simple (in some conventional sense) answer. Specific examples, however, can be amenable to striking *ad hoc* solutions as illustrated by the example of spherical triangles. The simple solution, accessible to the Greeks and even to modern people with scant mathematical education, provided here is also striking, maybe even more so, to a professional mathematicians who encounters it for the first time, unaware of the existence of such a simple approach. The proof above gives an explanation for why something is true, but not necessarily a deeper understanding. A step towards the latter is provided by the observation that angular excess (or more generally angular discrepancy) is an additive property which ties in with a fundamental theme in mathematics. People in the humanities often make a point of contrasting mere explanation, as in the natural sciences, with the deeper notion of understanding as being the loftier concern of those engaged with the human spirit, without even bothering to make a clear distinction between the words. Here, in one field too boot, possible illustrations of the two concepts, have been presented. Furthermore discrete examples of the continuous concepts are presented and the additive countings are generalized strikingly into the topological category[7]. I could not resist a slight elaboration on the notion, illustrating on one hand a so called 'functorial' approach common in modern mathematics setting up a way of computing defined implicitly by its desired properties (the axioms) rather than by explicit construction. In fact the former actually indicates how it could be defined as to give the axioms a firm ground, but the actual construction is of less import and can be, using a Wittgensteinian metaphor, disposed of as a ladder when it has served its purpose. More significantly it shows how starting from one piece of interesting mathematics one becomes aware of its wide ramifications and how it can lead to unexpected things. What strikes the mathematician is the interconnectedness of mathematics, how different seemingly unrelated pieces have mutually benefi-

[7]The standard way students encounter euler numbers is as the alternating sum of so called betti-numbers, whose definitions are rather technical and part of graduate education on simplicial homology. The present author being no exception. Only as a more mature mathematician did I accidentally come across, during an ordinary lecture for students given by an eminent German mathematician, the notion of euler numbers, or euler characteristics as it also is called, as a counting function, which completely changed my outlook. Chance remarks tossed off can have profound implications also in mathematical intercourse.

cial applications[8]. The oft quoted line by William Blake of seeing the world in a grain of sand, has its special relevance to mathematics.

Noteworthy is the 'proof' of the Gauss-Bonnet formula supplied. This illustrates the fact that by using the right definitions one may render proofs trivial, not to say tautological. Such slick steps are usually taken at the end when a subject allows a birds-eye view and provide some sort of closure. However, there is a large amount of cheating involved and students who are predominantly brought up on such fare get a skewed view of the subject.

Finally I included two technical calculations, which to the uninitiated may seem impressive, if intimidating and maybe even repulsive. As far as a mathematician is concerned they are routine and he may be called upon to perform repeatedly in a day's work. They do correspond to tools part of an inevitable tool-kit a mathematician need to possess, just as plumber relies on his. In the first case it does modest duty by providing a round about computation of the area of a sphere, a curiosity which only a mathematician may savor, by computing the area of an infinitesimal spherical triangle, hence those truncated Taylor expansions, which may have puzzled the intermittent reader. Those expansions really correspond to computing with infinitesimals of different orders in a formally impeccable way. The second computation, to anchor the proof of Gauss-Bonnet led to a dead end, the common fate of most mathematical computations. Such computations normally do not entail understanding, being in the nature of manipulations, but reveal facts which can serve as the springboards to further developments.

[8] This is not something that occurs in the study of chess from the professional point of view. One can pose questions of chess of a mathematical nature but they would have scant interest to someone eager to improve his game.