Dreams of Calculus

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Claes Johnson has a personal vision of mathematics. What it is, what it is good for, and how it should be taught. The book is ostensibly on the last part, although most of it is actually a survey of what mathematics is and how it fits into modern society.

Such a personal vision is of course, at least on the face of it, a good thing. I fear that many mathematicians if confronted with the question on the nature of mathematics would be rather evasive and refer to some standard cliches. If asked why they do it, they might become defensive, maybe refer to its beauty and to its important applications, although, if pressed on what they actually do, they would be forced to admit to be doing rather mundane things, with little if any beauty and with no practical applications whatsoever. So why do it? The cornered mathematician might at this stage squirm a little, mutter something about not being able to do anything else, that it is a pleasant activity, and you have to be doing something. One would hope that this is not a typical mathematician, although many of the traits may be typical. So when a mathematicians comes along with a clear picture and a well-defined perspective, we may either agree with him or disagree, but in any case we should respect him, and use his particular picture as a testing stone for our own views, maybe sharpening them as a reaction.

The author sees mathematics as ultimately computations. The big arching over-structure that has developed during its history and so fascinated many a mathematician basically only serves the ultimate object of computation and as far as it is irrelevant to that purpose, it may be charming but essentially pointless. Now with the recent advent of the electronic computer, the art of mathematics has been revolutionized, as its dreams have in a sense finally come true. In fact the awesome computing power nowadays available to us makes many of the traditional arts of mathematics, say spherical trigonometry, as irrelevant as the slide rule[1]. The author regrets that this fact has not universally registered among mathematicians, incidentally making most of them irrelevant. Using fashionable terminology he refers to a Paradigm shift, the nature of which it is his duty to publicise and exploit, in particular in the teaching of mathematics.

So what is the point of computing? This too is easily explained. There are of course traditional applications to physics, but also to its off-shoots like meteorology both its short terms predictions and its long range considerations like Global warming. There are also applications to medicine and more generally the biological sciences, especially

[1] There is of course nothing wrong with spherical trigonometry, it is a beautiful subject, in fact more so than ordinary trigonometry, but as a tool for spherical computations it has been superseeded, so those in former days forced to study it for its practical computational applications, can now safely sidestep it. Similarly the traditional means of estimating astronomical distances in the solar system was based on some clever geometry, with the development of technologically high-powered devices, like those sending laser-beams, this can now be accomplished using far simpler principles

to the 3-dimensional manifestations of proteins and enzymes, so crucial to their working in bio-chemistry. But maybe most important of all is the general possibility of creating virtual realities, which very well may revolutionize the way we live and think. In all of this computational mathematics is crucial.

To many mathematicians this view of mathematics may appear rather barren and technical. Not to disparage the ingenuity that goes into the developement of computational algorithms, but the role of a mathematician reduces to that of an engineer. Sharp and clever maybe, but somewhat myopic.

Johnson rightly notes that the distinction between pure and applied mathematics is of rather recent vintage, that the great classic mathematicians like Newton, Euler and Gauss say, took anything as feed for thought, but in so doing and pushing his computational perspective, he blurs the distinction between mathematics and science, a distinction maybe invisible to the general public, but still crucial.

What is mathematical truth and what is the rigor necessary to validate it? Those philosophical issues have stimulated much mathematical reflection and with the efforts of the mathematicians of the 19th century a basis for a mature understanding was laid, starting with people like Abel and Cauchy, through Weierstrass and, controversely so, to Cantor. If one wants to exhibit one feature of mathematics that sets it apart from the empirical sciences it is the contemplation of infinity. Infinity, whether actual or potential (the fine distinction tends to wear away on sustained reflection), cannot be physically realised. It, along with such idealizations like the completly straight, infinitely thin line, extending indefinitely in both directions, are clearly objects of the mind not of matter. Traditionally it is useful to refer to them as objects in the Platonic Universe of forms. And in fact there is no better, maybe no other useful realization of Platos vision than that of mathematical concepts, as his ideas of forms easily becomes more comic than instructive when applied to everyday objects. Such matters are clearly not amendable to empirical testing, their truths have to be taken on faith (axiomatics) and their relationships deduced from rigorous thinking. This, as indicated, was already understood by the old Greeks. Although the ultimate meta-mathematical object of proving consistency turned out to be a hopeless dream, this failure has not discouraged mathematicians in their ambitions of laying axiomatic foundations to their various disciplines, deciding that any subject which cannot ultimately be reduced to such, is in a crisis. That absolute truth in mathematics is unattainable, as it is in the world at large, is just something that most mathematicians easily have accomodated themselves to[2] .

Johnson proposes another kind of rigor, what he refers to as the constructive rigor rather then the formal. It is the kind of rigor that applies to the physical world of mechanical gadgets and electronic computers and everything else subject to the inevitable laws of

[2] Mathematics is a deductive science, still mathematicians are but humans, and often conviction need in addition to deductive reasoning, independant tests and corroboration, some of them numerical, to carry the day. I recall John Tate once entering the old common room at Divinity Avenue at Harvard, presenting two apparently unrelated integrals (no doubt having popped up in number-theory) whose numerical values coincided as far as accuracy of computation had allowed (this must have been in the spring of 1972). The conclusion was that the two integrals must be equal. Gleason remarked that a numerical verification probably would carry more conviction than a formal proof, but Mackey was aghast at that idea.

physics (whatever they are). The proof is in the pudding, attained by meticolous attention to detail, and validated by things actually working. This is a pragmatic approach to truth, in the spirit of Poppers falsiability criteria. How can you falsify statements involving ideal objects? Thus Johnson is forced to throw out much of mathematics, refering to it as ideal and formal. However, his attitude is not particularly original, it hankers back to the ideas of intuitionism and constructiveness from the previous change of century, which at the time came about as a reaction to a real crisis in mathematics. Although Johnson refers to this as an ongoing quarrel, although not in the nature of being subject to resolution, it has become apparent that the division is not that crucial after all. The patient work of logicians, taking a truly formal point of view, has convinced both sides that, except for certain existence statements, most of classical Cantorian mathematics also survives in the straight-jacket imposed by constructionism, very much like people can survive as vegetarians (to borrow a simile from the book) renouncing meat. (But one may ask, pace Johnson, why really bother?)

Thus Johnson renounces the Cantorian Paradise as being irrelevant and just a residue of old faith being rendered irrelevant by modern life, modern life in this case being the wonderful world of computation, just as traditional religion with its superstitious trimmings is being renounced by modern science. And of course no one faults him here, what possible consequences could Cantors flights of fancy have on the real world we live in? Does the axiom of choice[3] influence whether a bridge collapses or not? But their is a hitch. Just because we cannot physically represent all the integers does it mean that they do not exist in their entirety? The classical subterfuge is to think of the idea of the integers being the possibility of always going to the next step when challenged, thus the integers are in the nature of possibilities, (summoned from where?) whenever we need them. This idea of going through the integers one at a time is a very simple one, and which lies at the basis of the induction axiom of Peano. This thought experiment allows us to convince ourselves that a theorem about integers can be true, if it can be verified for each one separately in an infinite process, even if there may never be any way we can supply a formal proof of finite length. Such an idea is intuitive, and it takes place in our mind, and cannot be physically tested; yet to abandon such flights of thought, seems to impose on our thinking a censorious restriction that most of us would find intolerable. Of course we also recognise that such speculations have no bearing on practical computations.

But, and this Johnson never seems to consider, if we abandon the idea of infinity, we should also abandon the idea of very large numbers. In fact very large numbers are even harder to fathom, and in fact induce a keener sense of vertigo, than infinity herself. Should we renounce large numbers as well, because they too are as impossible to exhibit physically as infinity? In fact, one could make a case for contrasting the finite (the arithmetic or linear) with the infinite (the geometric or exponential) the latter refering to phasespaces of possibilities, and the former to actual choices of history. The closest Johnson comes to

[3] Admittedly none of Cantors responsibilities. The axiom of choice does have fanciful mathematical applications, obviously, one of them being the existence of non-measurable sets evocatively illustrated by the notorious Banach-Tarski paradox. The fact that the axiom of choice is indeed a matter of choice, has led Gödel to postulate the existence, of hitherto unknown axioms, far more intuitive and normal, testifying that even such a formal enterprise as set theory is ultimately motivated by compelling intuition

those ideas are in the contemplation of the Borges Library of Babel, which for all practical purposes is an example of an infinite set, whose members only potentially exist.

However, there seems to be one way in which infinity is actually physically realised, namely space itself, and more specifically the continum of the real line. This was already noted by the Greeks in what has been handed down as the paradoxes of Zenon. Zeno actually postulated an infinite number of events happening in a finite time by one of the first thought-experiments recorded in history. The physical hitch is of course the notion of a point, which has no physical counterpart, as measures of locations can only be approximated, and the notion of a precise location is physically meaningless (As we all know the possibility of indefinite simultaneous approximations is challenged by modern quantum physics). But of course so strong is our intuitive sense of the continum, that if we make the thought experiment of an infinite succession of coins tossed, and in each stage, we are to decide between a left or right division of succesively halved intervals, we believe that in the end there will be a point corresponding. Of course formally we may only recognise dyadic representations based on a finite constructive formula, but the real numbers remain uncountable nevertheless be it in a constructive way, thus illustrating simply the formal equivalence between the unlimited and the limited approach. The true ideas of Cantor are the simple ones of one-to-one correspondence and the diagonal principle, on which incidentally the Gödelian proof is based; whether or not 'all' subsets occur, is a moot point.

Thus the philosophical case that Johnson is making is neither original, nor does it in any way challenge the pursuit of mathematics as conceived in the ideal sense. Of course the advent of the powerful computer can challenge the way we think of mathematics and in fact provide new perspectives, and Johnson is of course not only entitled but encouraged to do so, but in no way does it invalidate it.

Johnson confronts the intuitive/democratic approach to mathematics with what he calls the formal/aristocratic approach as exemplified by Plato himself (which was of course an aristocrat and contemptous of the democratic sophists, with whom Johnson, supposedly unwittingly, identifies himself with), as well as Cantor and Hilbert. He regrets that the formalists (incidentally quite misleading a characterization of the majority of mathematicians his label would necessarily refer to) have dominated the teaching of mathematics to the detriment of the constructively inclined. Thus, making some sweeping generalisation, based more on fanciful speculation than fact, he proposes that the great majority of mathematics students are alienated by the traditional curriculum, that stifles their creativity and lust for experimentation, by presenting tricks and stratagems out of the blue. He makes a comparison between the set pieces of classical music and the improvisations on which jazz is performed. One wonders whom he has in mind when he thinks of mathematical education, obviously not the tiny minority that actually thrives on the traditional fare, exciting their sense of wonder and beauty, stimulating their creativity; but the less mathematically inclined, ready to serve as the engineers to report to computational duty. Johnson points out correctly, that mathematics is hard, difficult to learn, and knowing of no short-cuts; and also, more controversally that mathematics is irrelevant for people actually not involved in the line of computations, which means the overwhelming majority. The statement that his computerbased approach to calculus is actually more conducive to understanding has been hotly resisted by his detractors. Would a dispassionate study of this not be a suitable project for the didactics people, so eager to be useful and look for interesting and productive research?

To conclude on a more technical note. Johnson disparages so called analytic solutions to differential equations, the mathematical modelling par execellence of physical inquiry. It is no clear what he means by those, a natural guess would be solutions so to speak in 'closed form' employing special functions. What library of special functions you use depends on your interest and specialization. Traditional schooling supplies the student with the trigonometric and exponential ones, as well as inverses of the same and combinations thereof. The manipulations of those functions constitute a large part of the familiarity that most students acquire with Calculus. It is of course true that most problems that you pose have no such ready-made solutions, although each equation may be used to define a new set of special functions (their qualitive properties inherent in the equations themselves), but nevertheless whenever there are such exact closed solutions or approximations thereof, they greatly enhance your understanding of the solutions, in a way the mere plotting of numerical data would not (and Johnson concedes this point as well on the sly). Of course in real-life situations the elegant solution is never an option, but does that make them irrelevant as just contrived text-book problems? Johnsons quick introduction to calculus may instruct the already knowledgable, but would be rather daunting and confusing to the neophyte. It is one thing to understand something on a second or third encounter, quite another thing to do it on a first. After the third encounter, say, you may feel so confident that you can finally (if mistakenly) write down a definitive formulation intended as a short-cut for a first encounter. But unfortunately this is not the way things usually work. His discussion of the Navier-Stokes equation is the most technical and incomprehensible in the book. One wonders to whom it is addressed- The beginning calculus student or the professional, but misdirected, mathematician? But there are also some material that was very instructive for me to learn, namely that one can numerically simulate the motion of the celestial bodies, with only a linear growth of error, thus in practice making simulations up to say a million years. There is of course no indication how this is done technically, and this would also have been inappropriate in the context. Also the fact that many of the classical equations in physics, like the Einstein equation or the Schrödinger equation, do not readily allow themselves to be treated (effectively) in interesting circumstances, is also something that is important to realise, and may cast doubt on the feasibility of traditional mathematical modelling via partial differential equations, and thus incidentally on Johnsons grand vision of computation itself based as it is on differential equations. (Although he notes that maybe there would be other types of equations, if hard to find and formulate, and before being 'happy' you need to 'marry')

In conclusion the book is written in an easy-going and leisurely way, giving a version of mathematics and mathematical history 'light', with a commendable idiomatic touch[4]. Some may find the style charming (I have to admit that I enjoyed a few of the similes, some of them exploited above) and refreshingly irreverent, others may be put off by a certain flippancy. Although he does make interesting points here and there, the overall

[4] With obvious slips like 'debate article' appearing in the appendices, but maybe those were not penned by Johnson himself

impression of the book is one of disorganization (its rambling style indicating that it was conceived and produced in haste), after you have read it, it is hard to remember what you read. (Admittedly this may be due to an intermittent reading spread out over a long time interval). I obviously do not agree with the tenets of the author, but reading a book with which you disagrees may often be more instructive although not always as satisfying than reading one that confirms your views. Essentially I disagree with his vision of mathematics as it seems to leave out the rich and fascinating culture that mathematics provides. In fact it is not clear whether the author really adhers to the views he sets out, always keeping to a strict vegetarian diet makes a dull life for Jack. Furthermore one does get the impression that Johnson is not aware of large tracts of mathematics, many of which would provide some excellent vegetarian dishes to add to his mathematical smörgåsbord.

To the book there is attached three concluding appendices. In the first he makes some sarcastic comments on the formulated guidelines of the so called mathematical delegation. Given the vagueness of official prose this is indeed a very easy thing to do, the needle with which to puncture does not have to be very sharp. One interesting issue, often evaded, is whether mathematics really sharpens the mind, and if not, whether it is useful to subject students to it. The argument of sharpening your powers of reasoning was traditionally applied to the study of classical languages[5] but for a variety of reasons those subjects faded out. It is of course not automatic that excellence in mathematical reasoning translates into a similar excellence in reasoning in other fields, as any mathematician can eagerly testify to by simply pointing at some of his colleagues, but the same argument can be levied against education in general. Why educate people 'en masse' if so many of them do not 'get it'? Johnsons own attitude seems a bit confused, as noted above. On one hand most people should not be forced to study mathematics, as the subject is not congenial to them (an argument bound to curry a lot of sympathy) only a handful that is needed for the computational tasks of a modern society. That handful seems not to be chosen among those with aptitude, but instead geared towards those alienated by traditional presentations. That mathematics could be anything else than computation is of course the whole point of Johnsons outlook.

The second appendix deals with an outline of the triology 'Body and Soul' which together with appropriate soft-ware is meant as a total package, ostensibly tried out with great success according to Johnson. 'Body' obviously refers to computation, while 'Soul' appears to refer to the traditional over-arching structure of mathematics (or rather calculus). Although not having studied the package yet, one may be permitted the suspicion that the presentation is in fact rather conventional, except for an intermittent technical idiosyncarcy (like replacing the notion of continous function with Lipschitz conditioned ones). Johnson was brought up in a traditional way and is most likely to retain most of its structure. That theorems more naturally come at the end of proofs, rather than the other way, is of course hardly an original innovation to enhance readability and build up motivation. As to the computer packages, it is my conviction that if computers and math-

[5] It is rather remarkable that in the Tripos exams of Cambridge University in the 19th century there usually was a rather high corrolation between excellence in Greek and Latin and mathematics, and Gauss himself did in his youth, much to the consternation of his father, consider a career of studying the classical languages

ematics should be successfully interwoven, this should be done on first principles, rather than using commercial packages, however convenient it may appear. I am well aware that this may be an extreme view of mine, but everyones is of course entitled to an opinion. The third appendix, consisting of a translation of an opinion piece in the local newspaper, does add very little to the tract.

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