From Here to Infinity

A Guide to Today's Mathematics

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The ambition of the book is big, namely to give an overview of modern mathematics. This is a risky business as the author has found out to his peril, thus each subsequent edition has seen major changes. The present one is from 1995, just after the Andrew Wiles proof. One wonders what a 2010 edition would be like.

Now it is not so easy to make a good selection of modern mathematics, because a selection it has to be. There are certain areas, or maybe rather features, that cannot really be ignored no matter what the mathematical taste of the author, other subjects may reflect it though, and will hence vary from author to author. It is here one expects the writing to be at its best, even if the choices may strike one as somewhat eccentric. Needless to say any such attempt is bound to meet with criticism, because no matter what every critic will find something crucial missing, the present one being no exception.

Now in a work like this written after the mid-nineties, the proof of the Fermat Theorem is bound to be included. It has caught the mathematical imagination of the public, or at least part of the public that has any. It is hard to motivate, unless motivation seems already intrinsic. If any it is a *tour de force* and plays up with the romantic notion of a mathematician isolating himself for seven years in his pursuit of fame, because surely this most modest and unassuming of mathematicians was by no means by far not devoid of a passionate ambition to set a mark in the history of mathematics. Indeed as we all later learned, the ambition to solve that mystery was with him since the age of ten. Indeed what an ironic twist of fate that he would drift into the kind of mathematics with which there would be an unexpected link. Now to do the story justice you need to delve into elliptic curves, but that is inevitable, and lucky enough elliptic curves have also accidentally turned up in one other very topical application, namely safe transmission of sensitive data. This is of course something else that cannot be sidestepped, esoteric number theory, seen for a long time as being immune to applications, turning up in the big way. Primes you can never avoid, especially not nowadays when large prime-numbers are being commercial entities almost. This leads to primality testing, and the rather striking fact that a number can be shown to be composite without there being any reasonable way of finding an explicit factor, which is of course the very meaning of the notion of composite. Something that should give food for thought to those logical-postivists who claim that the meaning of a claim lies in its verifiability. (But no one any longer claims to be one, but usually not for that reason). Now nothing wrong with prime-numbers, but from a mathematical point of view it can easily lead to marginal digressions. Non-Euclidean geometry is a sweet topic which no author can afford to ignore, although few make justice to it given the usual limitations on space. Stewart has also included a chapter on the ghosts of departed quantities and writes actually a good and illuminative account on non-standard analysis. Maybe too good and illuminative to be really appreciated by the non-professional mathematician. If not, it shows that a text may actually be profitably consulted by both. Stewart brings up some excellent philosophical point in the context, which deserves to be deliberated upon. By removing the archimedean axiom, which from a formal logical point of view belongs to second order predicate logic, as it is really expressed by an infinite axiomatic scheme, the model for the reals is no longer unique. From an intuitive point of view this is unsettling, because most mathematicians has a very strong physical intuition about the reals. The physical world is described by it, and times really flows continuously. Yet from a formal point of view non-standard analysis is very convenient because it justifies, at least in some rigid formal way, certain types of handwaving arguments people have been using heuristically. Now given the formal structure, such informal proofs can actually be translated in classical $\epsilon - \delta$ proofs, and the ultimate justification being that one may then be able to find rigorous classical proofs one would otherwise not have had the imagination of discovering. Or, as far as proofs merely serve verificational purposes, those can be handled much faster. Thus non-standard analysis seems to be dealing with some ghost-like object, a kind of fattening of the real line, which gives, as the saying goes, more elbow room. Similar approaches are already familiar from Algebraic geometry through the considerations of nilpotents, the introduction by Study of dual numbers, being the most elementary example.

Now the four-color problem is also a must, although it illustrates more than anything else the danger of dead-ends in mathematical inquiry, and thus makes us surprised that it does not happen more often in mathematics. The eventual settling of the claim disappointed all mathematicians, as they learned that it was just a case of reducing the conjecture to a large number of special cases and then going through these case by case in a computer. Where was the illumination? The liberating idea? If anything a dead-end. Mathematical facts are not interesting if we cannot understand why they are true.

Graph theory, so easy to motivate, gives rise to topology, and Stewart choses to spend most of it on knots, yet another rather easy subject to motivate and explain, especially since much of it can be reduced to graph-theory via flat representations and different types of crossings. The subject once in danger of becoming yet a dead-end also has intriguing connections to far more sophisticated mathematics, such as von Neumann algebras. Stewart, wisely or not, makes no attempt to explain what this is all about, suffices it for his purposes to say that it exists, that it is very hard, and that the connections turned out to be very unexpected.

Then there is Newton and the many-body problem. A digression into symmetries and groups, with occasions to touch upon the truly romantic story of Galois, romantic in spite of, or perhaps rather because of its tragedy, would be criminal not to engage in. Now this, as the name of Galois indicates, may profitably be introduced by the solving of equations of high degree. Anyone knows the solution of the quadratic equation, or although it seems to be on the borderline of the mathematical competence of the general public[1]. Thus even curious school-boys are drawn to the possibility of solving the cubic equation. Eventually

[1] How could anybody who is introduced to the elegant notion of completing a square not be caught by it. How many mathematicians like me constantly go through the process, never bothering to memorize the formula?

the topic can be made to include the classification of finite simple groups, yet another triumph of modern mathematics, and a truly collective one at that.

Chance and probability is another must, and no book giving a survey of mathematics can afford not to touch on it, even if not delving into the fascinating philosophical aspects. Everybody thinks of chance as something extra, something mysterious, but what is it really? In a strict mathematical treatment the mystery evaporates in a mundane way, but only in the mathematical sense. The application to the supposed real world remains, and the interface between the two is seldom pursued.

Chance and determinism seem to be totally opposed, yet even a deterministic process may show many of the features of randomness and unpredictability. Maybe more of an epistemological problem than an ontological, yet the phenomenon has been hyped up as chaos theory, the name reminiscent of catastrophe theory. One should never make light of the importance of a name. What would have been the general cultural impact of relativity theory if Einstein had gotten his way and called it invariant theory. It is really much more about invariance than relativity. The relativity it proposes is in principle not different from the relativity inherent in geometrical perspective.

Now fashionable trends in modern mathematics are mixed with classical topics, such as the complex numbers. Complex function theory is really the first time a student of mathematics encounters pure magic. This is something that is seldom conveyed in popular books, maybe it is left as a delightful surprise to the serious student. In a way the magic of complex analysis only becomes apparent to a prepared mind. A reader who will look upon mathematics as the one incomprehensible thing after another, will be incapacitated by the noise of his or her own confusion to detect it.

Fractal geometry is another very popular entry into modern mathematics. And Gdel and algorithms and computability, are really applied mathematics. Mathematics applied to its own reasoning. It is fascinating, but to a true mathematician not as fascinating as say the discovery of the complex world, or multi-dimensional geometries, and such bread and butter stuff of main-stream mathematics. It is, however, an aspect of mathematics, few if any mathematicians can ignore, due to its meta-mathematical nature. It provides problems which it is very hard not to think of, just as you cannot resist touching a spot on your body, which hurts. Many popularizers does those aspects well, in a sense it involves ideas and objects which are easy to motivate, and the mathematician when expounding on it, is getting closer to the situation of a regular scientist.

Now the book contains one curiosity, and that is about areas and volumes, how you can subdivide some figures in a finite number of pieces and rearrange them into radically different. In a way this is a variety, be it a very sophisticated one,. of recreational mathematics. Apart form the Banach-Tarski paradox, involving the axiom of choice and non-measurable sets, the one restriction in the plane is the area, while there are, as Dehn discovered, other constraints in the 3-dimensional case. This is curious mathematics, and not really main-stream, but yet fascinating, showing the great variety of pure mathematics.

Applicability of mathematics is a contentious topic. Do applications provide the ultimate justification of mathematics? Or is it merely another instance of the power of mathematics and hence of thought? Those two things are very different. To claim the first is of course a matter of taste. Mathematics is a rather austere phenomenon, in many ways so inhuman. It can be seen as arid unless engaging in matters of more direct human concern. The debate will continue. Now the politically correct position is that there really is no difference. In a sense this is true, almost a truism. On the other hand there is a difference between identifying them both because it is impossible to separate them, just as two sets which are dense in each other (technically included in the frontiers of each other). What makes mathematics exciting is how it can be applied not only to the so called outside real world, but even more effectively to itself. Mathematics is a web in which everything is connected. Make a rip at one place, and it will have repercussions everywhere else. (At least in principle).

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