Ton och Klang[1]

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This is one of the many affordable science books which was readily available in my teenage years in Sweden. Consequently I found it in my library. I must admit though, that as with the majority of those books, they remain unread around half a century later. But this does not necessarily mean that they are dated. On the contrary if scientific advance has not passed through some drastic paradigm shift, or dismissed huge tracts as mistaken, they tend to be more readable and instructive than contemporary attempts. For one thing they do not assume that the reader is an idiot, even if they acknowledge deep ignorance. This one in particular is a book written for people versed in music, but not having the mathematical and physical background. Admittedly the latter is far from necessary. Music differs from intellectual pursuits such as mathematics and physics, in that you can appreciate it without any knowledge. Thus there are huge audiences for music, but none really for physics, and especially none for mathematics. You do not have to learn to understand why a note sounds beautiful, this understanding is given to you automatically. On the other hand if you are a physicist, as the author, it is tempting to combine the two domains. One as a professional, the other as an amateur. Preferably as a physicist and musician respectively.

It all started with Pythagoras. He noted that the pitch of a tone corresponded to the length of the string, and if a string was twice as long it harmonized so much with the original tone that it was experienced as the same. He also found out that if the ratio between two lengths of the string was in a relation 3:2 it harmonized as well, but sounded different. Another pleasant harmony was given by 4:3, which in fact was the inverse of 3:2. This can be seen as follows. Let A and B be two string lengths, such that B is twice the length of A. If we find a point C whose relation to A is 3:2 then it is not true that the relation of B to C is 3:2, in fact it will be 4:3 instead. Both those fractions, or rather proportions as the Pythagoreans, and later the Classical Greeks would have thought of them as, are examples of fractions with low denominators between 1 and 2.

The Pythagoreans believed initially that all lengths were proportional to each other as integers were proportional. Given two lengths A and B one assumed that one could find two integers m, n such that $mA = nB$, and if the smallest such pair (m, n) was exhibited, then it followed that they had no common factors. Which can also be thought of as the fraction m/n being reduced as much as possible. Another way of putting it is that if you divide A in n equal lengths and B in m equal lengths, those lengths are the same. Call it U. The point is that both A and B can measured by the same length U. We have $A = nU$ and $B = mU$ (and thus $mA = m(nU) = mnU = nmU = n(mU) = nB$). We say that A and B have the same measure, and that they are commensurable. This conception shows that the universe is governed by integers. And in particular pleasing tones are related to

[1] Horns, Strings and Harmony

each other via simple proportions, i.e. by small integers, and the corresponding quotients of lengths have small denominators, such as 3:2 and 4:3.

For this basic discovery Pythagoras has been noted as the first mathematical physicist and for many centuries, in fact until the dawn of the modern age, music, along with astronomy were seen as part of mathematics, alongside arithmetic and geometry. What the Pythagoreans may not have known was that sound could be seen as wave motion traveling at a fairly fast speed and that the wavelength was proportional to the length of the string[2]. Formally and mathematically if follows that the frequency, the number of oscillations in a time interval, is inversely proportional to the length. A shorter string a higher tone. The product of the wavelength with the frequency is the distance covered by the sound during the given time interval.

What is remarkable here is the correspondence of the subjective with the objective. Our sense of sound, as with color, is a purely subjective matter, while the length of a string, or a wavelength, or a frequency is an objective measure. More precisely, how we experience a sound is a purely subjective matter, but apparently the sameness of sounds are the same for all people, an experience that can be exported, and hence has to have an objective basis.

Basic for the experience of all sensory input is the law of Weber and his student Fechner. When we experience though our sensory organs, we can basically only do it in a relative way. What is important is not the absolute amount of sensory excitement, but the relative. There is no absolute unit, only relative. We can compare differences in intensity of lights, or pitches, or loudness of sounds. So while we in our minds fashion a linear scale, in which we accord the same length to two intervals if they correspond to the same quotient, the objective scale is geometric. So while the intervals in pitches given by the frequencies 100, 200, 400, 800 ... Hz[3] are sounded the same, their objective measures follows a geometric series. The phenomenon of so called absolute pitch is a rare one, and it is not clear whether it has any true musical advantage, while most people can only experience tones relatively. It is not the absolute pitch that matters, but the differences between pitches that is essential. Similarly, the same holds for luminosity, except that there is not this phenomenon of cyclicity as to pitches. The Greek classified the stars according to magnitude, stars of first, second down to sixth[4].

Thus if we would like to have an equal distribution of tones, we would have to look at

[2] The sound in air travels at a respectable speed of 340 m/s. This is fast, and it took mankind until the end of the Second World War to design a vehicle that went faster than the speed of sound. Projectiles from rifles and cannons could of course be made to travel faster at an earlier age. But it is not faster than we can experience an echo in an empty room, where the distance to the walls are fairly short, as we are able to distinguish time differences on the scale of a few hundredths of a second. And for larger distances, the lag between a flash and a thunder is quite noticeable and allows you to make a fair estimate of the distance involved

[3] 1 Hz is one oscillation per second

[4] In modern astronomy it has been standardized so that a difference of five magnitudes correspond to an intensity of luminosity by a factor of hundred. The classification of stars need to be made continuously, and a bright star like Sirius correspond to magnitude -1.6, which means that it is almost 1000 times brighter than faint stars on the threshold of our perception. While the sun at magnitude -26.6 is 10 billion

powers of 3/2 and its inverse 4/3 suitably modified by powers of 2 in order to fit into the interval $[1, 2]$. We would get the two sequences

$$3/2, 9/8, 27/16, 81/64, 243/128, 729/512$$

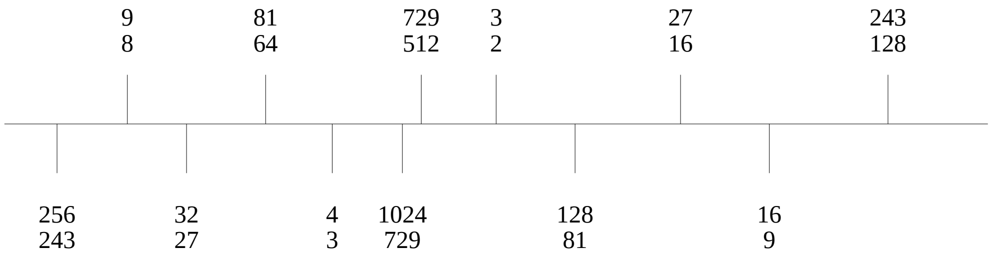
and

$$4/3, 16/9, 32/27, 128/81, 256/243, 1024/729$$

and if we would order them according to magnitude, the tones corresponding to

$$256/243, 9/8, 32/27, 81/64, 4/3, 1024/729 \sim 729/512, 3/2, 128/81, 27/16, 16/9, 243/128$$

which is represented by



If life would be perfect then $1024/729 = 729/512$ and in fact the fraction would be the square root of two, and correspond to a point C in the the standard interval AB above such that the proportion AC would be the same as CB. We would have in fact that if $a = 3/2$ then $a^6 = a^{-6}$ and thus $a^{12} = 1$ modulo powers of two. We would have a cyclic group of order 12, and if 1 is the smallest generator, then a would be the element 7 and its multiples would generate all elements modulo 12 in fact we would have $7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, 0$. But in fact $2^{19} = 524288 \neq 531441 = 3^{12}$ and how could it possibly be otherwise? A power of two equal to a power of three? The first has to be even, and the latter odd. In fact as the Pythagoreans discovered to their horror, not all proportions between segments are commensurable. In particular it is impossible to find a point C as above with rational proportions. As mathematicians put it the square root of two is irrational.

The difference $729/512 - 1024/729 = 531441 - 524288/729 \times 512 \sim 0.019164$ is referred to as the Pythagorean comma and is responsible for the unevenly spaced intervals above. Would they be evenly spaced we would have a picture as below

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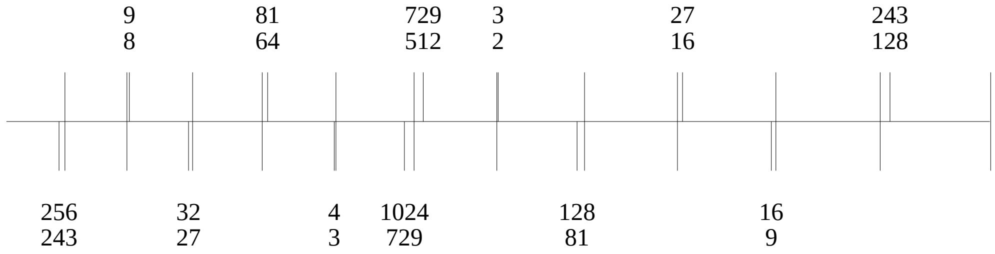
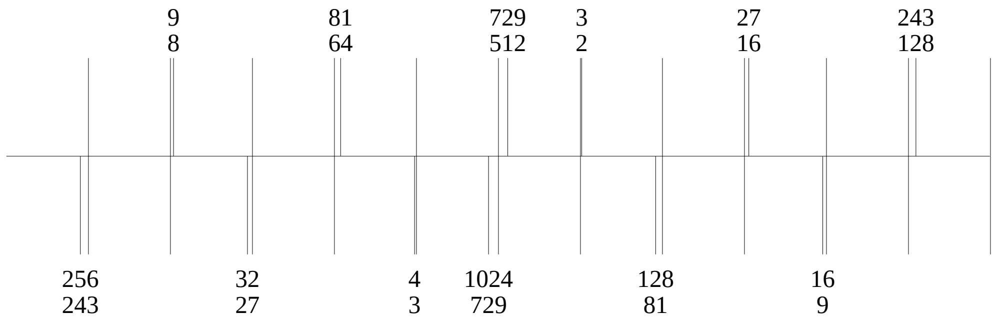
times brighter than Sirius. Such discrepancies can never be directly appreciated by the human sensory equipment if presented directly. It means directly appreciating the difference between 1 mm and 10 000 km comparable to the diameter of the earth. The square root of that number is roughly comparable to the quotient between the distances to the Sun and to Sirius.



We see in particular that the notes 3/2(quint) 4/3(quart) and 9/8 do correspond closely to the uniform scale, which is based on the powers of the twelfth root of 2 an irrational number approximately given by $1.059463\ldots$. We get a clearer picture if we take the logarithm



The Pythagorean scale not being uniform, means that it is not translation invariant. It means that if an instrument, say a piano is tuned according to the Pythagorean scale things will sound differently depending on what tone you start playing. A uniform scale, or in practice, a good approximation of one. Such a scale is referred to a tempered one.

Another drawback of the Pythagorean scale is that many of the tones correspond to quite complicated fractions, and was it not imagined that the simpler the fraction, the more harmonious the tone? $3/2, 4/3$ and also $9/8$ are such examples. But in a tempered scale, the intervals are even irrational, and non-commensurable. Would that not correspond to horrible dissonances? The key-point is approximative. Not even $3/2$ can of course be measured accurately. We have now the notion of finding an appropriate fraction to any real number. It will of course not be the one which closest approximates the number, as there is no such one. Being close to a simple fraction will of course count for more than being close to a more complicated fraction. A reasonable measure of the closeness of a fraction p/q to x is given by the relative error $|p/q - x|$ multiplied by q^2. Thus fractions with large denominators will pay dearly.

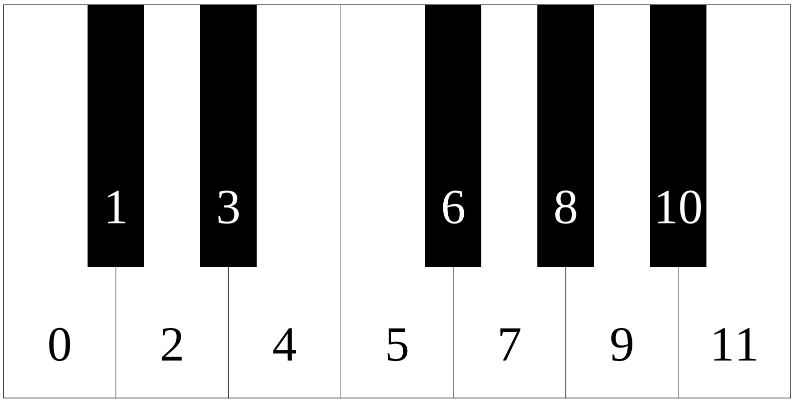
Let us find the best rational approximations of the tempered intervals.

tone	*value*	*error*	*fraction*
1	1.059463	0.777448	17/16
		0.184834	18/17
2	1.122462	0.999360	8/7
		0.162429	9/8
		0.031378	55/49
3	1.189207	0.972686	5/4
		0.269822	6/5
		0.024540	44/37
4	1.259921	0.960316	3/2
		0.660711	4/3
		0.158737	5/4
5	1.334840	0.660641	3/2
		0.013559	4/3
6	1.414214	0.343146	3/2
7	1.498307	0.006772	3/2
8	1.587401	0.349604	3/2
		0.314974	8/5
		0.241096	27/17
9	1.681793	0.727171	3/2
		0.136135	5/3
		0.012270	37/22
10	1.781797	0.508759	7/4
		0.455064	9/5
		0.325592	16/9
		0.062755	98/55
11	1.887749	0.815912	15/8
		0.092361	17/9
		0.062202	185/98

We find here that the tones $2, 4, 5, 7, 9, 11$ have good approximation with small fractions $9/8[55/49], 5/4, 4/3, 3/2, 5/3[37/22], 17/9$ with alternate in square brackets.

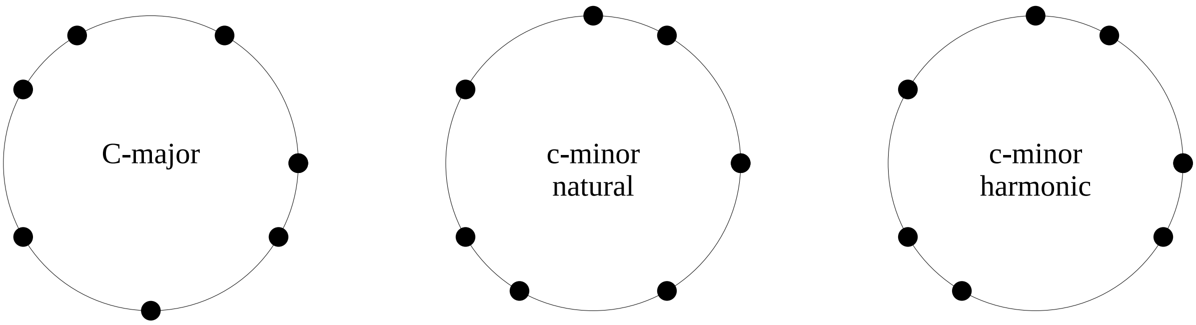
It is not too hard to show that any real number x allows rational approximations p/q such that $|x - p/q| < 1/q^2$ for an infinite number of q. For solutions to quadratic roots you cannot do better, in fact there will be a constant a depending only on x such that $|x - p/q| > a/q^2$. In the case of $x = \sqrt{2}$ we will in fact can chose $a = 1/2\sqrt{2} \sim 0.354253$ working for all but a finite number, in fact $3/2$ is in this sense the best you can get, but there will be an infinite number that will accumulate from above to the value above. In other words 6 is a tone that you should avoid, except for special effects, being the most dissonant tone.

The eight notes (including 0=12) made up of those above will make a nice scale. If 0 is placed at what is called C we will have the so called C-major scale, with the notes classically denoted by c, d, e, f, g, a, b, c' (or $0, 2, 4, 5, 7, 9, 11, 12$) will make up the octave. On a piano they will correspond to the white keys.



The sequence of intervals will be given by $2, 2, 1, 2, 2, 2, 1$. But there are other scales.

The natural c-minor scale on the other hand will consist of $0, 2, 3, 5, 7, 8, 10, 12$ and the sequence of intervals given by $2, 1, 2, 2, 1, 2, 2$, while the harmonic c-minor scale will be given by $0, 2, 3, 5, 7, 8, 11, 12$ and the corresponding intervals $2, 1, 2, 2, 1, 3, 1$. It can be helpful to represent them naturally in cyclic fashion.



From this we see immediately that the harmonic minor scale can never coincide with any of the other two by any rotation, as it contains a gap of three halftones, while none of the other does it. However the first two can be made the same by a rotation. By taking a $3/12$th of a full negative rotation of the c-minor scale we get the C-major one. And conversely a $9/12$th of a full positive rotation, the C-major scale becomes a c-minor. Thus a $d\#$-minor scale will coincide with a $C-$major, and a A-major with a c-minor, but only as to the choice of pitches, the beginning tone is important as it gives the start of an ascending scale. (If the tones are played starting not at the lowest pitch, the scale will obviously fall at some point).

One may ask why there are twelve tones, not any other number. One reason could be that 3^{12} is close to a power of two. Or that the quotient $\log 3/\log 2$ is relatively closely approximated by $19/12$. Could there be other approximations? Of course we would get arbitrary close rational approximations, would we allow large denominator, but if we put a premium on their smallness for practical reasons.

We get

$\log 3/\log 2$	*error*	*fraction*
1.584963	0.339850	3/2
	0.234600	19/12
	0.159665	84/53
	0.041881	1054/665

We note that only 53 tones would be an improvement, and definitely 665 if we would like to improve on the Pythagorean comma. In other words those would be closer approximations to a well-tempered scale. However they would hardly be very practical, unless you would restrict the tonal range. Furthermore the grainy character of music would be lost if all the shades of pitches would be used. It would also be an interesting exercise to check out what pitches, due to superior approximation with rationals of low height should be picked out. This has been done in the appendix.

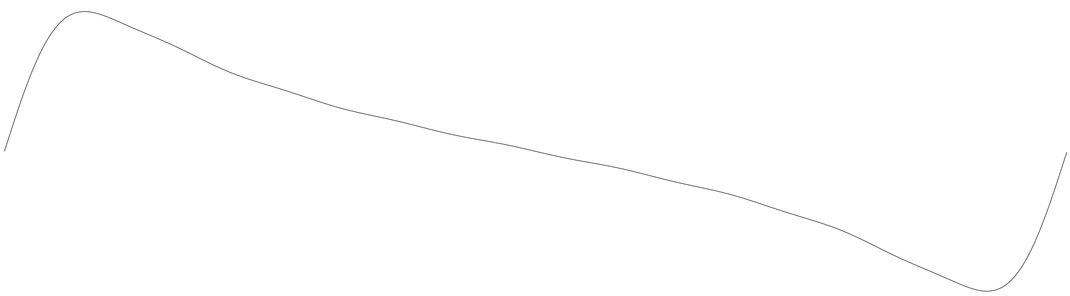
The tone that is given by an instrument is in general not pure. It comes equipped with many overtones that give the special character of the sound. A musical sound is differs from a mere cacophony in that its overtones are all integral multiples of its base. The particular combination of overtones give the timbre of the instrument producing them. In practice only a few overtones may be considered as they are fairly quickly damped. While the sound of pure tone is given by a single sine wave, more complex tome are given by a combination of such.

Given the overtones to H we get a sequence

$H, (3/2)H, H, (5/4)H, (3/2)H, (7/4)H, H, (9/8)H, (5/4)H..$

where only the $(7/4)H$ is somewhat dissonant, the others mesh nicely with the those corresponding to $3/2, 5/4, 9/8$ which explains why those overtones coincide with the standard ones.

We get a picture like this



Finally let us note that most of the book is devoted not to the formal mathematical representations of tones but the far more down-to-earth physical one of how instruments actually produce sounds and giving indications how to build your own instruments. What is apparent that in actually making instruments it is a matter as much of an art, meaning using practical experience, so called tacit knowledge, as a science.

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