A Mathematicians Apology

G.H.Hardy

January 23, 2007

> *I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.*

Those concluding words of Hardy have often been quoted. Often quoted in a spirit of shock and disapproval. For those who take offense, Hardy becomes the representative *par excellence* of an effete coterie of degenerate aestheticism seeking splendid seclusion in the proverbial Ivory Tower. However, one should keep in mind that those lines were written by a very bitter man looking back upon his life and its pleasures for ever beyond his reach. The actual argument he presents in his classical essay is far more nuanced and compelling, and even if you do not temperamentally agree with it, you should be compelled to take it seriously and respect his candor.

The book I got in October 1969 during my first semester at the University. It is a slight volume published by the Cambridge University Press and printed earlier that year. The dust jacket sports a picture of Hardy, one of the few extant snapshots, and between the covers the essay of Hardy is padded by a longish foreword by C.P.Snow, who provides an impressionistic portrait of the man himself, rounding out the snatches Hardy provides *en passent*[1]. And an essay it is, rather than a book, a quick leafing of its small pages covered with large print gives an estimate of about 70'000 characters. I have read it a couple of times in my life. Excerpts maybe in the Newman anthology of the World of Mathematics before I sampled it in the original in my late teens. As with many books you read in your youth it influences your thinking, often without you being aware of it, so that many ideas and opinions you may think are your own, can often be traced back to a more original source. This is something that was also brought home to me during my recent rereading. It is interesting to read a book during different stages of your life, apart from the opportunities of nostalgic revival (because in addition books read during formative years are often quite well retained in memory), those re-readings high-light different things.

The essay is engagingly written with a deft touch and certainly confirming Hardy's hunch, that had not chosen to become a mathematician, but had sought an extra-academic pursuit, journalism would have been closest to his heart and natural skill. But it is also written with a sadness bordering onto bitterness, which Snow is very careful to point out. Because even if it is ultimately a celebration of mathematics, it basically is an apology for doing mathematics and devoting your whole life to it. Mathematics itself needs no

[1] It is remarkable that no full-length biography has been written of him (but few mathematicians have been the subject of such), all what there appears to be available is in addition to obituaries and encyclopedic entries, this sketch by Snow

apologists, although it is not always so easy to explain why, and Hardy makes some sincere, if somewhat half-hearted attempts to do just that, namely to explain why, but even his half-hearted attempts are better articulated than most other attempts, and may indeed go some way in explaining the classical cult status of his book.

First Hardy explains that one should do what one is best at. Only a minority of people have some talent, and very few of those who are talented are endowed with real talent, i.e. being able to do something very well indeed. Such an attitude of course goes against the grain of modern egalitarian notions, but Hardy was born and bred in a time in which the celebration of talent and genius was generally accepted, not only the ephemeral appearances of such[2]. Hardy is not bragging, he is very clear about the limitations of talents, while a few may be blessed with some, talent is almost always very specialized, and doing well in one discipline almost always exclude rising above mediocrity in others. When Hardy speaks of talents, he does not only mean mathematical, or poetic or generally creative talents, but he also includes sports, being himself an enthusiastic tennis-player and cricketer. Furhermore he makes the rather remarkable claim that he has never known any person with a genuine gift for mathematics, who has not pursued a mathematical career, and for whom any alternative would have been unthinkable. As it is, it is indeed a rather remarkable statement, and even if it would be hard to prove in any sense, it does indicate a very strong moral obligation. A moral obligation strong enough by itself to obviate the need of any apology. A cynical observer may of course point out that this moral attitude of Hardy, was simply a consequence of growing up in a social class taking such familiar injunctions of the Bible to cultivate your gifts sincerely with the ambition of bettering themselves. Nevertheless it does provide one of the moral pillars of his arguments.

But wherein lies the moral imperative? In particular what motives drive people to do research? There is of course a variety of most admirable ones, but Hardy selects three crucial ones, presented in order of importance. The most important, without which no other motivation would make any sense, is curiosity, the desire to know the truth. Then comes the professional pride in doing ones best and avoiding the shame of not performing up to par, when ones talents are concerned. And finally ambition, the desire for reputation and the power such might yield. It is typical of the sardonic wit of Hardy to include the third, something more sentimental men would have abstained from doing.

Hardy himself was initially seduced into mathematics by his success, following the recognition of his talent and the admiration it engendered. I suspect that many mathematicians have shared this heady experience, and thus ironically receiving their most fervent accolades before they have even started to discover mathematics itself. The mathematical competition in the form of the Tripos pervaded mathematics at Cambridge still at Hardy's times, and when he himself came of age he worked to have it abolished. British mathematics at the end of the 19th century was in fact rather backward compared to continental standards[3]. The idea of mathematics was what we now would call mathemat-

[2] As to the great majority of people who posses no particular talent, the message seems to be that it does not really matter what they do, as they have nothing to waste anyway.

[3] Bertrand Russell was five years older than Hardy, but still of the same generation. It has been suggested by Monk in his biography that the uninspired teaching of mathematics at Cambridge steered Russell towards philosophy. It is hard though to believe that Russell would have turned out to be as

ical physics, where mechanics, especially hydromechanics, played an important role. The Tripos examinations were a rat-race, in which the candidates employed tutors to teach them the tricks of the trade, and thus to reduce the art of solving problems to a kind of obstacle course in which the object was to neutralize what the examiners had thrown in their ways[4]. Hardy admitted that he had a good teacher giving him the invaluable advice to read Jordan's *Cours d'Analys* and finally he started to realize what mathematics was all about. Hardy eventually become a Fourth Wrangler in 1898, something that rankled him, because he thought that he should have won, although the competition itself was ridiculous. So much to be said about the rather ignominious motivation of competition, although most successful people are far from being inured to it. Let us now instead turn to the ultimate motivation for any scientist - curiosity.

The main point Hardy makes is that is indeed the ultimate motivation also for scientists who are engaged in what the members of the general public see as most beneficial, medicine (or physiology as Hardy calls it) being perhaps the most obvious example. Those people may claim that they are motivated by a passion to alleviate the suffering of mankind, a very noble sentiment if any. But noble sentiments only carry you so far, if you are really to make some progress you need to be genuinely interested in the intrinsic aspects of the problem. This I think is a very crucial observation[5] Of course a medical man do not need to apologise to the public why he is devoting his life to medical research. This is of course very convenient, public relations are something medical researchers seldom have to worry about[6]. I will not belabour this point further.

This fundamental motivation is crucial in order to make sense of Hardys attitude towards applied mathematics, and although Hardy is known for his championship of pure mathematics[7] he basically did not make a distinction based on applicability but on intrinsic interest. Thus the division was not really between pure and applied, as between interesting and un-interesting mathematics. Then it is another thing that Hardy found pure mathematics generally far more interesting than applied mathematics, finding the latter plodding and pedestrian[8] . I believe that this is a sentiment shared by most mathematicians whether pure or applied, the real sweet problems are pure in character. Now

---

celebrated a mathematician as he eventually turned to be as a philosopher. Furthermore the same people who did well in mathematics also tended to do well in classics (and vice versa?), indicating a general ability of playing the game.

[4] This reminds me of the ambition of latter-day didactics people to teach problem-solving using the books by Polya

[5] This is an example of an idea which I have believed I had independantly thought of, until I realized by rereading that I must have read it in Hardy long before, and even if it might not have registered consciously it must have done so unconsiously.

[6] Of course if their prescribed remedies go awry, they will find themselves (temporarily) in the doghouse and incur public wrath to an extent mathematicians never have to experience.

[7] His Calculus book was appropiately called 'A course on Pure Mathematics' and ran into nine editions, the first stemming from 1908, the last in 1944, reprinted several times.

[8] Much of applied mathematics, or rather the application of mathematics to the practical world involves the fiddling with mathematical models in order to tailor theory to facts. And in fact ultimately most of such models are used for numerical simulation, the principles of which are pure mathematical. In physics there

applications may inspire problems in pure mathematics, but the real reason that those are solved are because of the intellectual satisfaction involved in solving them. Thus Hardy claims that even if you desperately want to solve a problem because of its applications, you are only succesful if you are interested in the problem as a mathematical problem.

Now, one should have no illusions about Hardy's disparaging attitude towards applied problems as formulated above[9]. It did not mean that he abhorred applications, (except of course those pertaining to warfare of which he had some pretty sardonic things to say during the First World War) on the contrary he saw applications as a manifestation of the seriousness and depth of mathematics, but as he noted, Shakespeare had a far more pervasive influence on the English language than say his contemporary X. but this was just a consequence of being by far the better poet. It is the poetry of Shakespeare that counts, the influence is just a consequence of it, and not its justification. To summarize: Mathematics need not be justified by its applications, although they are testimonies to its worth. What justifies mathematics is the curiosity it engenders in its practioners. This is of course a moral stand, but why do we live at all? The quest for discovery and truth is as worthy a reason for existing at all as anything else. What you chose is a matter of temperament and ability. For those who have no aptitude for mathematics, the mathematical quest may indeed seem incomprehensible, and in the absence of any practical applications also appear totally irrelevant. What Hardy is trying to do is to give testimony to the worthiness of mathematical creation, and even if that is an elitist ambition I find it eminently justifiable. The real hard task that Hardy confronts is to make the fascination of mathematics comprehensible even to the non-mathematician. It is a task that in principle could be impossible, because only a minority would be susceptible to it[10].

Hardy points out that in fact the worth and usefulness of mathematics is indeed recognized by the general public, and in fact at his time such a recognition would no doubt also have been spiced with a certain amount of admiration. Then he proceeds to claim that the public is in fact fascinated by incipient mathematics as testified by the general interest in puzzles and games. Hardy believes that this interest in mathematics is in fact more pervasive than in music, which at least on the face of it seems hard to agree with. One wonders what Hardy would have made of the recent craze for Sudokus, the solving of

---

is a two-way street, but not one in say biology, where biology seldom if ever present mathematical ideas. Of course there are very important problems in biology, such as to figure out the way that proteins configurate themselves spatially, a deterministic process crucial in understanding their bio-chemical functions. Such problems are very hard but apparently not amenable to real mathematical insights, and their solutions attained by simulation and ad-hoc reasoning.

[9] Hardy pointed out the biologist Hogben as a championship of the usefulness of mathematics, remarking that all Hogben knew was 'school' mathematics, and that he had no sense at all of the beauty and fascination of 'higher' mathematics. He grudgingly admitted that Hogben may after all have done a communal service by pointing out to the illiterate that there was more to mathematics than meets the eye. Hogbens book 'Mathematics for the Millions' was a big success, and it has in fact inspired more than one great mathematician, Mumford has testified to the effect Hogben had on him.

[10] As usual when an expert tries to reach out to make a case for his field, the most susceptible outsiders are those who may never yet have realized their intrinsic susceptibility, in practice this means the young (and still corruptible).

which has little intrinsic interest to a mathematician. The point he wants to make is that even if people may claim practical justification they do indeed show great appreciation of things devoid of any practical implications[11]. Chess is mathematical he explains, but only on a trivial level, just as mathematical recreations. Chess has no significance beyond itself, but mathematics has. And here he comes to the crux of the matter, namely to explain what is meant by mathematical significance, and how it really differs from the much more readily explicable practical applicability. Hardy decides to present two gems from Greek mathematics, namely the proof of the infinitude of the primes and the irrationality of $\sqrt{2}$. To appreciate such gems you need no mathematical education, nor any lengthy introductions, just a dormant susceptibility to the beauty manifested through the combination of surprise and inevitability that marks a real mathematical argument. In chess, Hardy remarks, you may sacrifice a piece to gain an advantage, in mathematics you sacrifice the whole game (he is surely refering to proof by contradiction) in order to gain the world. It is doubtful whether Hardy really succeeds, but it is doubtful whether any popular mathematical text really succeed at all, except to those that are destined to succumb anyway. As examples of trivial mathematics he picks more or less at random from Rose *Mathematical Recreations* There are just two four-digit numbers that are integral multiples of their reversals. Namely $8712 = 4 \times 2178, 9801 = 9 \times 1089$ something that may intrigue amateurs but leave mathematicians cold. It is not particular difficult to prove such things (one can always use trial and error for what is but a rather limited number of cases) and verifying the fact is not instructive. There simply is nothing that is 'going on'. The human activity of mathematics is filled with false leads, the ancient obsession with perfect numbers and such things, being obvious examples. It is not the uselessness that is fascinatining with pure mathematics, but the way it relates to other mathematical things. That mathematics constitute a multiply connected web, the realization of which surely being what seduces the mathematically attuned to mathematics itself.

The age old controversy on Platonism and Mathematics is of course unavoidable in any philosophical discussion on mathematics, and Hardy confronts it without dwelling on it. He makes a distinction between the real physical world and the mathematical, claiming that one can prove nothing about the former through the latter[12]. Furthermore '317' is a prime whether we humans exist or not. In general though he is not particularly concerned with the question, Platonic facts tend to be too abstract and general to be interesting. The fact that '$8712 = 4 \times 2178$' is as unchanging and Platonic that '317' is a prime or '$\sqrt{2}$' is not rational, but what is really interesting is our human relation to those facts, and with such a focus the Platonic character of mathematics becomes irrelevant, although by most mathematicians taken for granted[13]. To Hardy mathematics is an art, a creative art, where patterns are made out of ideas, and hence more endurable than any other

[11] The human interest in say jewellery and precious stones surely illustrates a general tendency to be fascinated by the useless.

[12] In particular that different mathematical geometries exist, and their existences are in no way affected by the particular physical geometric manifestation our (local) space happens to conform to

[13] Some people consider the Platonic persuasions of mathematicians to be naive and unthinking, but I fail to see what advantages are really gained by denying it. The remarkable convergence of mathematical development across cultural barriers is something even die-hard anti-Platonists are bound to admit. Ra-

human activity. Such an endurability comes with a price, namely the mathematical legacy is chillingly impersonal. In the works of a poet, even a philosopher, the personality of the creator is to some extent purveyed as well, but not so in mathematics. Even if you can be quite emotional about mathematics, it provides no vehicle to express emotions as such[14]. A mathematician is like a painter, he observes the mathematical world, he makes discoveries, but what is fascinating to the individual artist is the form he choses to render those in, and the significance he attributes to them. Thus mathematics is a humanistic, artistic endeavour, not really a scientific one.

The fact that mathematics is a creative pursuit if anything at all, makes it impossible for a mathematician just to contemplate the eternal mathematical truths, he has to discover new ones. It is not the fixity that fascinates but the fluidity[15]. To really do mathematics is really hard work, and your prime is but short and when you get old you inevitable lose the knack. Hardy may have been the one who coined the phrase 'mathematics is a young mans game' pointing out that the average age of election to the Royal Societry is lowest for mathematicians[16] . Hardy himself was a late bloomer, paradoxical for a mathematical progidy, claiming that he did not achieve his prime until his early forties. By his late fifties the energy and the originality were gone, and when he was writing his Apology he considered himself washed out, unable anymore to contribute significantly. Some mathematicians at the end of their careers may claim that they have never been as good as they are now. Such men I suspect are either extremly vital, or, what is far more likely, have never really tried to do mathematics seriously[17].

In spite of everything this rather melancholy book (as Hardy famously points out at the very start that*It is a melancholy experience for a professional mathematician to find himself writing about mathematics*) does convey a sense of guilt. Hardy had a charmed, priviliged life, effectively protected from the usual viccissitudes of normal existence. Hardy himself had been seduced by the charms of an academic life through the rather second-rate book *A*

---

manujam is in this respect a very interesting example. His mathematical strangeness is not really a social cultural one, but a manifestation of his singular autodidactic education. And even here, there is of course a convergence, otherwise there would have been no fruitful exchange.

[14] Hardy relates the question if a memorial would be made of you, would you then prefer to have your statue placed high enough so none of your features were discernable, or would you rather have it low, so everyone could recognise you. Hardy apparently would prefer the first, while most people would be more comfortable with the second. The point being that as far as enduring fame goes, mathematics is really impersonal.

[15] A mathematician repeatedly goes over familiar grounds, just as Hardy went back to the elementary examples he proposes, but here the saying of Heraclitus holds sway, namely of you never stepping into the same river twice. Each time you revist something familiar you learn something new, because you place it into a different context, if for no other reason than it becomes a matter of comparisons to previous contexts.

[16] As well as pointing out the outstanding contributions by those how died very early, such as Galois, Abel and Riemann.

[17] The mathematician Ruelle points out that most scientists have never ever achieved anything of value having quite before they have even started in earnest.

Fellow at Trinity[18], and myself must admit that in my youth life at Cambridge, as relayed by Russell and Hardy seemed to me to be the closest approximation of blissful heaven on earth I could imagine. It was a life of sherry and walnuts in the combination room, clever discussions at High table, serene twilight walks over well-manicured lawns accompanied by chimes from nearby chapels[19]. And it is this aspect of Hardy, the University Don spending (at most) four hours of concentration each day on mathematics, the rest lounging around, that C.P.Snow reports on with such fascination.

Snow had no deeper interest in mathematics, and what hence really made Hardy tick was totally opaque to him. It was a common interest of cricket that brought him into Hardy's orbit initially, an interest which in the case of Hardy was obsessive, in the case of Snow passing. To Snow Hardy was the excentric genius (although Hardy would deny such an exalted characterizations[20]) and he compares him to Einstein. The Hardy that comes across is the brilliant conversationalist, obsessed not only with clever word-games but also with cricket. In fact the latter obsession makes one wonder whether he did not after all have a strong autistic streak in him[21]. Snow also reports on his strange phobia for mirrors, and for being a novelist he displays a striking lack of imagination in attributing this to anti-narcissism[22]. But while Einstein tended to become stranger and stranger the more you got to know him [23], Hardy appeared more and more normal, the deeper you penetrated behind his stances. Could it be that after all Hardy was rather ordinary, just a very clever boy among the other clever dons, sparking with wit in a self-contained universe of esoteric mathematics and classical wisdom? As noted above Hardy matured late and it is tempting to speculate, as Hardy did himself, that the key to his success was his close collaboration with Littlewood and Ramanujam, the latter being the supreme romantic accident of his life[24]. As with many men who mature late, Snow explains, they stay young for a long time, but such extended grace make them singularly unequipped to face the rigors of ageing. Well into his fifties Hardy was a keen athlete, never strong he was on

[18] In literature as in mathematics, it is rather the second-rate that has practical applications

[19] Hardy professed along with political radicalism a militant atheism and a concomitant horror of organized religion, not unusual among those benefitting from a sheltered existence.

[20] At his best, he claimed that he might possibly have been the fifth best mathematician in the world, the identities of at least two people he must have ranked ahead of himself are obvious to guess

[21] The last thing he heard as he was dying, was his sister reading him the cricket news. Maynard-Keynes used to chide him that if he spent as much concentration on the stock-exchange columns in the mornings as he did on those devoted to cricket, he would have made a bundle.

[22] I can well imagine the phobia having deeper roots than a mere disgust for ones appearance, which after all is a very narcissistic feature. To look yourself in the mirror is to externalize yourself, seeing yourself just as a thing among other things, and then to rob you of your subconscious comfort of solipsism and to provide a reminder of the ephemeral nature of your being. To gaze at yourself is an act of self-reference frought with the usual dizzying paradoxes that such inevitably entail.

[23] did Snow get to know Einstein? or is he but vicariously reporting?

[24] Snow reports in his foreword that Hardy was not the first mathematician that Ramanujam contacted, two previous quite well-known (but not named by Snow, although they were at the time already dead) had received his unsolicited manuscripts, but chosen to ignore them, a practice Snow admits is rather understandable.

the other hand slim and agile, and played a good game of tennis. At the age of sixty-two he suffered a coronary thrombosis, he did recover of sorts, but the active life to which he considered himself entitled, was over, and it was at the beginning of those bitter twilight years he wrote his famous Apology. He lingered on for another decade before he finally succumbed, prematurely aged.

January 24-25, 2007 **Ulf Persson:** *Prof.em, Chalmers U.of Tech., Göteborg Sweden* ulfp@chalmers.se