Mathematics and Abstract Games

An Intimate Connection

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Is mathematics just a game, or rather is it game-like, and if so what features does it share with games, and more interestingly how does it differ from games? The book by Wells consists of two parts, one in which he discusses games and mathematics intended for the layman without any particular expertise on either[1] and one (actually consisting in a series of appendices) in which he digresses on philosophy. The first part is written with verve, if occasionally in a somewhat breezy style[2], in which he presents with commendable lucidity the nature of games and their relations to mathematics. The mathematical examples are to a large extent standard and conventional ones, and that is of course only to be expected, but in addition to those there are some quite new and refreshing examples, the nature of which I fear will only be fully appreciated by the experienced mathematician, even if they are in principle understandable to the layman[3]. One may of course always have some quarrel with the selection, and if so I think that one of the darlings of the author, a chaotic perturbation of the Fibonacci series is given too much space[4]. Then one may also argue as to how far one should bring each example. The author does not go very

[1] It is hard though for him to imagine that they may be people who are not familiar with chess nor with the basics of mathematics. As to the former he does not explain chess notation (admittedly it does not make any difference, but I as a reader would not have found it amiss), as to the latter he makes concessions to the ignorant by speaking of counting numbers rather than just integers, and defining what is a prime number

[2] Why write the great physicist Paul Dirac, why not be content with just physicist or better still just Dirac. Most readers one assumes are familiar with Dirac and the qualification would only be felt as patronizing

[3] One such example is the remarkable identity $1^3 + 2^3 \ldots n^3 = (1 + 2 + \ldots n)^2$ which I suspect many people accidentally discover. It is trivial to verify it by mathematical induction, but such proofs are of course deeply unsatisfying, only telling you that something is true, without indicating 'why' it is true. This particular example was thus also highlighted by Feynman in one of his books. I once tried to give some geometric explanation of the fact, using a 4-dimensional pyramid, in a popular lecture. I do not recall the details which suggests that it could not have been too successful. The author puts it into a wider context supplied by Liouville which most professional mathematician do not now about and hence are delighted to be instructed. The author also makes a conjecture (based on nothing really it seems) that whenever a sum of the cubes of a set of numbers equals the square of its sum, then it is connected to the situation of Liouville, namely being the set of numbers is given by the number of divisors to the complete set of divisors of a fixed integer.

[4] Obviously this would be a delightful source for an elementary paper, but maybe not significant enough to warrant such a lengthy inclusion

far, on the other hand he manages to say something essential about each problem without going into technicalities. This is no mean feat. It makes it easy to read the book quickly, your attention span is never ever challenged. For the professional mathematician this is no doubt a bonus, he or she is able to skim through, getting the essentials, and only pausing occasionally if provoked into deeper exploration. For the layman it might not work equally smoothly, even if he or she might also appreciate the constant change of scenery. There are some cases though when the presentation seems to be truncated. In the presentation of the Napoleonic theorem on page 59 using tessellation I am lost. I suspect I would not be the only one. Also in the discussion of the product $\prod(1-x)(1-x^2)(1-x^3)\dots$ considered by Euler I think most laymen would appreciate more detail, such as the terms of the product is given by a choice of term from each factor, with the proviso of the tail being an infinite sequence of $1's$. From this follows that the coefficient in front of a given term x^n say has to do with the sum of the number of partitions of n counted with appropriate sign depending on the parity of the number of summands. The professional mathematicians spots this more or less immediately, but the innocent one may be confused. Then it is not clear from the presentation of whether Euler actually proved his guess or just assumed it. Finally there is no tying of loose ends, what exactly does the second approach lead to? Still I must admit that the author manages to convey what he wants to convey.

The fundamental problem with a survey like this is that in spite of the ostensible lack of necessary prerequisites, a deeper understanding is only possible through a wider context. Thus the book works most effectively for those who are hardly in need to be instructed about mathematics not just being a matter of soulless calculation. This is also a dilemma in general for books on popular science, how to instruct without being condescending, and how to reach the intelligent layman[5].

Mathematics is fun, something we all can do, and very different from the deadening routines of school-mathematics. This is the upbeat message from educators. It certainly has much truth to it, on the other hand it is but a glib simplification. Even deadening school-mathematics cannot fail to fascinate the gifted child, and even in calculations, as Wells admits, clever tricks can be envisioned[6] It is true that a large segment of the population can enjoy and appreciate elementary games and puzzles of a mathematical twist[7] Hardy speculated whether this level and extent of appreciation was not actually bigger

[5] which in some sense is a kind of contradiction nowadays, because that creature should by virtue of his or her intelligence no longer be a layman. Now this can be countered first by observing that we are all laymen outside our fields of expertise, without losing our intelligence in the process; and more importantly that the most important layman is the adolescent. Many distinguished scientists can date their awakening to a popular book on science at the crucial moment. In this regard I fear that the popular books available nowadays are inferior to those of the past, by being more condescending and flashy, underestimating the perceptiveness and provocability of the gifted young mind.

[6] My mother does not tire to tell people about my early pre-school exploits in multiplying numbers exploiting ingenious factorizations and regroupings, Clearly, in the sense of Wells, I early on realized that mental calculation could be a fun game.

[7] The latter is an important qualification. The most popular puzzles, such as crossword puzzles and sudokos are not mathematical. Yes, even not sudokos although it has to do with numbers rather than letters.

than that which was accorded to poetry[8]. However, when it comes down to it, some people are much better at it than others, everything is not insight and discovery, muscles have to be developed as well, and the gifted mathematician will of course have no problem with traditional mathematical chores as presented at school, even if they would be bored and unchallenged by it. There is no such thing as an alternate math intelligence. One of the essences of a mathematical gift[9] is the ability of abstract reasoning, just as athletic prowess to a large extent is based of physical fitness[10]. This of course presents a major challenge for mathematical education, how to strike the right balance. Everything cannot be fun, in fact the point of school is to supply discipline and systematics to tasks that are intrinsically not congenial. Thus the build-up of skills belongs to the duties of an education, even if those skills may not be always meaningful unless exploited in an appropriate context.

Now to return to the central thesis of the book - the relationship between games and mathematics. An abstract game is based on rules, to be played it needs some physical manifestation, but of course the particulars of that manifestation are irrelevant as opposed to the need for it.[11] The rules are arbitrary, but in practice rather simple[12], and they define a miniature world. The game itself is constrained by the rules, but it is not in anyway directly forced by them, once a game is launched it lives its own life and becomes something independent from its creator. Chess has rules which can be learned in five minutes (quicker by children), but of course the rules themselves do not explain how the game should be played. This prompts a number of philosophical questions. What kind of object is a game. Is it something external to us, existing independently from ourselves, such as trees and goats and stones (on which we may kick)? Or is it some mental thing, such as dreams and other ghosts of our imagination? Or is it a a third kind of entity, not being properly speaking physical and external, nor being mental and private, such as a dream or our perception of color. Wells claims that it is of the third kind. It may be physically manifested, but of course games cannot be identified by its various manifestations. To what extent is it a product of the mind? The very choice of choosing a particular set of rules is of course a human act of will, those very rules can be communicated to other people, and the game may enter a social realm in which it is played and developed. In the process it will exhibit emerging features in no way apparent from the rules, and unlike the rules not subject to the discretion of the creator. In other words it acts as an

[8] When Hardy wrote that it was no doubt somewhat sensational. Poetry playing an important role in the education and culture of the Victorian times and its aftermath. Nowadays poetry is a most marginal activity, although its vulgarization, popular lyrics, seem as thriving as ever.

[9] the notion of such a one is not necessarily politically correct in the present climate

[10] I once read about some study that claimed that the abstract nature of mathematics taught in schools might turn away many future mathematicians. I found this about as absurd as to claim that the physical exercise classes are too physically demanding and liable to weed out potential athletes. Of course this does not mean that mathematics teaching cannot be too abstract, as little as it would impossible to imagine physical exercise which would crush the apt along with the inept.

[11] Chess could of course be played blindly without resource to a board and pieces. But even such an exercise, too demanding for most putative players, does presuppose some mental imagining of some standard set-up for most people.

[12] chess stands out by having such relatively complicated and ad-hoc rules

independent entity of which questions may be asked and sometimes answers discovered. In this way it differs significantly from fiction in which every 'fact' is freely to be made up by the author, and does not have a pre-existence to be discovered rather than arbitrarily invented.[13]. Thus it is tempting to speak about games, as defined by rules, exist in some Platonic realm. This is of course a metaphorical way of putting it, such ideas of games have no specific positions in space and time (their neurological realizations in different brains are very similar to their physical realizations as boards and pieces say), thus any spacial metaphor has to postulate a realm beyond our ordinary space and time. If you take this metaphor literally you get into the quandary of explaining how communication is possibly between our ordinary physical world and the exalted.[14]. Philosophers have different ways of rephrasing it. Platonism in its most 'platonic' abstraction, is to endow to each consistent system of thought an existence. Thus games correspond to rules, and rules can be identified with strings of characters using some conventional encoding, and thus every game exist, at least potentially, all of its consequences hidden but forced by its constraints. Now strings of characters can be encoded as integers, and than we are back to square one. C.S.Peirce claimed that the integers are more basic than logic itself, and there is indeed some sense to it[15]. Thus questions of games, or more extensively of formal systems are reduced to a study of numbers. Thus we reduce the question of games (and mathematics) to the subdiscipline of number-theory (albeit phrasing problems in a way that is usually not in the interest of its traditional practicioners), and the question of Platonism and mathematics does reduce to whether the integers all exist actually or only

[13] However, serious writers of fictions claim that their characters take on independent lives and the author has to take that into account. This is no doubt an apt psychological insight, and surely without this 'illusion' of 'autonomy' no good and gripping fiction can be written. But I refer to it as an 'illusion' because this sense of autonomy is clearly subjective (and purely mental) in the sense that it cannot be communicated. For one thing the writer 'knows' much more about the character than he or she is able to communicate, that knowledge is thus unlike the rules of a game mostly hidden. Without this ability to share we are unable to test whether two different people would extend the actions of a given character in the same way, just as they may come up with the same moves in a game.

[14] This is a trap into which many critics of Platonism fall, including Bancareff, and also, I am sorry to note, the author finds it fit to repeat. This is about as intelligent as to question the use of 6-dimensional configuration space, when we need to compute say the average distance between two points in a solid. Where is this 6-dimensional space located? It is a space of pairs of points, and does of course not have a representation in ordinary physical 3-space if we insist that every pair of points should be identified with a single point. You may argue that this 6-dimensional space is but a creation of the human mind, in one sense it is, but what about ordinary 3-dimensional space? The conception of which is tautologically created by human minds.

[15] Frege and Russell et al tried to derive the integers from logic, the result is hardly convincing. The great advance in logic, effected by Gdel, was to do it the other way, reduce logic to the integers. I am well aware that this is a glib simplification, the moral though is that the mathematicization of logic turned out to be more fruitful, then the logicization of mathematics. In the words of C.S.Peirce again, the former is but a sub-field of mathematics, while logic itself is a question of ethics. Thus the latter is an attempt to legitimize mathematics in a moral sense, i.e. the search for rigor to free it of contradictions and make it confluent with Truth.

potentially[16] . In my view the author has reinvented Platonism as regards to games, and his main philosophical contention is that mathematics due to its game-like nature, benefits a similar treatment. I will return to a discussion of those claims at the end of the essay in which I will discuss the philosophy of Wells at greater length. But before that let us once again more closely investitage the nature of games, and how it differs from mathematics. This is a rather concrete philosophical question and as such much more liable to be treated intelligently and satisfactorily.

As noted before, knowing the rules of chess or go, or any other similar game whether existing or yet to be invented, does in no way indicate how it will be played. From now on the word 'game' will refer to any of those, and sometimes for the benefit of concreteness be specified to a particular one. Formally a game is a string of moves adhering to the syntax of the game. Such strings of moves, mindlessly churned out by a programmed computer or by a single person, do not really constitute a game in its social and psychological sense. In a game there are two minds partaking, and each new symbol in the sequence, is the individual response to what has gone on before. Playing a game of chess is not a solipsistic exercise (which makes it frustrating to some of us seeing our well-thought out plans being frustrated by a malicious opponent) and in particular you cannot play chess or go against yourself, not as a real game[17]. When you play an interesting game you do not follow algorithms, you follow strategies[18]. Strategies are by nature impossible to convey, at best they can be suggested. Human strategies evolve in response to the rules of the game, and in any attempt to describe them they will involve secondary notions such as 'weak squares', 'pawn value of pieces' and others I have no idea of. Those notions are very vague, but also very crucial in forming and evaluation different strategies. Strategies are inventions of humans in response to the game environment, and they no doubt could be seen as adaptive evolutions. Strategies are related to each other, so in actual human playing through the games, there is a certain similarity. Chess openings are similar, and responses are also so. The natural meta-game question to ask is whether this is a social coincidence. Could there be other traditions of playing? Would other alien civilizations evolve entirely different strategies and ways of responding? Would those be superior or inferior to ours? The latter kind of question is normally not the kind which has a meaningful answer but in chess and go and such games it actually has. There is but one way of finding out. Pit the two players against each other! This begs the question about compatibility of play. Does transitivity work? If A beats B and B beats C can we be sure that A beats C? In other types of games such as soccer and hockey this is far from being true, but in chess and go, it is

[16] Clever notation, ultimately reducing to recursive definitions, allow us to write down numbers of mind-boggling magnitude, far harder to encompass (and manipulate?) than infinity. In particular such numbers go beyond any imaginable interpretation as cardinal numbers, out of which the number concept supposedly is extracted.

[17] Of course that exercise is often necessary when you actually play a game, trying out in your mind a multitude of possible moves and countermoves.

[18] What about computers? Clearly they follow algorithms unlike human players. On the other hand computers are programmed (at least ultimately) by humans, and they follow certain strategies in conceiving the programs.

true to a rather high degree, and truer the more definitive the beatings are.[19] Now is this mathematics? I would say no. The difference between doing mathematics with chess and playing chess consist in the kind of questions which are being asked. A mathematician would be interested in other questions than the player, the latter is only interested (as a player) in what can improve his game. Often it is in the nature of suggestions and vaguely phrased advice, which would be meaningless by themselves but assume meaning in the context of the players understanding of his game. A mathematician would ask meta-questions, as Wells insightfully explains, such as the number of games possible, whether there are ultimate strategies. Sometimes there would be an overlap. A certain position would enable white to mate say in three moves, no matter the responses of black. This is a purely mathematical statement, a precise statement derivable from the rules. (Most mathematicians would not be interested in that kind of theorem, unless for sentimental reasons in case they are intrigued by chess playing). Such a statement would be of interest to a chess-player. Once he recognized the proof of such a statement, there would be no longer a game. If he is black he certainly at this point would give up, and were he white he would lose interest. At this position the individuals have no longer any significance, the situation is forced and rigidified. It is no longer human, free choice is no longer an option.[20]

Mathematics can be thought of a game because after all it is restrained by certain syntax. Certain things can be done, others not. But what is meant by playing the game? Algebraic manipulation is one thing. It is a kind of solitary, in which starting from one position you need to reach another. Say in the simplification of trigonometric identities. There are no opponents involved, but otherwise it involves certain vague strategies to be successful. Practice certainly helps matters, but it is neither wholly necessary and far from sufficient. Then of course the game changes, because the choice of the final destination depends on a higher kind of game that is being played. A given expression can be transformed in a multitude of different ways, each one useful for specific purposes. Most often one cannot predict in advance what form will be sufficiently useful.

The analogy is clear, mathematics can be formalized by giving a certain number of axioms and syntactic rules (which of course could be subsumed among the former). This establishes the formal analogue with the rules of a game. Then in order to prove things[21], you need to actually play games, employing various strategies which you partly

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[19] In the most 'objective' of athletic games, where a result is not contingent upon a definite pairing, such as running. If A runs faster than B and B runs faster than C than A runs faster than C. This is more or less a tautology if suitably interpreted as assigning to each player a number. In chess one can also assign a number based on individual encounters, the precise way it is being done is of course a convention, as opposed to the principles that lie behind it. As far as I can tell it has a fair amount of consistency.

[20] Now it could be that chess is deterministic, i.e. there could exist a winning strategy, and a winning strategy is a mathematical concept, it is precise and based on an algorithm. Usual strategies are vaguer, and in a sense more interesting. If this is the case, and I guess one can prove its existence, without in anyway understanding how it would work algorithmically, then chess is only interesting to the extent that it has not been 'seen through'. Many simple games, however, have been seen through. This is an achievement of mathematics.

[21] Superficially similar to the manipulations of algebraic expressions, as in the mechanistic world of the

acquire through extended experience, partly through instinct and occasionally by pure luck. It is hard to convey those strategies, if it was possible, mathematical education would have quite another focus[22]. But certain general remarks can be adduced, such as the recognition of underlying patterns, the drawing of analogies, even the drawing of analogies between analogies (and so on?). The purpose of his examples are not as much as to instruct the reader mathematically (for this a more systematic approach is necessary) but to illustrate what those very vague and abstract notions actually mean when manifested in particular instances. A professional mathematician would of course know already what he is talking about, but that does not mean that he does not get pleasure from re-acquaintance. Just as a connoisseur of art, landscape or wines say, does not mind having his knowledge reconfirmed. The examples are elementary, would he dig deeper into mathematics, the interconnectedness would become even clearer, as well as the magic of certain concepts. True understanding does of course only evolve when you solve something yourself and understand it 'fully'. Mathematics, as the author reminds the reader, is no spectator sport, and in fact its audience more or less coincides with its practioners. Now the examples are elementary, that has as just noted its limitations, on the other hand many professional mathematicians may appreciate elementary examples more than complicated ones, even if from their own speciality, for the reason that while most of actual mathematics is churned out and contain no real gems, such are to be seen more often in the elementary kind due to pre-selection. Also, the professional mathematician rests his result on many other results, some of which he may not even understand but takes on trust, and this is from an intellectual point of view deeply unsatisfactory[23] This is more or less inevitable, the capacity of a single mathematician is very limited, just as our everyday life and its comforts and possibilities, rest on high-technology which we normally understand poorly if at all, a professional mathematician exploits technology of which he has but a superficial understanding. This leads to a certain alienation, as it also does in everyday life. To return to more elementary mathematics, in which everything involved is more or less understood, can be felt as a rejuvenation of an original enthusiasm that may have gone partly stale. When you understand everything our imagination is jogged in ways it might not necessarily be when you are pursuing something with a narrow set of techniques you have grown accustomed to use.

One example of a formal system is Euclidean geometry. Here the axioms are not

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logician, mathematics is a series of tautologies, trying to go from one to another, through small steps. Of course this is a travesty of what proving theorems actually amounts to.

[22] Some people in didactics, such as Mogens Niss, actually believe that eventually the discipline will have revealed the mechanisms behind mathematical thinking. Now in all fairness it is not clear in what sense this claim of Niss should be taken. If it means that mathematical strategies can be reduced to algorithms, this is certainly believed to be totally unrealistic, and the logically minded would be tempted to supply a diagonal argument. If it just indicates in an illuminating and surprising way the mathematical mind works, it is more reasonable, although I would very much doubt that the insights would be achieved my mathematical educators.

[23] A classical example is Hironakas resolution of singularities. How many people using it has actually tried to read the proof, let alone understand it? I suspect that only a minority has looked up the relevant reference.

arbitrary but in a sense empirical being in the nature of incontestable truths. Geometry is about to encode the spacial reality in which we dwell. This is in contrast to chess or go, where it does not make sense to ask whether the rules are true or not. We simply make them true, that is the metaphysical essence of the game. Now in addition to our logic we also have what the Germans call 'Anschaung', so to speak a visual overview, which means that in geometry we have an intuitive idea of what is or should be true at least when it comes to the larger questions. This is of course not the case with an arbitrary system of axioms, in which there will be no intuition, and hence no meaningful check. In geometry we are not blind, nothing really obvious evades us; while in a formal system we may overlook the most central things, because unlike geometry and our physical world in particular, everything does not immediately manifest itself.[24] As Wells reminds us, the perfect metaphor for understanding is seeing[25], this means getting an overview, understanding how different things relate to each other. In short to have the 'pop up' effect we experience when we suddenly make sense of a confusing set of data.

To be a good school-boy in Euclidean geometry you have to catch on, not so much to the axioms, which clearly become second nature to you, but the prevailing strategies. Those can be summarized as being based on the various congruence theorems, and than constructing figures with many triangles, and then using them as intermediate ladders to compare lengths and angles. This is inevitably such a vague description as to make no sense to anybody not already versant with it. This is the synthetic strategy, and to my knowledge there is only one alternative - the so called analytic approach initiated by Descartes. The former is intellectually more satisfying, because it is more like an art, and understanding is of an aesthetic nature; the latter is more like a mechanical method, far more efficient in establishing the correctness of theorems, but with the drawback of no real understanding[26]. Now the point of Descartes was to transcend the ad hoc reasoning of synthetic geometry and remove the necessity of ingenuity. This is what constitutes

[24] One possibly apocryptical example is that of the Oxford graduate who decided to investigate metric spaces in which the triangle inequality was reversed. It does not take too much thought to convince yourself that such a space could have at most two elements. The graduate had not realized this and as a consequence proven a hoist of theorems all of them rendered trivial by this observation. The example is cautionary, especially when pursuing mathematics as a formal game. There might always be observations that greatly simplifies the picture. If we only knew the physical world from some kind of axiomatization of it, many features we now find unavoidable might be unknown. There are many features of say Hyperbolic geometry which would be immediate to us would we live in it provided the unit of length was sufficiently small compared to our senses, but would not be so, would we start to derive theorems from the axioms.

[25] In fact as he points out the most common use of the word 'see' is in its metaphorical sense.

[26] the late educator Mats Martinsson, made in my opinion a useful distinction between something being evident and something being obvious. The latter relate to this gut reaction of understanding, it is a kind of conviction that is direct and available to the primitive; the former is the result of something akin to a calculation, in principle we know by adding many small steps that something is true, but the global understanding is missing. Obvious truths are very hard to convey, either you feel it or not. In that case it is very subjective, which does not prevent it from being even more convincing. Evident truths on the other hand are more objective and much easier to communicate. The conviction that $1+1=2$ is very hard to explain, however the fact that $1734 \times 2146 = 3721166$ is easy to explain, but much harder

progress in science as well as in mathematics. And of course analytic methods did not remove the beauty of mathematics, it just moved it to another level.

So if we want to summarize. Mathematics can be seen as a game with certain rules, but like all games, it transcends its rules. Furthermore, while chess and go are dead-ends, enclosed in themselves, forming closed miniature worlds with no applications[27] mathematics on the other hand form an interconnected web with ramifications actually going beyond the subject itself. It is true that a chess player may in his games involve more ingenuity in thinking than a mathematician does in playing his small ones; but the efforts of the grand master lead nowhere really beyond his own triumph, while the mathematician takes part in a huge collective effort, laying a brick on which others may use as support for further bricks.

So in what sense is the playing of games and mathematics Platonic? In what sense do they exist outside humans? The actual games that are played by humans form a tiny subset of all possible ones. There is nothing canonical about this subset, it is a more or less fortuitous consequence of random events and historical contingencies, admittedly to some extent restrained by rules and the drive for people to improve their games, leading to some kind of evolution. There is nothing Platonic about that. On the other hand meta-statements of the game, involving forced positions and such things, those are incontestable truths of the game, and would in principle be discovered by alien civilizations. As to mathematics, the actual theorems and statements of relations between different concepts, are true or not regardless of human desires, however the selection of such is an artifact of actual human history. Furthermore humans do not only seek to establish true facts, but also to 'understand' them. This is a truly human notion and has nothing to do with platonism, and plays a crucial role in mathematics as a human activity. One may compare the relation between mathematics conducted and the mathematical world, to a painter depicting a scene. The scene itself exists outside the painter, but what appears on the canvas is an expression of the painters visual exploration and its tactile interpretation, which obviously are part of the painter. There tends to be a confusion between those two aspects of mathematics, just as if there would be a confusion between the rules of a game and the actual playings of the same.

And then some final remarks. First there are a lot of books extolling the virtues and fascination with mathematics, but few if any on the wonders of chess and go? I have no idea of what it means to have an intuition in chess, what it actually means to follow a strategy. The author does not make any attempt to convey this. Maybe it is something that only be achieved through playing many games? On the other hand children who learn to play at a tender age obviously must catch on to something, otherwise there would

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to directly see, and is believed on the authority of the method, which we might understand, but whose output goes beyond our immediate understanding.

[27] There are always claimed collateral applications such as a training in thinking and extended concentration, just as mathematics and latin were considered to train the brain in classical school education, and just as nowadays the humanities is thought to enhance the moral sense of an individual and widen his perspectives, whatever that means. To some extent this might very well be true, yet it is marginal to the real issues involved. For one thing why do we need those training in thinking and widened perspectives. To play better chess, to appreciate mathematics more?

be not sufficient motivation to continue. What is it they are catching on to? Does this discrepancy between chess or go and mathematics depend on the former being closed activities, while the latter is an open one? And that all people have some rudimentary mathematical understanding, in particular they are able to express, or at least appreciate their rationality through mathematics, but most of us are barred so from games like chess?

Secondly the book does not touch upon programming. This is in many ways a kind of game, exploiting the fact that the spirit has become flesh. In programming you have an opponent of sorts - the computer. Whatever you do it elicits a feedback, which is sadly missing in mathematical exploration. Thus it may keep you going when you ordinarily would have stopped, because fixing a bug in a program seems always just to be around the corner. While mathematicians are usually (but of course not always) bad at chess, most of them are good at programming, if not necessarily idiomatic in their approaches[28], and may find great satisfaction in seeing their ideas confirmed (case by case). Computer games are supposed to be addictive, but what more addictive game there is than simply programming, preferably from first principles? It gives you a heady sense of control and power, when you manifest elusive thought in tangibilities.

Finally as to the authors philosophical conclusion. I must admit that I have a hard time following them. First he claims that language spans a wide spectrum, from the very formal to the vague and metaphorical, and that most modern philosophers, whom he characterizes as liberal modernists, are fixated on the formal side; while he as a rational modernist, understands the whole span? Is this not a caricature of philosopher, making them into mens of straw, the easier to demolish them? It is true that much of analytic philosophy seems rather narrow[29], that there has been a fashionable tendency to assume that anything capable of study has to be formalized. (Hence the notorious phenomenon of behaviorism, denying the inner workings of the mind). The natural question is to ask to what extent those are mere technical limitations, imposed by an ambition of making philosophy a science, or reflect wider metaphysical beliefs. Surely the ambition of Artificial Intelligence to explain the mind algorithmically, is a meta-physical statement. It is to most people very disturbing[30], but by its nature it is very hard to refute it, unless making an appeal to intuition[31] which is just one step below the desperation of appealing to God. Any refutation of the AI claim seems to involve some version of Cartesian dualism, which

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[28] using the computer for mathematics does involve another kind of thinking. Mathematicians (like myself) are liable to translate mathematical reasoning into codes, while in programming it is usually more efficient to be more primitive. If you want to systematically decide when you have quadratic residues with respect to a fixed prime say, it is far easier and more natural to simply list the quadratic residues in an array as a first step and then subsequently search when examples present themselves, than to implement the formulation of quadratic reciprocity.

[29] analytic philosophy has the reputation of being rather technical in a more or less misdirected effort of becoming scientific and thereby abandoning the traditional concerns of philosophy. Some philosophers do not agree with this, but sees analytic philosophy as just being the continuation of classical philosophy, while alternatives such as existensialism are but faddish perversions

[30] the present author of this review being no exception, along with the author of the book

[31] The classical response by Penrose is to invoke Gdels theorem, something logicians dismiss as being an egregious misunderstanding of its scope.

is another stand that is anathema to most thinkers. The most reasonable resolution , as proposed by McGinn, would be that even if the mind is reducible to the matter, the way this is done is beyond the capacity of men to make sense of. A mental representation of the essentials of the brain cannot be found inside the brain itself[32] .

As to the limitation of the physical as opposed to the mental Wells looks for a third category different from both, in which games would naturally fit, and hence by implication mathematics itself. Still he is not unaware of Poppers three worlds, the first one relating to the outside physical world, the second to our private mental ones[33], and the third consisting of the fruits of the latter, which has a feedback to the first, through the physical manifestations those can take [34]. What is wrong with this third world as an abode for abstract games? The contents of the third world are communicable and shared among many minds. They are obviously not physical, nor private incommunicable phantoms of the mind, such as our perceptions, they are of the mind, yet independent of it.

Russell and Frege had a common goal, namely that of deriving mathematics, or more specifically arithmetics from logic [35]. One must say that the attempt failed. This is not noteworthy, not necessarily damning, most attempts fail, and by failure much can be learned. I diagnose the failure to the assumption that logic was more basic than the integers. The definition of integer that they supplied, and which later was refined by von Neumann, is intuitively perverse, although the form it was given certainly served some special purposes[36] . One may be dismissive of those who no longer can respond. However, one risks appearing like a crank. One may well learn from Poppers criticism of Plato and Marx. It is easy to hit the weak points, but the real way to deal with opponents is to

[32] This is in a sense a bit misleading, one should not expect a single individual to understand how his brain works, any understanding would be of a more instrumental kind collectively held. Still such an understanding ought to involve some new elementary ways of looking at things, which by themselves would be accessible to individual minds, such as the diagonal trick of Cantor. This in a sense is reminiscent of Gdels proposal that the unsatisfactory state of set-theory would be resolved would some new natural axioms be discovered, whose truth we all as rational beings would instinctively understand. In fact, as Plato puts it, remember as things we have always known but forgotten

[33] Clearly Popper is in some sense a dualist, or more accurately as we will see, a trialist

[34] Penrose in his 'the Road to Reality' describes a similar kind of loop. The brain is just one small part of the universe, mathematics is just one (small) part of the contents of the mind (brain), and the kind of mathematics that is employed in physics to explain the universe, is but a small part of mathematics.

[35] As the Swedish philosopher Wedberg points out, this can be seen as taking up the challenge of Plato to provide a dialectics in which the unproven axioms of geometry should be grounded. A task more exalted than mathematics

[36] In the same way one should interpret the formalism of Hilbert. It was not necessarily so that he thought of mathematics in a formal way, only that he had a very specific goal in mind, to show the consistency of mathematics, so we no longer needed to worry about it. In order to show that he had to encode mathematics in a certain way, it did not mean that he identified mathematics with its formal codification. It did not reveal the deeper aspects of mathematics, just as a set of simple axioms do not immediately reveal their secrets, for this human intervention is needed. The consistency of mathematics would of course not be the last word on the subject, rather the first. What Gdel showed was that the form in which Hilbert approached the task was bound to fail

concentrate on their strong points and if necessary even strengthen them. Popper admits to a deep admiration for Plato, as well as lauding many aspects of Marx[37]. On the other hand he minces no words expressing his total distaste for Hegel. It is true that some thinkers one is out of sympathy with, if so, they should be ignored, because little will be gained by engaging them. Others inspire a strange mixture of sympathy and revulsion, their views are disturbing but not easily rejected. Sarcasm may be a tempting weapon, but it is counterproductive.

Leaving Russell aside, as in the case of hardcore philosophy, Frege covers him[38]. We note that Frege was abhorred at the idea of basing the notion of number on psychology, the least developed of all sciences, pretending to give the foundation on what seemed the hardest rock[39]. In what sense did his putative failure to appreciate the game-like nature of mathematics disqualify him? I must admit that I cannot follow the author here.

As to the attack of Hersh this makes more sense, if for no other reason that he is alive and can respond, and provocative statements may have a salutary effect. Now Hersh claims that mathematics is a humanistic subject not a scientific one. I am in sympathy with this claim, but I do not see how it invalidates the Platonic persuasion. As I have noted above, a central aspect of all mathematical activity is understanding. We do not just want to know what is correct or not, we also want to understand 'why' something is true. And thus understanding, and this meaning of the why, are human features, having nothing to do with mathematics objective existence. As noted above a painter expresses his visual understanding of a scene by putting it on a canvas. The painting of the canvas is part of that exploration, while the scene that inspires him, is outside himself. The objective existence of relations between mathematical objects, relations we have no control over, does in no way detract from the possibility to speak about mathematical beauty, in fact it is rather a prerequisite. Now Hersh makes the strange, and thus potentially interesting, claim that mathematics is objective as far as the individual is concerned, but not from the point of view of mankind as a whole. This may seem to make no sense. But on the other hand there are plenty of phenomena which fit that bill, the most obvious ones being human institutions taken in the widest sense. I am thinking of language and morality, phenomena which would not make sense without humans. Another, less deeply entrenched institution is that of money. Clearly an individual cannot flaunt the conventions of money, but society at large? The value of money hinges on a shared convention, in times of severe inflation this shared commitment is being eroded, but this erosion is only manifest on the level of society, although each individual imperceptivly contributes to it[40]. Thus the natural question to ask is to what extent is mathematics like language, or morality, or even money? As to language both the author and I agree that mathematics is not. It is a fashionable misconception nurtured by taking metaphors like that of Galileo literally, or the fact that

[37] In fact my appreciation for him was rather enhanced than demolished from Poppers critique.

[38] Among philosophers I have learned, Frege is considered more interesting than Russell. For readers of biography and collectors of anecdotes, the reverse is clearly true.

[39] Some people such as Brian Davies, wants to base number on neurological activity, this seems to me to be to carry the awkward step of Russell-Whitehead, to an even more impenetrable level

[40] The phenomenon of inflation can in a sense be seen as natural phenomenon, given the rules of the game of financial exchanges.

mathematics is used to model the physical, and lately also the social world, and thus being reduced to some kind of tool (and as all tools likely to be eventually superseded by more congenial ones). Finally mathematics has a history and evolving tradition, those are also obvious examples of something which is objective at the level of the individual, but has no independant meaning outside humans. The standards of rigors in the mathematical community is an example of such evolving traditions, and hence what is considered to be correct and established. The Platonic nature of mathematics allows one to make a distinction between a theorem being true or just believed to be true, a distinction which cannot be made in the context of fiction.

In conclusion the book is a refreshing read. It is aimed at a more elementary level than that of similar books by Ian Stewart and Marcus de Sautoy, who have the ambitions of treating aspects of modern mathematics in some depth. The level of Wells book is that of an ambitious book on puzzles, staying with rather elementary material, but treating it from a philosophical point of view. This of course makes the meat of it more accessible than would be the case of Stewart and de Sautoy. However the concluding chapters on philosophy is somewhat disappointing. The division of philosophers into liberal modernists and romantic ones, is a bit too obviously self-serving, and as such inviting ridicule. (See here what fools those great men were compared to me!) Clearly the author could convey this idea in a more subtle way, allowing the readers to draw the conclusion instead. A humbler and more respectful attitude towards Russell and Frege might also get the points across more effectively. To be original in philosophy is very hard, especially if you want to do something profound and valuable as well. Many of the ideas of Russell and Frege are now hackneyed, but the point is that they were not at the time. I think they deserve some respect, which of course does not preclude fundamental disagreement, as this review has illustrated.

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