What is Mathematics Really?

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The philosophy of mathematics is a dead subject ossified since the crisis of foundations at the turn of the last century. Mathematics is a rich subject with a fascinating history and a multifarious practice. In short it is a human activity on par with language and art, and should be treated as such. Thus Hersh rejects the standard philosophical approaches of Platonism, formalism and intuitionism (or in its modern incarnation - constructionsm) and instead champions what he refers to as a humanist approach. He sets out to present a few criteria of what a succesful philosophy of mathematics should fulfill, some of which are deemed essential, others merely optional, and then at the end of the book he grades the performance of his own philosophy and invites the reader to do the same with the standard approaches. A substantial part of the book is devoted to presenting more or less in chronological order the mainstream philosophers and the so called maverics respectively. Such an activity of collecting all the great minds of the past as if into a class-room and then give them grades is of course a very pleasurable one (and one is reminded of the very same delight at mince-meating his opponents Frege displays in his work, and on which Hersh comments with a mixture of disparagment and sympathy). In addition to that, in a later aside, Hersh also classifies his philosophers on a political left-right spectrum, concluding somewnat triumphantly that almost all the mainstream philosophers were right-wing (Russell the old aristocrat is grudgingly accorded inclusion among the left), while the champions of humanism are almost all left-wingers (including Aristotle who is opposed to Plato). Hersh clearly is a liberal American with a soft spot for such politically correct phenomena as feminism, ethnomathematics, and social constructions, and sitting on the throne at the Last Judgement and separating the leftist sheep from the rightist goats, does somehow diminish his achievement, reducing him to a cantankerous old man with an axe to grind. This is a pity, because what Hersh does well, he does excellently with aplomb, and the book turns out to be a veritable page-turner hard to put down, even in the middle of the night.

Now I thoroughly agree with Hersh that the standard philosophy of mathematics has very little to do with how mathematics is actually practised by mathematicians, and that its focus on foundations and logic is sterile; that a serious and fruitful philosophy of mathematics has to come to grips not only with the history of mathematics, how it has evolved, meaning how new concepts have arisen, but also with the intuitive component of mathematical thinking, without which its history and concepts would be unintelligible, and to realise that paradoxically, deductive thinking is only one aspect, and by far not the most important, of mathematics. Such a philosophy and history of mathematics can only be conceived and written by living mathematicians, not by academic historians and philosophers, who, in the words of Hersh, have only been treated to the public face of mathematics, never been privy to its backstage machinations. Furthermore it is important to note while there is a remarkable consensus in mathematics as to what is true there is less so when it comes to what is interesting and what is beautiful, and those more subjective aspects are in a sense also the most important ones, which should, and often do, play a far greater role in what is accepted in journals than what is merely correct, because ironically while there may be no place (no permanent anyway) in mathematics for ugly mathematics, according to Hardy, there is always place for what is strictly not correct[1]. Thus mathematics is not this inhuman monolith, which Platonism makes it out to be, but a living, throbbing human activity, with all the weaknesses and faibles the living inevitably is riddled with. That mathematics is a human activity, is of course a truism that it would be impossible to deny. That personal ambition and the desire for social confirmation not to say admiration may play as an important role as the discovery of timeless truth in the career of a gifted individual is no secret. But in putting too much emphasis on the obvious, there is a danger to failing to distinguish between the practice of mathematics and mathematics itself. Of course to Hersh such a distinction may very well be illusory, there is a practice of mathematics, but to envision a mathematics beyond its practice, is to fall into the Platonist trap, postulating an eternal, inhuman realm, and thereby committing the grave sin of mysticism and obscurantism. It is with this I want to argue. Hersh falls into a similar trap as the materialist who proudly proclaims there is no metaphysics, thereby not realising, as Collingwood points out, that they are making a metaphysical statements, or the logical positivists who are unaware that their own principles cannot be subjected to the same, and thus logically to be discarded. To argue that the practice of mathematics is the same as mathematics itself[2] is a kind of behavourism that discard consciousness as being unobservable.

Why is Platonism such a repulsive idea? In many ways it is naively assumed by most mathematical realists who in their everyday working life subconciously display a Platonist attitude, but when challenged and coming close to actually articulating one, they invariably make a disclaimer that they do not want to be seen as Platonists. This phenomenon of being temperamentally a Platonist, but not wanting to admit it, can also be observed beyond the mathematical ken, in biology as well as in theoretical physics,. The suspicion that we are dealing with anxiety about political correctness is inescapable. One way of making Platonism respectable, and simultaneously getting rid of it, is the notion of formalism. In many ways this is a materialistic conception of mathematics, which is seen as being built up by 'atoms' of primitive truths (so called axioms) subjected to some universal deductive rules. Mathematical statements are nothing but strings of meaningless symbols mechanically generated from the atoms. The whole thing is uncannily like the vision of Laplace, in which a superior intelligence, given the initial conditions of all the particles in the Universe, instantaneously would no all of the future as well as all of the past. Such a view of mathematics cries out to be mechanized, and in fact the computer is the perfect medium for such mechanization. The universal deductive rules constitute the hardware while the axiom system are the varied inputs, out of which unending sequences of theorems will be mindlessly churned out. The idea is chilling to most mathematicians and seen as a monstrous travesty of the practice of mathematics. A mathematician is not primarily inter-

[1] this of course reminds one of Weyls quip that if he would be forced to choose between what is true and what is beautifull, he would choose the latter

[2] Similar to the definition of intelligence as what is measured by intelligence tests

ested in what is true, but why it is true, to achieve 'understanding' a concept transcending any formalization. Now after over-coming the first shock of this, understanding that this simply is an attempt to show that mathematical truths are independant of us humans, the result of mechanized calculations, and that Platonism only consists in realizing that arbritary strings of symbols all exist somehow (show me one explicit which does not exist!), just as Mozart is reputed to have said that all music exists, it is just a matter of writing it down, the mathematician realises that by making all mathematics meaningless, the formalists have introduced concrete objects (axiom systems) and formulated precise rules for their manipulations, and hence precise questions to be asked, such as internal consistency, rendering the whole susceptible to mathematical treatment. In short, all of mathematics has been mapped (and this not only metaphorically) into a very special sub-discipline of mathematics, namely one of a strong combinatorical flavour, whose objects are far less interesting that the classical objects of mathematical inquiry. Formalism in mathematics has a long pedigree (and Leibniz is surely one of the great forerunners) and certainly was part of the ambitions of Frege and Russell to reduce mathematics to mere logic, yet it is in its perfection associated with the name of Hilbert. This is very unfair because Hilbert was no formalist in temperament, the formalist stratagem was simply a tool to once and for all settle the disturbing question of the consistency of mathematical thinking. Mathematics has to be ultimately correct, otherwise what is it? But this kind of correctness is only the very beginning, it is this that makes it real, putting it into logical space so to speak, a kind of analogue (Platonic or otherwise) to physical space-time. Hilberts program was not philosophical, it was technical and mathematical. But what happened was unpredictable. By mapping mathematics into a tiny subset of itself, the temptation of self-reference was irresistable. It was Gdel who performed this Cantorian Diagonal trick on mathematical thinking (as conceived formalistically). The diagonal trick to my mind being the manifestation of free will. The result is well-known to us all, and in particular it made the formalist project collapse. I do not think this affected Hilbert too deeply, in many ways it was a mathematical project (be it meta-mathematical in the context), and like most mathematicians he surely was used to seeing many promising mathematical ideas collapse. The formalist project was dead, at least philosophically, but not mathematically. After all the kind of undecidable problems Gdel showed the existence of where mathematically speaking very contrived. And the formal proof-theory, which Hilbert had developed (only to be discarded unsentimentally when it had done its job?) became simply a branch of mathematics, shedding all ambitions of metaphysically justifying the whole. Nowadays logicians are just specialist mathematicians, some of them finding a congenial niche in computer science, while analytic philosophers find themselves high and dry after the tide has retreated. But formalism does live on in the idea that informal mathematics is just an approximation, and what gives mathematical reasoning its ultimate justification is that it in principle can be reduced to formal language, and then mechanically checked. This is a gross and hence misleading simplification, yet it has it points. Mathematical results can often be tested by calculations performed by computers. Such confirmations are often gratifying, ironically no matter how much lip-service is made to the deductive method, mathematicians are but humans, well aware of their fallability, and conviction, yet another one of thus transcendant notions like 'understanding', is not usually sufficiently aroused by mere chains of arguments, but how well the result accords with other results, mathematics being seen as a weaved web of many disparate threads where everything must harmonize with everything else, this being what is meant metaphysically by the unity of mathematics[3].

But if formalism is thoroughly discredited, does that not mean that Platonism is as well, being so wedded to it? No, Platonism was wedded to formalism, just as a rider is wedded to a horse, if the horse is shot down under him, he simply choses a new horse, and to that I will return below. And as to the death of formalism, it is, for better and for worse, greatly exaggareted. It is dead as a viable philosophy of mathematics, but not as an important aspect of it. Playing formal games can be quite entertaining, and not seldom instructive, and as Dieudonné has pointed out, axiomatics has it advantages, making for economy of thought and effort, and the discovery of formally 'isomorphic' areas in widely different parts of mathematics is always exciting, although this is not, contrary to the opinions of those with a formal temperament, the main business of mathematics. When it comes to the influence of formalism on the teaching of mathematics, it has been rather unfortunate. Set theory, an obsession of mathematical foundationalists has proved to be a convenient terminology in which to phrase many mathematical concepts, a view especially propagated by the Bourbakists[4], but when taught to school-children is merely puzzling[5]. Yet, to many people this formal way of presenting mathematics has seductive advantages, although in many cases all too seductive, leading to a perverting charm of insipid generalization[6] . And as noted obliquely above, formalism is often a powerful tool, enhancing the human ability to reason by long strings of arguments (somehow a kind of calculation), and much progress in modern mathematics would never have been made without such formal tools, the discovery of which constitute an important aspect of mathematical research. Finally a formal attitude, dispensing of meaning, can at times be liberating, allowing hidden connections to be seen. Although the reduction of mathematics to formal calculation is seldom feasible, the ability to formalize if needs be, belongs to the skills of a professional mathematician.

Any philosophy of mathematics, as Hersh rightly notes, has to take into account two fundamental experiences of the working mathematician. One is of course intuition, and the other, closely related to it, the tangible sense of reality, fully comparable with that of the physically external world which we all share. In doing mathematics you are constrained by circumstances, inconvenient facts kick back at you, and you cannot simply will them away, as you can when writing a novel or discussing philosophy for that matter. Without this most mathematicians would not be mathematicians at all, the activity would be

[3] Hersh refers to the fragmentization of mathematics, implying that its unity is but a myth. This is clearly a matter of confusion of categories and should not be taken too literally.

[4] A group G is a set on which a composition law (i.e. a particular subset of $G \times G \times G$) with some special properties. Be it in additional topological, it means that it has a distinguished collection of subsets, called open, closed under arbitrary union and finite intersection, compatible with the composition law etc

[5] Any presentation of set-theory, that does not go into infiniet sets and cardinalities misses the point of sets as an exciting mathematical idea and becomes a rather pedantic exercise.

[6] category theory is one such egregious example of mathematics going haywire, although one cannot deny that it still has its enthusiastic proponents claiming they can put it to good use.

sensed as meaningless. Traditionally Platonism provides a plausible explanation. Mathematics is real, and intuition is a direct means of perceiving this mathematical reality. With Platonism Hersh has no truck, it is to him simply an outdated superstition, whose preserverance into modern life is a kind of intellectual fossile. To believe in Platonism is akin to believing in the acts of God, even after God himself has disappeared. In the old days when the belief of the existence of God was not only acceptable but imperative in polite society, a Platonistic viewpoint was fully in accordance with everything else. Just as God created the world, he also created mathematics, all of it residing in his mind, and thus the study of mathematics was ultimately nothing but the celebration of God. But now in a secular atheistic world? Surely God is not let in through the back window, in his new incarnation as mathematics? Is mathematics just in the mind of the individual? This idea, known as psychologism, was passionately refuted by Frege[7], and since then, Hersh notes, it has never dared to rear its ugly head again. Hersh proposes instead that mathematics is a social phenomenon, this he claims assures its objectivity, as it then transcends every individual mind. On the face of it this hardly seems to be an explanation, except as a kind of collective solipsism, yet as he notes, we as individuals are in fact iron-bound by many social institutions, which would never make sense, let alone exist, outside a human context. Money is of course a rather abstract concept, but for most people a dollar bill is far more concrete than many things it can buy[8] . The legal system in a nation, is likewise a social construction, yet it can in many countries kill you as surely as a jump from a high cliff. Popper, a maverick in the eyes of Hersh[9], speaks about the three worlds[10],one - World 3 of constructs of the human mind, is particularly close to the conception of Hersh. To Popper World 1 is the outside external world, while World 2 is that of our individual minds, including our most subjective so called qualia, but as such nevertheless a product of World 1, just as World 3 is a product of the individual World 2's and very much affecting World 1. Collingwood expresses similar ideas, taking thoughts as basic objects, freely to be communicated in it their objective aspects from one mind to the other. Language and Art are other denizens of World 3, and in order to discuss the objectivity of mathematics it would be instructive to compare it to other, less controversial classifications as social conventions. Unfortunately Hersh does not do this. How does mathematics differ from say language? Language has rules of course, some learned others somehow innate. Yet when we want to really pinpoint correct usage of language we come up against the nebulous concept of accepted usage. Words do not have definite meanings, they have meanings we attribute to them. Language changes if slowly over time. Those changes are usually not the acts of individual wills (although some people may be in fortunate positions to coin

[7] How could the idea of number, Frege writes in his 'Grundlagen der Arithmetik', be based on the flimsiest of sciences?

[8] This can to some extent explain the outrage that many people feel in face of inflation. Money is tacitly supposed to have some intrinsic worth, and when this is being hollowed out, the ground under our feet may literally be shaking, as if some great deceipt is being levelled against us, which in one sense it really is

[9] in mine a commendably lucid and traditional philosopher

[10] Incidentally ratherv similar to the three, in each other nested worlds, presented by the self-appointed Platonist Penrose in his 'Road to Reality'

new words and get them accepted) but as the result of statistical decisions so to speak within masses of communicating people. In fact languages may be studied as objects, independant of humans. There are such notions that important features of it are evolved and thus part of mankinds biological heritage as opposed to its social. How does language compare to mathematics? Some people claim, not only metaphorically, that mathematics is a language of science. I do strongly disagree, for one thing mathematics is not a natural language (especially its formal codification, which is the closest it comes to language in my opinion), and thus we need to properly understand an extension of the notion of human language to intelligently being able to discuss that issue. Yet the acquisition of language as opposed to the acquisition of mathematics are profoundly different experiences. Mathematics involves conscious reasoning, while language, when properly appropriated, involves no such thing at all. Conventions are legio in language, while they are notably absent from the essence of mathematics. Basically though there is a very definite difference in the feeling of objectivity, the more you probe into mathematics, the more apparent its truth nature, the more you probe into lingustics, the hazier the notion of language truth. Linguistic may be a fascinating subject closer to the heart of most people (including many a mathematician), yet its depth seems only within human grasp. When we come to art, and especially the discussion of art, arbritariness and convention seem to stare us in the eye wherever we look. We have the feeling that we can bend things to fit our whims, and the intellectual reward in such discussion are secondary to the objects of the discussions themselves, while in mathematics, it is the other way around.

Finally when it comes to social institutions, those are subject to the vagaries of revolutions and political upheavals. As is well-known the impeccable legal institutions of Germany were quickly corrupted by the Nazis[11], and more omniously, consensual morality changed. Would they have prevailed militarily and say effected a world-conquest, would that have changed the general sense of morality world-wide, retroactivelly sanctioned atrocities we now find unimaginable. And to what extent would such social conventions of morality have overridden individual consciences? Those are very disturbing questions to ponder, and they would give pause to philosophers of the post-modernistic bent, who are so eager to exaggarate the arbritariness of the findings of hard science, one seldom sees them applying the same zeal in questioning the basis for common morality. Plato at least put goodness and beauty on the same level as truth, according them divine forms as well. In our modern secular world, we may still accord truth a status beyond human, but certainly not ethics and aesthetics, which we now believe only make sense in a human context and have no wider cosmologcal significance. Clearly we feel that mathematics is more stable than social institutions subjected to political whims. Revolutions appear in mathematics as well, but not as whims, and more importantly not as repudiations, but as clarifications.

It seems to me that the social context is insufficient to explain the stability of mathematics (as opposed to the conventions of mathematical practice, involving who gets appointed and receiving prizes, and which disciplines get support and which are discouraged). One may the resort to an evolutionary explanation, similar to the one proposed by Chom-

[11] This is though a more subtle question than one may initially think, laws were changed, yet much of the going-ons in Nazi Germany would have been illegal also by their standards, and in the beginning such legal standards were actually enforced, if only occasionally

sky pertaining to the deepr structures of language use and acquistion. Now it is tempting to materialistically try to explain mathematics either bottom-up through neuro-biology or top-down as an evolutionary adaptation. But not the simplest thoughts can be explained neuro-biologically[12], and it is far from clear that such an explanation of a thought, would it ever be effected, would in any way compete with our intuitive grasp of meanings of thoughts conventionally conveyed, As to the evolutionary explanation, evolution is often misunderstood by people who should know better, as the fashionable discipline of evolutionary psychology testifies[13]. Yes, many features of organisms have been fine-tuned by adaption, it is this that explains their stability as to errors; but that does not mean that everything evolution produces are the result of adaptive pressures, planeed so to speak, to use a metaphor, many things are just fortuitous combinations, which once being brought into existence follow their own logic. Incidentally a very Platonic notion. Clearly there have been no evolutionary pressures to produce minds that eventually would conceive, as did Darwin, of the evolutionary principle, providing order out of chaos. The fact that our minds can conceive of reality in terms that go far beyond mere survival, is a fact as well as a mystery, which may never be satisfactorily explained.

So what does all this has to do with Platonism, except the Platonism supposedly lurking behind evolution itself? What is Platonism in its abstract intellectual sense, not just as its particular manifestation by the historical person of Plato, and various interpretors and vulgarizators? Essentially it is that the world of the senses makes little sense in its bewildering variety, that sense is only made by imposing more abstract principles, in short that there is a world conceived only by the mind which affords explanations. In fact this has been the guiding principle in all scientific work, not to seek explanations in what is apparent, but in terms of what is hidden. What is physics but explanations via equations and such mathematical objects, whose shadows cast make up the sensous world as we know of it? It is a well-known philosophical observation, particularly noted by Frege, that while we cannot directly compare the direct sensous worlds of each of us (Worlds 2's in the terminology of Popper) with all their quale, when it comes to abstract entities, extracted from the material world, comparison is more apt. Thus what we share mind by mind is the world of abstract entities, (the World 3 of Popper). You may if you want place those in a world beyond space and time, because althoiugh they are manifested by material onjects, as to enable us to extract them, they clearly cannot be put on par with the same. But for us to conceive of those abstract entities, does not presuppose some kind of Extra-sensorial-perception, pace Bencareff. There might be something mystical about it, but nothing ad-hoc to provoke ridicule.

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[12] which is actually digressed upon in Hersh

[13] In fact this discipline is nothing but a collection of modern versions of Kiplings 'Just-So' stories.