The Language of Mathematics

Making the invisible visible

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In what sense is mathematics a language? Obviously not in a very specific sense with a specified vocabulary and syntax, to be learned as any other human language, but rather since the time of Galileo to be understood in a metaphorical sense. Mathematics is about patterns, mostly invisible patterns, which only become visible when we employ mathematical formalism. And here we are already encountering a basic confusion, namely between mathematics itself and the formalism it employs, the one not being so easily distinguished from the other, because formalism itself can be made into an object of mathematical study, a study engendering its own formalism. While the metaphor is certainly popular it is nevertheless somewhat misleading especially to working mathematicians of a Platonist bent. The metaphor suggests that mathematics is a mere formality, a convenient way of looking at the world, but who knows, liable to be superseded by something else, hopefully more congenial to the great majority of people.

How do you dress up mathematics to the reluctant public? How do you convince it of its power and beauty, and more importantly how do you do this in an entertaining and undemanding way with a minimum of instruction (there are enough text-books of mathematics as it is)? It is a tall order, but that does not mean that not many writers have tried to rise to the occasion. One of the most popular and sustained efforts were those of Martin Gardner in his Mathematical Games column. The original perspective was one of recreation, Gardner himself had been an enthusiastic magician himself, and many of the first pieces drew on classical puzzles from the 19th century, such as those of Samuel Lloyd. But more and more they tended to present real mathematics, without losing the 'hands on' approach, thus not merely describing but trying to engage the imagination of the reader and challenge his or her ingenuity. Thus highly technical branches of mathematics were off limit, on the other hand there is much in mathematics, which if presented in the right way, can be made rather accessible. Invariably there was an understandable emphasis on combinatorics, elementary geometry and number theory, but also some simple topology made its way. I know from my own early teenage experience that many of the pieces stimulated me, especially what I learned of Platonic Solids, instilling in me the ambition of trying to generalize them to higher dimensions.

The example set by Gardner has been emulated by many others. But while Gardner had the advantage of a column running twelve times a year for many a year, a modern popularizer has far less space at his disposal (but of course if successful from a publishers point of view, one book can be followed by another one inductively). Devlin choses a number of topics in order to both display the beauty of mathematics, but perhaps even more significantly, how mathematics all hangs together, that connections between seemingly unrelated parts of it, are constantly being discovered, this being one of the prime excitements of mathematical inquiry and discovery. Now, given the circumstances it would be pointless trying to criticize his choice of topics, after all whatever book he writes will just be one possibility of several, and there is no canonical way of making a choice, especially not when it comes to in what order topics should be presented.

Most of the topics chosen have been or could have been topics for the Gardner columns, except there are a few treating rather technical and advanced subjects, such as Weyls Gauge theory, Donaldsons invariants and Seiberg-Witten, whose treatments belong to the least satisfying in the collection. To take a topic from top down and strip it of all technicalities, tends to be a journalistic approach, and is bound to in the end become as incomprehensible to laymen and experts alike. Yet a book for the public cannot be restricted to simple, elementary mathematics, this would be unfair, in other sciences cutting edge work is presented to the public, giving a sense of excitement and importance. If mathematics is always reduced to the elementary level, it is bound to be seen as an antiquated subject in which everything of importance has discovered, and its justification reducing to that of an auxiliary role, as a formal language to organize the data of more serious endeavors. Yet, the essence of any mathematical presentation is understanding, and no mathematician can ever resist simple proofs such as the irrationality of $\sqrt{2}$, the existence of an infinite number of primes; neither does the author to his credit, which necessarily makes for an emphasis on elementary topics. Still there is not so much regular mathematical instruction to be found in the book, let alone any technicalities, but aside from descriptions more of an emphasis of making philosophical remarks, some of which could be quite interesting.

First when it comes to very basic mathematics, such as counting, any extensive treatment is bound to become tedious, and here the author is commendably brief and to the point. He claims interestingly, yet I find this personally a bit controversial, that the abstract concept of a number is something social which is being taught and transmitted, and that there are still tribes in the world whose members have not yet made that basic transition. That there is an innate sense of number relating to very few pieces, seems to be something that cognitive psychologists have established, this is pre-counting (i.e. not using any tricks like grouping together) and the human capacity for such does not seem to exceed that of crows and rats. But could it be that the notion of number needs to be taught, that it is not innate even in gifted individuals. Would somebody like Gauss, if born fifty thousand years ago been stymied by it? It is as hard to believe as it is impossible to check. Devlin gives the intriguing story of how numbers of a certain stock were represented by pebbles (or some such thing) in a sealed urn, and in order not to have to break the seal, some notation were scribbled on the lid. After a few generations it was realized that the pebbles were actually redundant, the notation sufficed, and clay tablets were born. A good story, but as with all good stories one has to ask is it true, not only ought it to be true. Yet the story has the ring of truth to it, how often to we oversee something obvious staring at us, and once we have seen it it is impossible to be blind to it. Thus once we have caught on to the number concept we cannot conceive how we ever had been unaware of it. Yet the interesting question to ask is whether there are such simple ideas, that once presented they are immediately grasped and absorbed. Not innate, but almost so, in the sense of the mind being prewired for their conception only needing that little spark. One can wonder about how to write the history of mathematics. Mathematics abounds in long technical proofs. Any one who learns some formal machinery is able to produce long and contorted arguments, which seem very deep and complicated, and who require a lot of effort both to produce and to understand. Mathematics is a hard technical subject, and most mathematicians are both unable and unwilling to delve into fields not their own. And it is a well-known fact that most math lectures as soon as they get a bit technical lose most of the audience. But still could it be that the progress of mathematics do not depend on those feats[1], but on some far simpler ideas, giving a totally new perspective, and once seen immediately grasped. The positional system with a zero, is one such idea, what are the others? (Of course mathematicians are judged not only by what they prove, but by what new simplifying ideas they provide, but most of those ideas are rather technical, what about truly non-technical ideas?)

Calculus is one example, this constituted the real advance upon classical Greek mathematics, although the Greek were very close to inventing it. Eudoxus was not only anticipating Dedekinds cut, but also by his method of exhaustion, being very close to the modern theory of integration. As Devlin suggests calculus is about dynamics, i.e. dependence on time, yet by the formalism of mathematics, by adding another variable, the dynamic becomes static[2], and he uses this to explain the modern conception of limit, of which the founders of calculus had only a vaguely articulated idea, although a firm intuitive grasp. Thus calculus made progress in spite of the fact that its rigorous foundations were not laid until the 19th century. Mathematics is more than a formal game.

In fact a formal game becomes an object of study itself. From being an active agent in pursuing correct reasoning, it is turned into a passive subject. (And this is a bit reminiscent of the trick of turning the dynamic to the static, as explained above.). Devlin is to be congratulated for his originality of presenting Aristotle's 256 cases of syllogisms[3] and showing how further formalization given by Boole made nontrivial inroads[4]. It is of course a mystery how logic can be studied by logic, but by exploiting the fact that the integers are more basic than logic[5] rather than the other way around[6] Gdel was able by skillfully mapping the thinking itself into what was thought about, making a kind of ultimate application of the Cantorian diagonal principle (the manifestation of free will as I see it) to give the death knell to all attempts of reducing mathematics to formal reasoning. Although one should be careful not to read more into Gdels proof than there is, the temptation to do so being strong, as many mathematical ideas it provides excellent metaphor, and metaphors should be thought of as stimulations to thought, not simulations;

[1] One is reminded of the computer proofs by Appel and Haken on one hand, and that by Hales on the other. None of which has really made anyone the wiser, nor contributed to any mathematical progress

[2] As bothered Karl Popper, relativity theory is timeless once time is incorporated in it. An entity which would have delighted Parmenides

[3] As remarked by the Swedish philosopher Wedberg, this was the first deductive treatment to be found in the world literature, antedating Euclid's. Of course the subject of Aristotle was far more narrow than that of Euclid, more introspective as it concerned thought processes, not the external physical space of which we all are part.

[4] such as spotting a few mistakes by Aristotle which went unobserved for two thousand years.

[5] as proposed by the American philosopher C.S. Peirce

[6] Frege, Russell and Whitehead

in fact when taken too literally, as when mathematics is seen as an ordinary language, turning just silly. This is all of course very interesting, subtle yet accessible even to the layman. Devlin does not milk it for all its worth, and a professional reader would happily have seen more detail, on the other hand the mark of a successful popularizer is to know when to stop.

Non-Euclidean geometry may be thought of as another simple but revolutionary idea. One effect of it though may have been unfortunate, while initially axioms were thought of as self-evident truths, basic bedrocks of reasoning, the freedom unleashed led into the temptation of making axioms more or less arbitrary and thus tended to reduce mathematics to frivolous formal games (which of course ties well into the formalism of deductive reasoning). It is not clear that Kant thought that Euclidean geometry was the only possible, he simply wrote that it was the way our intellects were disposed of organizing space. Its abuses apart, it certainly liberated mathematics, and by pointing out that Euclids geometry was about physics it anticipated Einstein. General relativity is on the other hand a case of making physics into geometry. The treatment of relativity is good, Devlin makes most of the basic points, including an explanation of the twin paradox (the fact that geodesics locally maximize length in the Minkowski metric is to be taken on faith, and that is fair enough). Had he included the fact that the world lines make up a hyperbolic 3-dimensional space, and classical aberration as discovered by Bradley translates into hyperbolic parallax, I would have applauded. But once again, it is the prerogative of an author to draw his limits. Furthermore I am a bit puzzled by his remark that all frames of reference are equivalent, not only those in uniform motion with respect to each other. When it comes to two frames accelerated in terms of each other, the bending of light-rays become very clear in a thought experiment, a bending which can be interpreted as the effect of gravity. This too would have been very motivating in the account. The bending of space-time is an intrinsic property, nothing relative about this. So only by Einsteins theory is it really possible to make precise the fact that the Earth revolves around the Sun, and not the other way around. When it came to circular movement Newton had a famous argument of talking about absolute movement [7] using the rotating bucket filled with water. How does this tie in with relativity theory?

There is pure mathematics and applied, and although sometimes the borders between the two are argued to being one of convention rather than for real, after all the great mathematicians of the past did both applied and pure mathematics, there is a real distinction once you start to understand wherein it lies. In one sense mathematics is a case about mathematics being applied to itself, meaning that it becomes all connected each part having ramifications on all other parts. There is nothing wrong with application, it is all a case of revealing the invisible. Also the mathematical applications to physics are spectacular, to many naive people there is little difference between mathematics and physics, and naivety is not always to be disparaged, as Devlin notes, in recent years there have been renewed applications from physics to mathematics. It means that a physical intuition can be very helpful in solving ostensibly purely mathematical problems. Mathematical models of the physical universe have had far deeper ramifications than were put

[7] Believing in an absolute space as opposed to Einstein, the notion of whether the Sun moved around the Earth or the other way had a precise meaning, as there were a canonical reference.

into them, so in that sense one should not perhaps speak about the Maxwells equations as a mathematical model (such as the one by Ptolemois, which had after all great predicative value, and which furthermore like all mathematical models had potential for refinement and elaboration, further enhancing its powers) but as a theory (in the words of Manin, belonging to the aristocracy of models). After all the velocity of light was an emergent feature of it, as was in the longer perspective the theory special relativity, in terms of the Lorentz group of symmetries. That the universe is basically mathematical and thus if not ultimately explainable, a source of wonder. Just the same kind of unexpected connections you find in mathematics, also exist in physics, and furthermore between the two. Both share common patterns to use the language of Hardy, Devlin and countless others. With other applications it is very different.

Devlin brings up the intriguing example of taking fingerprints of texts to identify authors. It turns out that relative frequencies of vocabularies (or grammatical constructions) are surprisingly stable for different authors, at least when they stick to special topics. This has repeatedly been applied to test for authorship (on the other hand an individual privy to the types of test would be able to provide fool-proofs forgeries given those criteria, and could only be revealed by adding more criteria to the arsenal. This is somewhat reminiscent of Gdels incompleteness theorem!) It is intriguing, but is it mathematics? Of course not, it uses numbers, but the patterns that are made visible in this case are not mathematical patterns. Numbers are useful for encoding in real life, but what is revealed here is nothing intrinsic. This does not mean that it is not worthwhile or interesting, only that the interest is non-mathematical. In similar ways mathematics has provided models for many phenomena in the real world of society, not only economics but also biology, political science. But those are models not theories, they may make certain patterns indeed visible, but those are not mathematical patterns, and they are seldom if ever beautiful in the way mathematical patterns are. There is a lot of pressure on mathematics to prove itself useful and justify the (admittedly rather limited) resources invested in it, and mathematics can indeed point to its crucial role in many features of the modern world, from the error-correcting transmission of data and securely encrypted financial transactions to whatever you can think of. This is of course very wonderful and important, and it testifies to the fact that mathematics is important, it is an activity with ramifications beyond itself, but is it really needed for its justification? The kind of thought and intellectual daring that goes into the solving of a real mathematical problem (I am not talking about the routine stuff that fill up most of the pages of mathematical journals) is not at all present in applications. The real social world with its ad hoc nature is not an integral part of the basic world, it does not exhibit the kind of beauty and inevitability and striking unexpectant phenomena. While mathematics has applications to the so called real social world (including biology?) it seems to be one-way street. There is no biological intuition which is helpful in solving mathematical problems, nor an economic. The close interrelatedness as seen between physics and mathematics is simply not there[8].

When it comes to mathematics most people think of statistics and probability. In fact

[8] Chemistry is a border case, much of the mathematical applications to it seems to be in the nature of vast computer simulations. This goes even more so for biology. It is reminiscent of computer proofs in mathematics, so vast and so intricate, involving the checking of so many cases, that no human mind

almost all the the applications to the social world are in terms of probability[9]. Probability is Janus-faced. It has a purely mathematical side, as axiomitized by Kolmogorov in some sense reducing it to a part of measure theory. A typical cheap way of getting a probability is to divide the cardinality (or measure) of a subset with that of the set. To calculate those may be a purely mathematical problem for which probability theory per se gives no guidance. Thus not surprisingly many problems in probability theory, such as percolation, are in fact purely mathematical problems. Of course probability theory has developed its own concepts, which although mathematical, nevertheless makes it transcend being part of measure theory. On the other hand it has a 'physical' side. In real life, including the experimental scientific one, we need to make sense of what is meant by a probability. To some extent it is, as Devlin admits, a question of a priori probability versus a posteriori, with the determination of the latter being a major problem and solved up to some probability itself (a priori or a posteriori? And with what probability are those estimates computed? An infinite regress?). But there are also more subtle subjective features such as utility as opposed to expectancy, the latter being a purely mathematical entity, once a distribution is given. And in fact it might be even more subtle than that? In inductive science, especially say in medicine, what margins of errors are we prepared to accept? What degrees of correlation do we need to claim a certain vaccine to be safe and effective? Those are fuzzy thinking, a pole apart from the crisp certainities of mathematics.

People have not evolved any correct intuition for probability, Devlin claims. This is indeed a somewhat misleading statement. People do not have any intuition whatsoever, mistakingly believing that if something is said to have a certain probability it actually means something specific, just as people tend to believe that there is only one mean value, when there are several depending on the context. To take an example similar to the one in the book. Assume that there are 25 taxi cabs twenty black five blue. What does it mean that a person identifies the right color of a cab with probability $\frac{4}{5}$? That he identifies every cab as black, or that he identifies one blue cab as black and four black cabs as blue? It certainly makes a difference. Once the precise notion of the meaning of probability is made clear, it is just a matter of formal logical reasoning to work things out. No mystery whatsoever once the notion of probability is divested of a spurious general significance.

The book is written by a mathematician and not a journalist, so once expects it to

can get a real grasp. (One wonders whether this will be the future of mathematics as well). Much of biochemistry hinges on the exact spatial configurations of the atoms in complicated molecules such those of enzymes and proteins, such can in principle be simulated on the computer, but is it mathematics, even if all the steps are mathematical? Does it give any kind of mathematical understanding or even inspiration? Mathematicians are encouraged to involve themselves in this. It would be stupid to claim that this is not worthwhile or important, but is it mathematically interesting? Do not those who engage in it lose their mathematics, and become something else? Important engineers, but hardly the disinterested seekers of beautiful patterns they once were. Mathematicians have in the past seen beautiful mathematics in biology, such as in the genetic code, in the last analysis, those beautiful explanations have turned out to be wrong and irrelevant.

[9] Almost all, but of course not all, the most intriguing recent examples of mathematics has been number theory as in secure encryption, but this is an almost mathematical problem after all, not one of modeling social realities, but nevertheless very important to them

be competently done, and those expectations are not disappointed. He obviously knows what he is talking about, except when it comes to Seiberg-Witten and related matters, the account is too vague and abstracted to allow any such definite conclusions to be drawn. One can make a few remarks. As trivial slips one notes the fact that Frege has become a Swede. This is news to us all. Frege had no connection whatsoever with Sweden. That the pre-eminently German mathematician Minkowski is Russian is likewise news, but in this case this mathematician of Polish extraction happened to be born in Lithuania, at the time part of the Russian empire. Such things proper editorial work by the publishers would have resolved. More seriously though the author makes a big thing about there being fourteen different cases of lattices in 3-space. What is meant by that? On what is this classification based? Without indicating this, the whole thing might be intriguing (what does the author means?) but meaningless. Similarly for the five types of Dirichlet domains for plane lattices. Dirichlet domains is one way of constructing fundamental domains for the group of translations. If the Dirichlet domain exhibits further symmetry, that will also become a symmetry of the lattice. Ultimately this ties up with the 17 wallpapers, but how? Once again much information is given that is suspended in midair without supporting anything else, and thus could be either removed or complemented.

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