Mathematics as Metaphor

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This is a collection of essays addressing various meta-issues, such as the formal nature of mathematics, its similarities and differences from languages, especially natural languages, its relation to truth and reality (and somehow those two are not identical if obviously congruent), especially its role in elucidating and inspiring modern physics (somewhat cynically one could claim that modern physics may have been more of an inspiration to mathematics than the other way around). In addition to this the author nurtures a deep interest in matters not usually connected to mathematics, such as linguistic, the study of myth, the Jungian concept of collective unconsciousness, the paleo-historical origin of language and consciousness and its neurological underpinnings as well as how the former can be restricted and warped, all of which are closely intertwined.

There is a great danger of expounding philosophically, a danger that a more technically oriented mathematical activity evades, because the demarcations between the profound and the silly, the potent and the vapid, are hard to draw. Philosophical reflection needs a shared experience to become meaningful to the audience, it is not instructive by itself, it does not generate knowledge as it cannot be manipulated, its purpose is instead that of expressing emotion, a purified emotion of intellectual satisfaction, and with all such emotive motivations, their satisfaction stemming more from closure than from providing a point of departure.

Manin does however differ rather profoundly from most people expounding on the philosophy of mathematics, his presentation is not predictable but goes beyond the expected homilies. Part of the reason maybe that he is Russian. 'The intelligentsia' is supposedly a concept that was coined by Russian intellectuals of the 19th century, and as such it admittedly carries mixed reviews, but one does connect to the Russian intellectual tradition, more perhaps than to any other, a down-to-earth passionate interest, cutting across many a discipline, into the meaning and purpose of activities of the mind (and its relation to matter?), even if the conclusion may be that there is none. Thus one pictures the Russian to be a pole apart both from the formalistic elegance of a Frenchman and the systematic penetration of a German, something that actually is born out by the tenor of mathematical papers produced in the Russian tradition. Yet it may very well be a sentimental illusion, but if so there is with Manin at least a romantic exception to a mundane truth.

Mathematics is about solving problems. A mathematician proves his mettle by combining great technical skill with startling creativity in order to make a successful assault on a problem which until then has defied the efforts of generations. However, most mathematician do not engage in the attempted conquests of generally acknowledged challenges, but are perfectly content by making up their own molehills and scaling them successfully. Manin does not belong to that school of thought, although he obviously does not disparage the necessity of problem solving, after all it constitutes the bulk (as well as bread and butter) of everyday mathematical activity, and in education there is no viable alternative to the immersion in problems for anyone ambitious enough to acquire a taste as well as a deeper understanding of mathematics. But ultimately what makes mathematics worthwhile is the exploration of new unchartered territory, of engaging in wide and fundamental research programs, and he refers in particular to the program initiated by Cantor and his transfinite arithmetic, forcing mathematicians to rethink the notion of infinity, mathematical truth and provability, and the role of formalization and computation, a program with many an unpredictable consequences, one of the most generally known being the incompleteness result of Gdel as a consequence of a skilled mapping of meta-mathematics into mathematics, simultaneously interpreting a sentence as to real meaning as well as formally. This pre-occupation with logic eventually found an outlet into the emerging phenomenon of electronic computation, turning the spirit into flesh, and greatly enlarging on the limited human capacity for tedious and mechanical thought recursively generated, thus greatly enlarging (for better of for worse) the potential as well as actual mathematical influence in the world at large.

Number theory constitutes Manin main mathematical interest and much of his technical work has concerned itself with its geometrical connections. But physics is another magnet of compelling interest, and Manin claims that no mathematical development of the 20th century has matched the revolutionary change of paradigm characterizing the fate of classical physics during the same period, a fact that should come as no surprise, except possibly to a few 'autistic' mathematicians. Mathematics and Physics used to be intimately connected since the time of Newton until the end of the 19th century[1], but then there was a split and a divergence, neither camp being particularly interested in what the other was preoccupied with. Paraphrasing the words of Manin, the mathematicians became obsessed with our relation to thought, while the physicists with our relation to reality. The former led to a kind of neurotic introspection and a subsequent hygiene of precision and formal reduction, while the latter led to flights of fancy, at least temporarily unfettered by precision and rigor. A mathematician tends to be aghast at the vagueness of the physicist pragmatic instincts, not so much bothered after all by the lack of rigor, as Manin points out, but the refusal of the physicist of allowing himself to be pinned down by mere definitions and hence the inability on part of the mathematician to understand what they are really talking about[2]. A cultural rift that survives to this day. Consequently, although mathematics grew remarkably during the 20th century, physicists were slow to adapt to the new concepts and instead tended to develop their own mathematics[3]. And in fact, the intuition of physicists have recently proved more fertile in solving mathematical problems than the techniques of mathematicians, something the latter, to their credit,

[1] Much of the contents to be tested on the Mathematical Triposes in Cambridge would from our perspective be classified as mathematical physics

[2] Russell famously characterized mathematics as a subject in which no one ultimately do not know what they are talking about. The charge could much more appropriately be laid at the feet of physicists, who in spite of their suggestive metaphors, fooling the general public, is at dark as to what their concepts really refer to

[3] A classical example is the rediscovery by Heisenberg of matrices, known to mathematicians since the middle of the 19th century, some of that, if in the disguise of quaternions, had actually found its way in the formalization of classical mechanics during the late Victorian era.

have freely acknowledged. In fact, as indicated initially, some parts of physics, or at least string theory, has had more significant applications to pure mathematics than it has had to physics and its ostensible subject of reality. Why this should be so is still something of a deep mystery. Some parts of modern physics, such as relativity theory, have been successfully mathematized, and can, but for the rigid boundaries of academic disciplines, be as well headed under the mathematical banner; while quantum theory, far more influential and 'useful', is still intellectually marred by internal contradictions and an incompatibility with general relativity, something that does not really bother the general physicist. The closer we view the material world, the less solid and commonsensical does it appear. It is a world which can only be approached and described mathematically, i.e. in the sense of using mathematics as both a language and as a method of manipulation. The reason for this growing effervescence, according to Manin, is that in quantum physics the observer can no longer be separated from what he observes, unlike the classical paradigm, in which one could (at least as a thought experiment) isolate pieces of the world and view them as closed systems, in which one controlled all the variables (i.e. its position in an idealized phase-space) as well as their number without affecting them. It is clear that when it comes to pure adventure, modern physics has provided a far more thrilling ride, but Manin considers himself a mathematician at heart, and thus only being able to claim an amateurish interest in the subject, piously hoping though that his mathematical expertise as a number theorist could be able to supply a new angle.

To give a birds-eye view of mathematics is impossible, while it is actually possible to give such an exalted view of physics, or at least Manin tries. Once again the natural question is to judge how far he succeeds. For one thing nothing meaningful can be conveyed to a neophyte in just a few pages, however if you already possess a fair amount of prior knowledge, the exercise of pretending naivety can be quite rewarding, dispensing with technical crutches. (But once again, only if you to a large extent master the technicality of a subject can you really appreciate a non-technical description of it.) Does he succeed. Yes and no. His attempt is definitely readable and original and it particularly appeals to a philosophically minded mathematician. One of the greatest intellectual satisfactions lies in finding underlying unities between seemingly disparate phenomena, and this clearly is the modern version of classical Platonism. Concomitant with the pleasure, not to say exhilaration of unification is the dual satisfaction of making unsuspected distinctions. This is the other favorite tool of a professional philosopher, and some out of a passion of pedantry, even seem to make a living out of that alone. A text which skillfully interweaves the two is bound to excite, and Manin does not disappoint. On the other hand there is no substitute for honest work. A reader who feeds entirely on philosophical fare is bound to starve, however sweet and exciting it may be. As noted above in order to really appreciate a philosophical account you have to have an intimate and personal acquaintance with the concepts involved, only then can you prosper from it.

Gdels incompleteness theorem has inspired a lot of silly hype. It used to be known only to mathematicians and logicians, but in recent decades it has, partly through the popularizing efforts of a Hofstader, become more generally spread. One can either see it strictly as a technical theorem about formal systems with no applications beyond its narrow domain, or as a profound statement on the human mind transcending algorithmic limits, as Penrose prefers to exploit it as. When I as a teenager first encountered it I was puzzled as how you through logic could determine its own limitations. Were you not caught in a vicious circle? My understanding was hazy and I had not even made a serious attempt to understand the proof, yet my perplexity was an honest and pertinent one. And indeed you do use a hook to lift yourself up, namely the appeal to a metaphysical principle that lies behind all those Gdel type insights. (More transparent proofs and instructive angles have developed since Gdels original work, and as usual in mathematics a variety of perspectives adds greatly to understanding. It is more to it than just a clever, or maybe, ultimate application of Cantors diagonal trick, which some of us may view as the manifestation of a free will.) That principle being that we can imagine infinite verifications, against which finite methods not surprisingly are shown to be no match. A mathematical proof is in some sense a physical phenomenon, in principle formalizable and implementable on a computer (the spirit turned into flesh). But there is no way a theorem, say Fermat, can be verified by checking all the infinite number of cases, although we can well imagine such a procedure which lies behind our conviction that the theorem is true or not, regardless of whether there is a finite proof of it. Formalisms scoff at this naive idea, and certainly infinity is a very big thing once you start looking at it physically. Thus somewhat paradoxically very large numbers are more liable to induce a sense of vertigo than infinity itself. Still the faith in such a procedure is unshakeable and by non-platonists seen as one of religious delusion. (Yet even such a statement is actually based on a similar kind of metaphysical reasoning if not so obvious).

All thinking seems to take place in a metaphysical universe. It is this universe which provides the context without which the thinking would not make sense. Yet there is a distinction between what you can think 'on' and manipulate, and what you can only think of and be vaguely aware of. Once you look hard at metaphysics and start to subject it to thoughtful analysis it ceases to be metaphysical, and often reduces to something silly, as the grand schemes of 19th century metaphysicists such as that of Hegel amply illustrate. Gauss certainly, or at least apparently, thought of the numbers metaphysically in the sense of not recognizing an actual infinity only a potential, thus in particular the infinity of numbers could not be subject to a mathematical analysis. Cantor changed this, and thereby reducing metaphysical objects to mathematical ones amenable to mathematical manipulation. The result is to most mathematicians deeply disturbing. On one hand the theory is very simple and beautiful, in fact there are few mathematical ideas which are so simple and accessible to the layman, on the other hand it seems sterile with no deep applications beyond itself and any really exciting revelations. To the mathematician anything beyond the cardinality of the continuum seems silly and worthless, although the difference between the cardinalities of the reals and the integers respectively is essential in modern analysis, otherwise measure theory as we know it would not be possible[4]. But could it be that if we would be beings capable of doing infinite proofs (thus being able to settle all kinds of number theoretical conjectures through mindless verifications) then higher cardinalities would provide a most fertile ground? But when we as mathematicians encounter axioms of ever higher inaccessible cardinals, we get a sinking feeling that there

[4] and the distinction allows all kinds of existence proofs, such as that of transcendental numbers which so impressed Cantors contemporaries

is no real substance to it, just frivolous games with marks on paper, having no significance beyond itself. It severely shakes our Platonic belief in the solid objectivity of mathematics, and becomes something of a rift in our mathematical fabric through which it threatens to be totally unravelled. Yes Cantor's excursion into the never-never land of thought did bring about a crisis in mathematics, at least to those of an introspective mind, a crisis ostensibly about its foundations and consistency but more deeply about its meaning and ultimate significance. Is mathematics but a human invention, a kind of strange language, which does not really makes sense outside the human community? That the theory of Cantors infinities are not really about anything, least of all about cardinalities and sets, except possibly an artifact of the the propensity of human thought to feed on itself. A mere convoluted reflection of mirrors in themselves.

This leads to another of Manins preoccupations, namely that of language. Are humans defined by language, in fact are we most basically social creatures to whom individuality is just a consequence, in other words that individual consciousness would be impossible without a shared language? Popper claims that sociology is not applied psychology, that it is in fact predates psychology and is a prerequisite for it. There is a natural evolutionary explanation for this, as social cohesion exist among other mammals, and it is this shared sense of socializing which bonds canine to humans, based on a profound misunderstanding to which both parties are seduced, namely that humans sentimentally view dogs as humans, and dogs certainly understand humans as dogs. Clearly modern humans evolved from humanoid creatures in which a social cohesion was already present. Organisms are not just genes they are part of cultural traditions and contingencies[5]. Language enhances social contact, but social contact is also a prerequisite for it. Thus through the concomitant conservation as well as evolving of language, we can talk about a continuous human community extending in time. It is this community to which Poppers schemes of falsification applies[6] and which was anticipated by the pragmatist C.S.Peirce, who suggested that truth was what the community asymptotically accepts. And of course it is very hard to argue against it. It is also this sense of language being something autonomous, subjective maybe from the point of view of humanity, but objective as far as the individual, which makes sense of Jungs ideas of collective subconsciousness, ideas that are usually scoffed at by Western intellectuals, but which it apparently takes a Russian to sympathetically appreciate[7].

How are languages learnt? As Chomsky has famously suggested it is not a question of

[5] The woman that gives birth to a child is not only connected to it by genetic makeup, but actually endows it with a bacterial flora as well, to refer to a somewhat prosaic fact. And the upbringing of a child is far from entirely reducible to that of instincts genetically encoded, but is a continuation of an extra-genetic tradition, call it cultural if you want. It is this which makes the reconstruction of ancient extinct species (such as dinosaurs) unfeasible, even if we would be able to get the complete genotype. It takes more than genes to make up an organism. The extinction of a species also involves the rupture of a continuous tradition with roots going back to the beginning of life (and beyond), a tradition that cannot be encoded, which just was.

[6] Popper claims that science cannot be conducted by an individual, it only makes sense as a communal effort.

[7] When I discovered Jung in my youth I was very much seduced, his ideas of subconsciousness seemed

a regular learning process (maybe there are no regular learning processes apart from the mindless cramming many students engage in before a test?) but seems to be intrinsically hard-wired. The ideas of Chomsky are very charming but so far I believe impossible to pin down in any systematic way (even his notion of a universal grammar seems to have resisted codification and formulation) let alone to a neurological level. This does not mean of course that it is wrong, but is more of a metaphor in the sense of Manin, than an actual theory. One thing seems clear to me, language is born out of a common non-verbal understanding, to which clues are supplied by the specific context[8]. The grand philosophical problem (epistemology versus ontology) is how language can transcend itself and supply the human community with truths beyond that of itself, even if those are not compatible with our survival as a species[9] ,

Languages are products of brains, and ultimately to understand languages, we need to understand brains. We are indeed very far from doing so, probably even further away than we realize. There is however one very seductively interesting theory of the brain that has caught the public imagination, including that of Manin himself, namely the split up of the brain in a left and right hemisphere, with radically different ways of functioning (although I believe that on the anatomical down to the neurological level the two hemispheres are indistinguishable, which pertains to the metaphor of the brain and the mind being like hardware and software), and that our thinking processes are actually an integration of both ways. This is also something that has excited Arnold, who in his reflections on Leibniz and Newton, classifies the former as a formalist left-brainer, while Newton is a

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to have a much wider scope than the reductive schemes of a Freud. But not all infatuations mature into permanent love and it is now many years since I read anything by Jung. Yet the reference to the collective subconsciousness strikes a hidden chord.

[8] To take a trivial and anecdotal example. Recently in a Moscow subway I became apprehensive about being on the right line or having missed my stop. My Russian fellow passengers sensed my worry and reassured me all in Russian. The words by themselves made no sense to me and hence I do not recall them, only the understanding of their reassurance. I believe it is through shared experiences like this that language is imparted. It is not a question so much of learning individual words (by pointing?) as learning entire flows, through which individual words may artificially be cut out. In fact in speech there is often no separation between different words and that does not seem to bother us, but a written text which would dispense with spaces between words would be very hard to decipher, testifying to the very different neurological pathways involved in natural spoken language processing as opposed to the artificial written one. Finally it is a widely observed phenomenon that we tend to pay more attention to the perceived meaning and intention of an utterance than the actual words. So a formally negated statement is perceived as the positive version if this is what we are led to expect by context and tone of voice.

[9] The evolutionary paradigm is a very simple and powerful idea whose impact on the popular imagination at its time was far more pervasive than even more radical materialistic ideas of classical mechanics that had been around for at least two centuries, and an idea that has only grown stronger and more central in the century that followed. According to this paradigm an organism only sustainably acquires concepts which are compatible with its propagation. Thus humans find truths not in any ontological sense, but in a pragmatic sense, something that no doubt influenced thinkers like William James at the turn of the last century.

right-brainer with his more synthetic geometric approach[10]. One peculiar and seldom appreciated feature of human brains (as well as other brains?) is its stability. A computer can go haywire through a single mistake (and set off a nuclear holocaust?), while similar madness seems absent in humans. When madness is manifested it is compartementalized and do no interfere with the working of the organism as such[11]. The most pervasive form of cerebral dysfunction seems to be advanced Alzheimer in which eventually the demented brain no longer is able to direct basic functions. And even in dementia, core properties of the human psyche seems to be intact for a very long time. One explanation is the great plasticity of the brain, its ability to regenerate and allow psychic features to be reestablished at alternative locations, and in fact maybe render the notion of a brain geography moot, pace the classical attention such an idea has exerted. The metaphor of the human brain with the computer seems to be extremely misleading, although it is hard to think of an alternative one, let alone a better one.

Finally we are of course encountering an impasse. The brain itself, or at least the idea of it, is a creation of the human mind, leading to a loop out of which we seem unable to extricate ourselves from. As Penrose notes. Only a tiny part of the brain is concerned with mathematics, only a small part of mathematics is applicable to the physical world, only a small part of the physical world is brain, and the loop goes on. But this is inevitable and we come against the kind of mysteries we may never hope to resolve. Science and rational thinking do not demystify the core of reality, it just points more accurately to where it is located,

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[10] Arnold makes fun of Leibniz first attempt of derivation being multiplicative, something Newton and his British colleagues never would have entertained.

[11] One neurological pathology, that of autism, seems to be due to some imbalance between the left and right hemisphere.