The Millennium Problems

The seven greatest unsolved Mathematical Puzzles of our Time

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As Devlin rightly remarks, the popularization of mathematics is the most difficult of all the sciences. One explanation may be that mathematics is abstract, that it goes on in the heads of people, and that it has no tangible physical manifestation as opposed to abstract. The number three can of course be manifested in any number of physical ways, as three pebbles, three owls or three stars, but the abstract notion of three transcends all such exemplifications, and can as such not be represented by any particular object, tangible or not[1]. More strikingly the set of all natural numbers, the integers, can easily (far too easily?) be fathomed by the mind, but has no physical manifestation at all. So why a scientist can usually refer to some key objects of his study as known and understood (never mind misunderstood), the mathematicians cannot, so in fact before he can take off from the ground, as the scientist can do from the very start, he has to engage in lengthy definitions, halfway through of which the minds of even the most dedicated audience have started to drift. The task to explain to the general audience (and in practice fellow scientists) what the rather esoteric problems at the forefront of mathematics really mean, is next to a hopeless task. Most often, it would be like describing an elephant, by starting on a lengthy description of tails, because that would be something that most people could respond to, and never mention trunks, or that elephants are big animals, not even that they are animals. A concept only makes sense in a context, just as the meaning of a sentence only is to be found in the context in which it resides. How can we convey the meaning of three without ever given instances of three objects? Thus the default of any such attempt is failure, so any attempt that goes beyond what is expected represents some kind of success, however minute. Thus failure being the norm you cannot fail. Maybe this is what finally motivated Devlin to take up the challenge.

Now it is easy to criticize any such attempt, especially if you are a mathematician. Such attempts are bound to be incomplete, and thus almost everyone may make a good case for including his or her favorite aspect. Yet it is inevitable, and I will now for each of the seven sections make some comments, most of them on based on the mathematicians quibble, that this or that is not made precise or not, and hence misleading or simply vacuous, or that this or that fact should have been mentioned, without which the account becomes incomprehensible, as if it ever could have been understandable in any real, hands-on way. This book is clearly not meant to be a text-book. A text-book has as the ambition

[1] The definition of three as say the set $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$ suggested by Frege and taken up by von Neumann serves some purposes, but of course for the very same reason fails to capture the abstract notion of 'three', and does not differ essentially from the old notion of say $|||$, which after a quarter turn is turned into the designation of the number three in Chinese.

to make things understandable on some basic technical level, thus after each section there are normally exercises, which the reader ought to be able to handle, had he or she properly understood what it was all about. This is technical understanding, or if you prefer, a technical skill, which is the basis for a more holistic understanding, seldom addressed by text-books as being beyond their assignments, and instead being something that it is the responsibility of the reader. Now without that unsolicited reflection, provoked by the curiosity of the reader, no real learning has taken place, because understanding is something that happens in the mind of the learner, and although it can be both stimulated and inspired by a text-book or a teacher, it can never be formed without the independent initiative of the learner[2]. Now the big misunderstanding in much of modern education, or at least in its vulgar form, is the idea that the higher understanding, the understanding of basic concepts as it is often termed, can be achieved without the basic exposure to skills, and in elementary math education, this means calculation. The vulgar idea is based on the fact that by modern technology, computations can be automated as well as the ferreting out of basic facts. How many educators have not claimed that the point of schooling is not to learn facts, facts which are bound to become obsolete anyway, but learn how to learn them when you need them.

Those short essays certainly do not teach any skills to the reader, although as a mathematician at heart, the author cannot resist the temptation to add appendices in which elementary but elegant mathematical proofs or definitions are presented. The purpose is indeed to show that mathematics exists, that it has mysterious connection to the real world, that it develops and that it engages some of the most subtle minds with a passion that surpasses most if not all that with which other human projects are pursued. If it manages to convey at least one of those things, it must be termed a success, and if manages in addition to convey the last, it is an astounding success.

But before making a blow by blow commentary on the seven sisters, I would like to dwell on a philosophical statement of Devlin claimed in the introductory section. Until some 150 years ago, he explains, mathematics was considered, even by pure mathematicians, to be about computation, Dirichlet, Dedekind and Riemann changed all that, after them mathematics was ultimately about conceptual understanding and finding patterns. If this is true, should not those three mathematicians be considered the major mathematicians of history, those who had the most substantial influence and who brought about the major revolution of mathematics. In short that their influence would surpass even that of Gauss? When it comes to Riemann, he certainly belongs there up in the pantheon, as original and profound as Gauss, although not as productive due to his frail health and short life-span; but Dirichlet for all his dazzle and brilliance, he does not stand out among a host of similarly brilliant mathematicians, perhaps mainly to be remembered for being a teacher of Riemann, to say nothing about Dedekind, who in comparison with the cream of mathematical geniuses, appears as somewhat of a mediocrity. Devlin acknowledges though that the Greek certainly were conceptually oriented, and as single example for his thesis, (at least he presents one) takes the case of the formal definition of a function from having been a formula, algebraic or not, with which one was able to compute a value from

[2] There is much talk about everyone being entitled to an education, as if education would be similar to having access to clean water and medical services.

a given one, to a simple formal correspondence between different objects, allowing also non-mathematical uses. I do not want to claim that the observation is wrong, or more to the point if wrong useless, but I would like to take exception, and in doing so you are forced to reevaluate the history of mathematics with this in view, which would be a very fruitful way of so doing. One may ask e.g. about Euler. Was his interest in mathematics mainly oriented towards calculation, especially calculation with numbers?

Now what to do about the Riemann Hypothesis if you encounter a reader who is not familiar with prime numbers, nor have had any previous exposure to complex numbers, let along complex functions? The task is not only daunting, but for all intent and purposes impossible. But you are forced to do it, so you better make the best of an impossible situation, mathematicians are supposed to cherish a challenge, be it of whatever kind. The temptation to give a crash-course is irresistible, and Devlin predictably does not resist it either. Definitions of prime numbers are given (should this not be a prerequisite of every high-school graduation), as well as the unique factorization and thus the often made remark that they constitute the basic elements of the chemistry of numbers. Furthermore complex numbers are introduced (should that not likewise be familiar to educated people? I fear that only mathematicians, physicists and maybe some engineers do generally know of them.) as well as complex functions. In an appendix he even proves Euclid's theorem on the infinite number of primes (to which I will return) and even more surprisingly the infinite euler factorization of the harmonic series, the latter being heavy going for someone who has to be told that 2 is a prime number. The zeroes of the Riemann zeta functions are presented as solutions of equations, a collection of buzz-words, which probably makes sense to the general public, but hardly to the mathematician in any but a formal sense. (Every statement in mathematics can be turned into an equation of sorts.). To talk about the distribution of primes is not so hard, at least not compared to most of the concepts that go into the statement of the Riemann Hypothesis, but the author already throws up his hand at natural logarithms. The innocent professional mathematician simply asks what is wrong about talking about the number of digits of a number[3] ?To explain analytic continuation in any technical sense is of course unthinkable, yet the idea can actually be conveyed, and it is a very powerful idea, and it plays a crucial role not only in the Riemann Hypothesis but also the Birch and Swinnerton-Dyer Conjecture. There is a common misunderstanding that given a segment of a sequence of numbers say $1, 4, 9, 16, 25, 36 \ldots$ it has a canonical continuation, in other words behind a fragmentary manifestation there already lurks the general idea. For a mathematician there is an infinite number of ways of generating any beginning sequence of numbers, and hence they can be continued arbitrarily, although for designers of intelligence tests things appear to be different. Analytic continuation is just this. Knowing the function in a tiny neighborhood (or for the mathematician, knowing all of its coefficients in the Taylor expansion) it has a unique extension. It is as if the function (or mathematics) has intelligence, no matter how small a piece of the function which is known, it can be reconstructed in its entirety (just as the DNA of an organism can be recaptured from any of its (undamaged) cells). Thus although there

[3] State the frequencies of primes up to 10,100, 1000 etc, and multiply them with the corresponding number of digits, and a clear and surprising pattern appears. Any mathematician can of course fudge the thing a little bit, to make it come out even clearer.

may be a formula that generate the function somewhere, the function transcends this formula and exists even where this formula does not make sense. This is truly magic, and a student of mathematics does not really encounter magic until he or she starts to learn complex analysis. And this is of course the magic of the Riemann Hypothesis, this uncanny relation between the discrete and the continuous, pioneered by Euler and Dirichlet. I am not saying that such a lengthy explanation should have been included, the point is that some esoteric concepts can actually be conveyed, once one abandons the technical approach. Of course the reader learns nothing specific and would never be able to recognize analytic continuation when encountered, yet I believe, as opposed to the short explanation given by the author, it is not vacuous. Now mathematics builds on mathematics providing ladders and ladders, and the poor reader is by this time probably quite dizzy and confused. What to do? Explain why prime numbers are important, and here the RSA codes makes perfect sense, not so much to relate to the Riemann hypothesis, but to show that mathematics can have applications not only by providing simplified models of a complicated reality (the partial differential equations of physics being prime examples). Or to present the biography of Riemann. Riemann was probably one of the sweetest and smartest of all mathematicians. His unpresupposing exterior, if any a 'nerd', belying a most daring originality. Physical convention and timidity is no indication of a mental such, on the contrary, social flamboyance that usually goes for a flaunting of conventions and a manifestation of originality is more often than not just a cloak for a conventional mindset. If this message could come across, the section would serve a most useful purpose even if the reader would be as confused as ever of what it is all about.

Finally as to Euclid's proof. I am not familiar with its original (Greek) articulation, but the presentation given by Devlin is unnecessarily contorted and centers around the claim that $N + 1$ is prime[4] which is misleading, bit of course 'true' in the counterfactual situation. Why not simply argue that given any number of primes, a new one can be concocted (as a factor of $N + 1$). Any reader who has been motivated to read so far draws the inevitable conclusion that there must be an unending supply without having to be confused by a hypothetical situation, which he might think of as a trick. Mathematicians are used to the construction of hypothetical worlds, only to be collapsed at the last minute, the general public are probably less enamored by them, and definitely more likely to be confused by their logic.

Yang-Mills theory and the Mass Gap theorem is rather easy to write about, because it gives an excuse to delve into physics at length, on the other hand of all the problems discussed, this might be the one furthest from being given a precise statement. Devlin presents a view of modern physics familiar to most mathematicians, who, without any particular expertise knows of course about relativity theory at least to the extent that the special one presented back in 1905 is physics without mass, and states the equivalence of frames of reference in uniform motion with respect to each other, and that the general one involves mass, and maybe makes any frame of reference equivalent, with the consequences that light rays are bent by gravity. The velocity of light is of course constant in all frames. The general mathematician also knows that quantum theory is a messy thing, that while

[4] N of course being the product of all the primes. When I encountered the proof as a child I was at first confused why it did not give a proof of the inifinitude of prime twins by considering $N - 1$

relativity theory is a closed and self-consistent mathematical theory, quantum theory is in conflict with it, and only works in spite of itself. Thus the grand project, the holy grail of modern physics is to unite the two approaches and in particular give a unified field theory, the most ambitious attempt being string theory, whose applications so far have only been mathematical and which appears immune to experimental testing. All of this is of course gossip for most of us. But somewhere in this mess, there is some mathematical problem lurking. Devlin gives the gossip, the general context, but as to making anything precise is of course not to important, the reader has gotten a lot for his money already and feels that he is almost on top of things (especially if he is a mathematician). The only criticism I can level, might be a trivial one yet in my opinion an inexcusable one, namely the one of plagiarism, a charge maybe mollified as it concerns a case of self-plagiarism from his previous The Language of mathematics. Of course no legal issue, given the circumstances, yet somehow a moral one. Plagiarism is a sign of laziness, writing should be such a pleasure as to making the pasting of old material anathema. Also the claim that Einstein rejected Germany because of its militarism is somewhat controversial. For one thing in its long history, the German nation(s), taken as a whole, was (were) up to the beginning of the last century one of the least militaristic nations in Europe, at least as far as a tradition goes, only Prussia exhibited normal European belligerency. Secondly Einstein had no problems accepting a position in Berlin when one was offered. To say it once is questionable, to say it twice, and in the very same words to boot, is bad. What is worse is that the self-plagiarism continues further on in the book, when discussing Newton and Leibniz and their claim to the invention of calculus.

The P versus the NP problem is by far the best in the book. This is something that is the most accessible of all the problems, maybe not necessarily in solving, but in understanding and appreciating. As Devlin somewhat up-beatedly claims, even an amateur may conceivably crack it, all what is needed is one good idea. Unlike the other problems the author has actually made a go at it, spending a week attacking it until having to admit defeat. There is not so much to criticize, maybe perhaps noting that we need the integers in Gdels theorem, to have the infinite enter, (this is the key to the distinction between truth as established by the thought experiment of an infinite check, and provability which needs a finitely presented argument.), and maybe also to remark that it is not difficult to conceive of problems that take an exponentially increasing amounts of steps (as to write down the digits of 2^{2^N}). The crucial point of an NP problem being that a solution can be checked in polynomial time. (It is not clear to me how a putative solution to the traveling salesman problem can be checked polynomically.) Now if factorization could be solved polynomically would that be such a disaster to security if it would turn out that the degree of the polynomial has to be very big?

The Navier-Stokes equation necessitates a crash-course on calculus (including, as noted above, some self-plagiarism), especially in several variables, as to make sense of the equation. Now unlike the Maxwell equation, Navier-Stokes appears rather *ad hoc*, it is not clear why the viscosity term introduced is in any sense canonical. The Maxwell equation yielded much more than was put into it, and thus by Manin deserves the designation of being a theory. The fact that the Navier-Stokes so far has no mathematical solution does not surprise me. To most applied mathematicians, this might appear to be the most interesting problem, but as some numerical analysts claim, what is supposed to be a solution is up to discussion.

The Poincaré Conjecture, invites a thumbnail portrait of Henri Poincaré himself, and that alone justifies the piece. It also gives an excuse to delve into the classification of surfaces and some elementary yet striking topology. (Devlin is fond of the London subway map, and makes once again some mild self-plagarism from his previous book on the Language of mathematics.) It is a good section, but of course as to 3-manifolds it says almost nothing, giving no indication of the complexity involved. Some spade work that would later have become useful for the final chapter on the Hodge Conjecture is not done, such as the introduction of (co)homology groups. The fundamental group is indicated, but the simpler and higher-dimensional analogues may justifiably been introduced, both to explain the notion of Algebraic Topology, and also to give some indication of a general theme that runs through so much of modern mathematics. Homology is symmetric (for manifolds), due to the so called Poincaré duality, and the proper way of expressing it in all dimensions is to refer to simply-connected homology spheres. Still this might be to quibble.

As to the Birch Swinnerton-Dyer conjecture, the account might have been significantly improved, if he had stated at the beginning the parametric solution to the Pythagorean equation $x^2 + y^2 = z^2$ and then showed how the problem he starts out with, namely Pythagorean triangles with a given (integral) area leads to a cubic diophantine equation. It would not have been amiss to present some alternative problems also leading to cubic equations. Then if he had indicated some striking rational solutions of great height he might have provoked the curiosity of the reader. A few more numerical examples, and the reader might be a bit hooked. Then it would be time to consider solutions over finite fields and the L-series. The account is basically sound, but could have been enhanced, with not too much work. Big numbers always fascinate, at least the kind of reader who might be tempted by the book.

Finally as to the Hodge Conjecture the author throws up his hands in despair. This is indeed a tough thing. Yet it might not been the very level of abstraction that is daunting as much as its formality, in a way, this is not much harder to make precise than some of the other problems (and in fact it is the only problem really in which the author gives a precise formulation) it is simply that it is very hard to fit it into an attractive context. There is no way around it, one needs to take up (co)homology groups, something which could have been prepared already in the Poincaré chapter. Hodge theory maybe dispense of Harmonic forms and functions, on which the author fasten his attention for want of other footholds, but not the further refinement of decomposition and the interplay between rational and complex cohomology. Still, one must admit that any attempt to make it come alive is bound to fail. It is going to be too primitive for the expert, and far too esoteric for the neophyte, and with regards to the Hodge Conjecture, there are indeed very few people who are neither. Had it not been for the Perelman solution of Poincaré, happening a few years after the book was written, I would be tempted to believe that the Hodge Conjecture would be the first to be settled (in the negative). But that is but a hunch.

Given together what impression does the presentation of the various problem give of mathematics to the general reader? Obviously one cannot hope of any kind of real understanding, the most that can be expected is a kind of fascination. A fascination similar to the kind afforded modern physics, where the general public is at least at much at large as in mathematics, but when there is no question that the matters are deep and significant and somehow transcends practical applications, the iconic status of Einstein being a testimony to the physicists great success as to public relations. Admittedly the second section in the book, although strictly speaking it is nothing but gossip is probably as far as exposition and fascination engendered the most successful. On the other hand the very gossipy notion is probably the key to its success, mathematical activity on a deeper mental level being nothing but a kind of sophisticated social intercourse, the difference being that it deals with mathematical concepts and their relations rather than with humans. To physics the public is already exposed, so there is already a basis for gossip, much harder in mathematics. Who is not a mathematician has ever heard of Riemann[5]? Devlin makes a big case of mathematics being so difficult because of its level of abstraction. This certainly has some undeniable truth to it, but I suspect that it is only part of the story. What makes mathematics difficult is the complicated structure, abstraction by itself can be rather vapid. Ironically in a sense the NP-problem is the most abstract of all the problems, at least in the sense of being a meta-problem. Rather than delving into a fascinating detail of mathematics, it wants to take a birds eye view and consider mathematics as nothing but computation trying to say something very general about it. The effect is, at least to me, slightly depressing. It is a very materialistic view of mathematics and its ultimate purpose is akin to effecting a kind of closure, i.e. death. While the Riemann Hypothesis is about striking and unexpected connection in mathematics and open ended and thus much more of a living vibrant part, stimulating to the mathematical imagination, and giving the impression that mathematical concepts have souls. General theories of computations are like considering the infinitude of all the integers in one go. Taking abstractly there is nothing but one, two, three and so on, while the sheer magnitude that individual numbers can attain, is far more staggering than simple, countable infinity itself. One could compare it with contrasting the statistical fact that there are 6 billion people or so on Earth with an imaginative description of a small sample of people.

So what could have been done, or more to the point, what could one do? Is the book too short, would it have been more effective if longer? Or is the very brevity essential, it being geared to people who just want a quick indication of what it is supposedly all about, and do not want to be bothered with more than they are prepared to know? Who reads it? Other scientists, not the general public that mythical notion central to the political concern of modern democracy? Sometimes, as in the last section on the Hodge Conjecture one gets the impression that the author himself is part of that general incomprehending public as he obviously struggles himself to discern any meaning in it and to whip up some kind of personal interest. How can you expect the reader to become excited about something the author himself has great difficulty even feigning excitement about? On the other hand by placing himself in the same boat as his reader a bond of sympathy is established between the two. But what does that sympathy has to do with the ostensible subject itself? To an expert it might be rather interesting to get an idea of what an outsider makes of the subject, but that kind of reader is bound to be in a minority (or maybe actually not?)

[5] A colleague once tried to do a short piece on him on National Radio in Sweden, being rebuffed by the response that he was only of internal scientific interest.

It is of course very easy to criticize an attempt such as this, and also grossly unfair to do so, as the attempt was solicited and not volunteered. In fact it takes a not inconsiderable amount of courage to do so, especially since the author is sensitive and well aware both of his shortcomings and the fact that the book is bound to be read by very critical people. Instead one ought to congratulate the author as well as regret the fact that not more mathematicians try their hands, or perhaps more accurately are allowed to try their hands as publishers are for obvious reasons much more willing to put the money on proven writers than on untried, knowing full well that what sells a book is the name of the author. On the other hand if commercially viable, such attempts may inspire publishers to cast their nets wider and thus allow the popular mathematical literature to expand and diversify. It is all about publicity.

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