Incompleteness

The Proof and Paradox of Kurt Gdel

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Everyone knows of Einstein, but who knows about Gdel? Ask the question to the man in the street, and more likely than not, there will be a blank stare. Or at least until the early 80's, when Hofstader popularized the man, followed by Penrose a decade or so later. After that he has been hyped up, a hype which seems to have somewhat overwhelmed the present author[1].

I learned about the man in my mid-teens, i.e. in the middle of the sixties. I am not so sure how I found out, it could have been through the anthology 'The world of mathematics' but even more likely a Time-Life books on mathematicians that I came across in the mid-sixties. At least I have a vague memory of having seen an august portrait of the man, the very epitome of the mad genius dwelling in a pure realm of thought[2] I certainly knew what he was all about in my late teens, somehow being perplexed by his supposed ability to show the limits of logic by logic. Was he not somehow going in circles. Obviously I had never studied his celebrated proof, because this is what he almost does, and therein lies the amazing technical feat of the construction that underlies his proof. When I came to Harvard in 1971 I once met a friend of a room-mate at my dorm, and that friend once removed had studied logic at Princeton, and had some stories to tell about Gdel and his awesome power, not confined to logic alone, but also to mathematics as a whole. Among other things he had supposedly solved a differential equation, and people had no idea of who he had done it. I did visit the Institute a few times while he was still living there, but the chances of me running into him would have been as great as running into the bona fide ghost of Einstein. Gdel did not need a ghost, he was his own already in life.

The author - Rebecca Goldstein, is a student of philosophy. Not just of philosophy in general but the supposedly hard-core variety known as the analytic kind and especially predominant in the Anglo-Saxon world. She obviously entered the field because of its intellectual glamour, at least to the initiated. This meant that she studied some logic and related mathematics in her formative youth. She met Gdel once, she reveals to us, that was in the early 70's when he uncharacteristically appeared at a party, almost holding court. She was sufficiently educated philosophically to appreciate the event. Since then she seems to have more or less permanently abandoned her early forages in mathematical

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[1] A good antidote to the present Gödel-hype is the short book by the recently (2006) deceased Swedish logician and computer-scientist - Torkel Franzén.

[2] The same very book also contained a picture of the ex-Polish mathematician Eilenberg, supposedly doing his mathematics in the subways of New York City. He, I would later meet, and actually become a colleage of during my time at Columbia University. He bestowed on me the epithet 'clean-up man' due to my ingestive abilities at colloquium dinners.

logic and involved herself in fiction, whose own intrinsic logic should not be scoffed at she cautions. Is she the right person to write about Gdel?

Of course there are many who are called to do so, and there is no reason to cull them all out but one, and she does bring her own particular experience and viewpoint. Still her gushing admiration for the most important mathematical theorem in the 20th century, is bound to grate on the nerves of a mathematician, but on the other hand she does get some important points right. One is further a bit annoyed by the pointless sloppiness of her writing, defects which easily could have been spotted and rectified by a competent and conscientious editor. I am in particular thinking of such seeming trivialities as getting the year of Gdels death wrong, except at the very end.

Central to the book is the friendship between Einstein and Gdel, if friendship is really the appropriate designation. Yet the two men regularly walked to the Institute together and there are some pictures of them standing side by side. By her highlighting this relation with Einstein, of almost being on equal terms with him, surely some of the grandeur of the latter is rubbing off onto the former, justifying her book. And one gets the feeling, at least from the preserved letters of Gdel to his mother, that he was very proud of this intimacy with Einstein, feeling no doubt that some of the greatness of that man was bestowed on himself, the most insecure of individuals,

There is the pathetic human being that was Kurt Gdel. The paranoid hypochondriac who eventually died from malnutrition, brought on by his own illusions. Logical yes, rational no. The two aspects should be kept apart, logic is about compulsion, rationality about choice. And there is the logician towering above all other 20th century logicians. Still the two aspects are of course intertwined[3].

Both Gdel and Einstein have been severely misunderstood and hyped-up, their work brought to carry much beyond their proper domains. The sexy name of Relativity Theory has suggested modernism in its emphasis on relativity of notions, how everything depends on your point of view. Supposedly the name Einstein initially has in mind was more like Invariance Theory, stressing the invariance of the speed of light. Einstein was not, as Goldstein is at pains to demonstrate, a relativist in any social-constructivists sense. The relativity of his theory is no more startling than the relativity of perspective is in any attempt to project 3-dimensional geometry to 2-dimensions. This is not just an analogy but can be made rather precise. Space-time, with time included as a variable say, can be

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[3] Gdel was compulsively logical, and in this respect he chose not to be rational, or rather common sensical, the distinction, at least in everyday life being fine. His hypochondria is a nice illustration of his self-sufficiency. As Goldstein recounts, having been struck with rheumatic fever as an eight-year old, he was convinced that he had developed heart failure, a conviction to stay with him for the rest of his life. He always envied the robust health of Einstein, comparing it to his own detrimentally to the latter. But Einstein lived the last years of his life knowing that his abdominal aneyurism, could any moment prove fatal, but making light of the fact. Gdel did not make light of his mortality, literally he ran into the embrace of death in a vain attempt to evade it. Thus typically when he encountered the medical problems of his friends he made light of them, as when Morgenstein was being eaten up with metastasies, he reassured him that those would simply go away, while at length recounting his own imaginary problems, which seemed to be stable over long time. Of course not a very sympathetic trait, but one his few friends put up with, as there were no other options, Gdel being the man he was.

thought of yet another instance of a time-honored mathematical trick to make something independent of time. It is as out of Parmenides complained Popper to Einstein. Similarly Gdel has been appropriated by those unsympathetic to his driving motivation and persuasion. At heart he was a Platonist, and according to Goldstein his ambition in life was to provide mathematical proofs with meta-mathematical implications. Thus combining the power and precision of mathematics, seldom applied in philosophical settings, to address deep and inescapable issues, which constitute the essence of philosophy. If ever any one came close to give a mathematical proof of Platonism, it was Gdel. Maybe he was totally convinced that he had done it, and was exasperated that people did not really understand. Technically he disproved the possibility of the project initiated by Hilbert, namely to show the completeness of mathematical reasoning, at least if pared down to its formal skeleton. In particular the impossibility of showing the inherent consistency of arithmetics and hence any piece of non-trivial mathematics. The consistency of arithmetics is a matter of faith, or at least transcendental understanding. To Gdel this was obvious, thus his negative result was not seen as the ultimate undercutting of the supposed privilige of mathematical certainty, only that it was not within our reach to support this certainty. It did not come from us, it was imposed on us from the outside.

Now the incompleteness that Gdel exhibited was seen, originally by John Lucas, and later expounded on by Penrose, as a proof that the human mind was no mere machine. That human intelligence was not just the working out of a master algorithm. The idea being that the human mind, as in the proof of Gdel's theorem, goes beyond the computable, is able to fathom something simple yet utterly beyond the necessarily finistic grasp of an algorithm. Mind is not matter, at least not classical matter subjected to mechanical rules; Penrose disdaining evoking purely mystical explanations resorts to quantum mechanics. Naturally becoming so technical and down-to-earth subjects the enterprise to all kinds of snags and objections; but of course this is what being technical means. Still one feels instinctively a bit wary to accept those kinds of reasonings, no matter how attractive their motivations and goals may tend to be. In recent years there has been a backlash against those exaggerated claims. Gdel is about formal systems of a certain powerful kind. There are other formal systems, looking almost as formidable and infinite, at least potentially, to which Gdel does not apply. The messy world around us does not easily lend itself to precise formalizations, something which is necessary to even start contemplating Gdel. Of course Gdel can be thought of as a metaphor, and as such it is very powerful and evocative. But poetry is not science, not even mathematics. Gdel did have a deep influence on logic, and as a consequence logic has tended to diverge from mathematics, just as mathematics has tended to diverge from physics during the 20th century. Consistency is not the same thing as truth. Add to the axioms of Peano, the statement that the axioms are inconsistent. This is a false theory, but nevertheless a consistent one. Goldstein sees Gdel as a mathematician, philosophers usually do, while mathematicians see him as a logician, i.e. a philosopher. The practical implications of Gdel on main-stream mathematics are almost nil. Although of course some natural undecidable problems have been identified, yet on the whole, the dictum of Hilbert seems to reign true. Central problems in mathematics tend to have solutions. Others are in a sense not really well-posed.

Hilbert is seen as the father of formalism, and furthermore formalism is seen as opposed

to Platonism. This is echoed by the author, yet she does have the commendable sense to see through this hackneyed version of the truth. Formalism is really not opposed to Platonism. Reduce mathematical concept and reasoning to formal systems with no intrinsic meaning. Yet those formal systems become objects, in fact rather concrete objects which incidentally can be encoded by numbers. You can ask all kinds of hard questions about them, such as their consistencies, which become real meaningful questions about real objects. Meaning-laden mathematics comes in through the backdoor, which Hilbert clearly understood. To him formalism was just a tool, a kind of boot-raising device, to relieve mathematics of some fundamental and irksome problems, after which mathematicians could go on exploring the paradise it constitutes. Hilbert was no formalist by temperament only by reasons of expediency.

To encode mathematical reasoning into logical formal proofs has seldom been done, except in the most elementary of toy-examples. To me this ambition is the same as that of representing a picture pixel by pixel. Great for certain purposes, but disastrous as to understanding. Now as to the axiomatic reasoning there are also some popular misconceptions. Traditionally, i.e. going back to Euclid, there was a distinction between axioms, some, actually referred to as axioms pertained to principles of reasoning, others were designated postulates, and referred to the objects of studies, their properties and relations to each other. Of course when it comes to principles of reasoning those are far more hard-wired than the actual objects. As to the latter one can indeed think of them as meaningless objects and by fiat made to satisfy all kinds of arbitrary things. The old-fashioned terminology referring to the axioms of a group clearly shows what is going on. Are the axioms of groups true? What a stupid question, they constitute the definition of groups. In fact Euclid would never think of them as axioms but as postulates. As to the truth of the geometrical axioms? You can play the same game. Formally and mathematically they are implicit definitions, only if you assume that they describe a physical reality, does the question of truth enter, and of course this becomes a little bit problematic as the idealized objects of say straight lines have no direct physical counterpart. The revolution of non-Euclidean geometry can only be understood in light of the confusion between the physical and the mathematical, which persisted for so long as the intuitions were so similar[4]. But principles of thought are different. They are the rules by which we reason, and here there seems to be a remarkable consensus. Those you do not tamper with, somehow they belong to the invisible context, which is always there in all kinds of situations, and once made visibly becomes known as metaphysics, and when tamed by terminology and subjected to cerebral manipulation, ceases to play the role of context, and becomes embedded in a bigger one[5]. When it comes to principles of reasoning we recognize them immediately when we see them, as being 'true'. The sense of conviction may be hard to explain but it does not mean that it is not experienced. The claim of Plato that knowledge is something we have always known but forgotten, acquires particular pertinence in this case. The

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[4] The unique model of the reals with the Archimedean axiom, has such a strong physical feel to it, that even modern mathematicians, may confuse its mathematical meaning with its physical model.

[5] That famous saying of Wittgenstein that concludes Tractatus, is not only vacuous (in some sense) and portentous, as quoted by Goldstein, it is also poetically true, and would indeed have had much more force, had Wittgenstein himself heeded the advice. As he actually tried to do.

problem with formalization is that it cannot be subjected to closure. There may always be new principles of reasoning which we have not yet remembered. I personally find the most intriguing opinion of Gdel is his suggestion that we have not yet stumbled on the right axioms of set theory. When we do, all those anomalies and all those arbitrary facts depending on choices of tentative axioms, will disappear. This is really a kind of strong Platonism.

Any biography has to put the protagonist in a context. The formative context of Gdel was Vienna and the Vienna Circle, to which he was invited as a very young man[6]. Gdel attended but did not participate. Logical Postivism as they called their program, has in retrospect gotten a bad name, although their no-nonsense influence is still very much in effect. Popper who was not included in that charmed circle, later denounced their rejection of metaphysics, as based on a metaphysical stand[7]. Amazingly though, those hardnosed thinkers idolized the greatest philosophical con-man of the 20th century - Wittgenstein. Gdel did not think highly of this man, and Wittgenstein, whose ignorance of mathematics did not stop him from expounding on its foundations, was incapable of understanding Gdel's proof. He referred contemptuously of it as some kind or irrelevant conjuring trick, which of course in a sense it was, and with little relevance being too caught up in paradoxes. Now, as have been indicated above, there is something to the remarks of Wittgenstein, con-men have basic instincts, otherwise they would not succeed.

The second part of Gdels life was spent at the Institute[8], where apart from his pedestrian association with Einstein, he was basically very isolated, an isolation that only grew with the years. He published nothing after 1958, although there is a large Nachlass, which has probably not been fully mined. His mathematical colleagues found him disturbingly strange, especially as he refrained from joining their expeditions against the current leadership[9]. The pure mathematicians at the Institute are portrayed as a rather ridiculous bunch, who spend their liberal free-time thinking up all kind of mischief[10], haughtily claiming that no one else can judge them, but that they are perfectly capable or judging others. Maybe a symptom of the combination of arrogance and ignorance that blesses a mathematical life.

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[6] Hahn, of Hahn-Banach fame was also very interested in logic, and thus had brought along his most brilliant students.

[7] This is reflected in Coolingwoods clip to the effect that anyone who rejects metaphysics takes a metaphysical stand

[8] The story of which is patiently explained more or less in the same factual terms as it was explained in her first novel - the Mind-Body Problem.

[9] One explanation being that Gdel always deferred to authority, there being a very logical reason for such rigidity.

[10] Freeman Dyson is a recent obituary statement for the late Kaysen, quotes Einstein and the war of the frogs in the 30's, when there was a mathematical row about the editing of Mathematische Annalen, clearly taking exception to the former antics of his mathematical colleagues.