Mathematical Lives

Protagonists of the Twentieth Century

C.Bartocci et al

June 14-16, 2011

When I was fourteen my father suggested that I read Eric Temple Bell's 'Men of Mathematics'. I did so and was enthralled. Few if any books I have ever read have had a more lasting influence on me. It provided me with a new set of heroes - intellectual heroes, thereby provoking and focusing a slumbering ambition and becoming instrumental in making me chose for better or for worse a mathematical career. One should indeed be careful as which books one reads as an impressionable adolescent. Naturally Bell's book is open to criticism, strange it would be otherwise, but his potted biographies remain fondly in my memory, and indeed it would be very interesting to have an update including the modern giants of the 20th century, because as I recall, Bell's book ends with Poincaré. The title of the present book as well as its preface promises exactly such an update. I am sorry to report that as a reader I am sorely disappointed. On the other hand given the expectations, I could hardly have reacted otherwise.

Bell was a real writer and his work was inspired and imbued with a single unifying authors voice. The present work is a hodgepodge of contributions by amateurs ranging from the admittedly interesting to the excruciatingly bad. Not surprisingly it is culled from contributions to an Italian journal by name of 'lettera matematica'. It is symptomatic for the collection that the pieces on non-mathematicians such as Queneau and Musil are among the most interesting, although they would properly belong to a general anthology on mathematics and culture. The contributions as original articles are self-contained, still the ambition has been to make it into a book, and many of the articles do refer explicitly to the book in which they are a part. But why do we need to be told in almost every single article that the Fields medal is the most prestigious award to be given to a mathematician? The selection is furthermore biased towards Italian mathematicians, which is fine and understandable given the provenance, but which is not at all apparent from the cover of the Springer edition which advertises a medley of Mathematical Lives from Hilbert to Wiles, who happen to be the first and last in the sequence. Admittedly the material on which Springer decided to issue an English rendition is not the most promising, yet one would nevertheless expect some more conscientious copy-editing from the publisher. (But this is admittedly and regrettably a complaint one may levy against publishers in general nowadays.) I was particularly irked by the garbled references in German, obviously the writers themselves may not know a word of that language, but one would suspect that such expertise would readily be available to a German publisher. Furthermore why refer to Michael Freedman as Steve Friedman?

To whom is such a volume directed? To the general public or to fellow mathematicians? Springer books are not addressed to the general public, they are not prominently displayed in general bookstores nor available for browsing in airline terminals. I doubt that such a book would ever reach the eyes of an innocent teenager, incidentally the most important component of the general public as far as popular science is concerned. If it is directed to the professional mathematician one would expect the biographical sketches to penetrate a bit deeper and not merely being regurgitations of the well-known. Hardy is one obvious case in point, there could hardly be any mathematician who is not familiar with the Hardy-Ramanujam romance or who has not read 'A mathematicians Apology'.

Disregarding such, perhaps petty criticism, perusing the book will give the reader an overview of 20th century mathematics, appropriately starting with Hilbert and his problems. It is a common misunderstanding that Hilbert gave one of the so called key-note talks at the international congress in Paris in 1900, in fact his talk was obscure, presented at a marginal section and to a small audience; and besides he only had time to present ten problems, the full list of 23 problems only appeared retrospectively as he was encouraged to write up for the proceedings. Hilbert's impact on 20th century mathematics is incontestable, yet to many people his most pervasive legacy may be his popularization of the notion of mathematics as a formal game starting with an arbitrary system of axioms. As long as a system is consistent, he tells us, it makes mathematical sense and the objects implicitly defined do exist. Frege, by the way, took exception to this radical approach, where the axioms took over the traditional work of definitions. As a mathematical philosopher, Hilbert is classified as a formalist. From a personal point of view this is misleading. Hilbert had a passionate interest in mathematics, and he held the optimistic view that whatever we want to now, we eventually will know. So typical of the belief in progress prevalent in his youth. Thus formalism was just a stratagem in order to finally present an absolute proof of the consistency of mathematics, enabling mathematicians to remain with a good conscience in the Paradise created by Cantor.

The subject of the foundations of mathematics is probably the most easily popularized of all mathematics. The problems are easy to understand and appreciate, and they speak, as does cosmology, to something deep within us. And, unlike most mathematics, it is possible to get the main ideas across, without getting bogged down in technicalities, the desperate evasion of which dooms so much of popular mathematics.

Rigamontis piece on Bertrand Russell is delightful and an illustration that one can write on foundational matters succinctly and to the point and not cheating the reader. The subject is of course the Russell paradox, which removed the rug underneath Frege's project. Mathematicians are of course familiar with it since their teens, are in no need of instruction. Poincare, who held mathematical logic in low regard, is reported to have exclaimed, that at least this sterile subject had produced contradictions, while reporting that it was of course old hat, and referred to the Liars paradox of Antiquity. According to Poincare the difficulty lay in the use of impredicative definitions, which ought to be banned from mathematics. But as Rigamonti points out we use them all the time in ordinary life without ever getting into problems. The point is of course that mathematics draws the ultimate conclusions. (Just think of the inductive step every human has a mother who is human, which if taken literally implies an infinite descent of human ancestors.). Russell came up with a solution to the quandary by introducing a stratification, in which the forbidden questions no longer could be legally posed. What may not be so familiar to most readers is that this reasonable step leads to much silliness, such as different kinds of numbers (in particular different kinds of empty sets!).

The 30's saw the most spectacular advances of logic ever. There is of course Gödel, a precocious attendant of the meetings of the so called Vienna Circle. This was a collection of philosophers, so called logical positivists, dedicated to precise language and the abolition of metaphysics. Karl Popper, who is often thought of as a positivist, remarked that the whole project was in fact based exactly on what they wanted to abolish. It came to a dramatic end when their leader Moritz Schlitz was assassinated on the steps to the University hall by a disgruntled student. Gödel having proved the completeness of propositional and predicative logic turned to more advanced systems, such as that presented in Principia Matematica. The rest is history. Gödels achievement was to make a formal language able to speak about itself. Thus to make sense of the statement 'This statement cannot be proved'. In a consistent system false statements cannot be proved, thus it becomes both true and unprovable (provided of course the system is consistent). In fact it shows, as von Neumann was quick to point out, that a system cannot prove its own consistency. In particular it dashed the hopes of Hilbert's program, and von Neumann, whose thesis had been on the foundations of mathematics, admitted that it shook his personal faith in the absoluteness of mathematics. While Frege and Russell had tried to base mathematics on logic, Gödel turned logic into a branch of applied mathematics, where it has resided ever since. Consistency of mathematics thus is in the nature of a metaphysical faith. Kant was right after all, and mathematics is a matter of synthetic and not analytic knowledge. Gödel's theorem has lent itself to a lot of hype, and a useful antidote is provided by the book 'Gödel's theorem and its uses and abuses' by the late Swedish logician and computer scientist Torkel Franzen. Neither Russell nor Wittgenstein were able to absorb the revolution in mathematical logic brought about by Gödel. Russell seemed to labor under the illusion that Gödel somehow had shown that arithmetics was not consistent, and Wittgenstein wondered how on the earth you could prove that something could not be proved. Gödel remarked that either they were stupid or pretended so to be. Russell had lost all interest in mathematics and logic after having exhausted himself on the matter, while Wittgenstein, probably the most overrated thinker of the 20th century, did not need to pretend. Gödel next tried to settle the first of Hilbert's problems - the Continuum Hypothesis. He managed to show that adding it to the standard axioms of set theory it did not lead to contradictions, by establishing a minimal model of sets. In the sixties Paul Cohen settled the question, literally by a 'tour de force', by showing the relative consistency of its negation as well. Already in the 20's Skolem had shown the existence of countable models for any system of axioms, the seeming paradox being resolved by a paucity of bijections. Maybe this shows that after all there is only one kind of infinity, and the hierarchy of Cantor's cardinals has no real ontological significance, only being a matter of epistemology.

A parallel development during the same decade, and one which to have more momentous practical consequences, not only for mathematics, was what came to be known as the Church-Turing thesis. Namely that there is only one way of making calculations. The easiest description was given by Turing and his Turing machine, and he proved that the $\lambda-$calculus of Church was equivalent to it. Gödel reportedly was very skeptical about such a thesis, his own definitions of definability and provability was always in reference to a given formalization, but he eventually got around. The Church-Turing thesis is if anything a metaphysical statement, not only about the limits of the human imagination, but maybe even of calculation itself, as far as that can be divorced from human activity. The notion of the Turing machine is so to speak the ideological basis for computers and computer science. In particular the notion of the universal Turing machine has led to the distinction between hardware and software, without which the modern computer revolution would have been impossible. Noteworthy is also Turing's resolution of the so called Halting problem, which shows that in general it is impossible to predict in advance whether a given program will run forever or come to a halt. The inconsistency of a given set of axioms could in principle be checked mechanically, through an algorithm that systematically produces all the formal implications. Such a program would halt if the system is inconsistent, consistency on the other hand would require it never to stop. Mantiyasevich negative solution to Hilbert's tenth problem gives another specific example of it in the setting of diophantine equations. Turing's proof of the general impossibility, is yet another application of the diagonal principle of Cantor, and can be understood by a high-school student. More specific examples need far more technical machinery.

Another theme of 20th century mathematics is the rise of probability theory. Its foundational controversies centered around on one hand those who advocated a subjective notion of probability and those who favored one based on frequencies. Its theoretical underpinnings, were given definite form by Kolmogorov, who set up an axiomatic approach that is familiar to all beginning students. Kolmogorov along with von Neumann, are examples of mathematical universalists of the 20th century. The contributions of von Neumann to computer science are well-known, as well as his forays into all kinds of applied mathematics, much of it defense related but also encompassing social science mathematics, i.e. economics. von Neumann was one of the pioneers of game theory, the point of which is to explore the notion of rational behavior in face of limited and, because of the feed-back mechanism, unstable information. Game theory leads to Nash, although his most impressive mathematical achievements during his very short career are to be found in more technical mathematics. Kolmogorov was one of the first to appreciate the mathematical fertility of Shannon's theory of information, connecting it to complexity in general and entropy. Kolmogorov, along with his student Arnold, explored the mathematical implications of classical mechanics in the spirit of Poincaré. Celestial mechanics gave rise to dynamical systems, and the well-known KAM theory. Kolmogorov's interest in turbulence and his theorem on the invariance of scales, connects to such highly fashionable themes in modern mathematics such as fractals and chaos.

One of the great scientific revolutions of the 20th century was Quantum theory. Paradoxically, although involving far more sophisticated mathematics than ever before in physics, it led to a deep split between mathematicians and physicists. In the words of Yuri Manin. Mathematicians were interested in thought, physicists in reality; and according to Manin, they were taken on a far more exciting ride. Dirac is a quintessential example. Dirac famously claimed that the mathematical beauty of equations is more important than their fits to experiments. As it would turn out, Dirac would be vindicated more than once. In fact the prediction of the positron was based on playing with equations. This unreasonable effectiveness of mathematics, to refer to one of the most quoted sayings in the subject, brings forth deep and intriguing philosophical questions as to the relation between mathematics and reality, which transcends the mechanical view proposed by axiomatization. Dirac emphasized the notion of mathematical beauty, which to him was as elusive as to formal definition as the one pertaining to art. In particular it was not directly linked to mere simplicity, and especially not to formal rigor, the latter being a sore point of contention between mathematicians and physicists, with the latter taking a far more pragmatic attitude. The physicists reinvented classical 19th century mathematics, and quantum physics has led to intriguing mathematical questions. von Neumann was among other things also a pioneer in this field providing mathematical foundations such as Hilbert space theory. Atiyah, finally, should also be placed in this category of physically inspired mathematicians, having championed Witten and string theory. Atiyah started out classically as a projective geometer, under the influence of Hirzebruch and Grothendieck, he delved into more abstract theories such as K-theory which led to the Atiyah-Singer index theorem, crucial input being supplied by work of Dirac.

As to pure mathematics one of the basic themes is the emergence of abstract algebra, the roots of which can also be traced back to Hilbert and his influence. In the 19th century there was active research in invariant theory, with lots of wonderful and complicated formulas. Hilbert in a sense killed the subject by giving a very elegant abstract proof of its central theoretical problem, the finitely generatedness of the ring of invariants. The nestor of the subject - Paul Gordon, exclaimed that this was not mathematics but theology. Emmy Noether, probably the most distinguished female mathematician ever and definitely the most influential, started out with a classical prodding thesis in invariant theory then to become the champion of modern abstract algebra, popularized by the books by her student van der Waerden. Modern commutative algebra, was introduced into algebraic geometry by Zariski, who claimed that although he had a hard time following the lectures of Noether, he was smitten by her enthusiasm. The rise of abstract algebra led to another perspective on mathematics as the science of structures. This was taken up by the Bourbaki school, whose ultimate ambition was to give a unified presentation of mathematics. Commutative algebra provided classical algebraic geometry with rigorous foundations, and brought to new levels of abstraction almost singlehandedly by Grothendieck. The thrust towards greater and greater abstraction, as testified by Category theory and topoi, ideas which incidentally also play an important role in mathematical logic, is not totally uncontroversial. Although it is one thing to be driven towards greater abstraction in grappling with central concrete problems, it is quite another thing to take it as a point of departure, where a lack of anchoring is likely to lead to nothing but mere inanities.

As examples of more concrete mathematics one may take the examples of Schwarz and Smale, both incidentally linked to each other by their very visible political action of a leftist bent, which initially brought them trouble, but eventually heroic nimbus. Schwarz, along with Hadamard, were almost banned to attend the ICM in Cambridge in 1950, where the former was about to receive his Field Medal (Smale incidentally arrived too late for his, for similar reasons). The whole affair had a lucky ending due to the intervention by Truman. In the case of Schwarz we are talking about the revolution in PDE brought about by distributions, while Smale worked in modern algebraic topology enhanced by the work by Rene Thom, whose championship of so called catastrophe theory with its exalted claims as to applicability degenerated into something of a fad and tainted his reputation. Smale is foremost associated with the resolution of the Poincaré conjecture in dimension five and higher (work incidentally also done independently by Stalling and Zeeman). The technically far more challenging cases of dimension four and three done by Freedman and Perelman respectively. Smale is also known for his work in dynamic theory, especially the Smale Horseshoe, and for having made some original forays into the mathematics of economics.

In addition to the global survey of mathematics one can extract, there is also a purely Italian one to be discerned. The foremost representative is Bombieri, but the piece on him, as opposed to that on Gian-Carlo Rota, is a bit of a disappointment. On Rota we learn a lot, such that he was also holding a chair on philosophy in addition to his mathematical one at MIT. As a philosopher he was very critical of modern analytic philosophy and its divorce from metaphysics and its concomitant limitation on speculation. Much of modern analytic philosophy is axiomatic and mathematical in character, but according to Rota, superficially so. As a philosopher he was much influenced by Husserl and in particular stressed the difference between the factual text and its meaning. In particular that meaning cannot be captured formally. This does of course tie in with the lessons of mathematical logic and the limitation of logical positivism, which we have already touched upon. Rota revived classical invariance theory by stressing its combinatorial nature, and can be credited with the creation of algebraic combinatorics, which put at least a part of the subject on a systematic basis. The article ends with some intriguing interchanges Rota had with Feynman and his contributions to combinatorial mathematics. The piece on di Giorgi written by a former student has its charm and makes a few good points, while the case of classical Italian Algebraic Geometry should have deserved a far better treatment. The lives involved are merely sketched, as in a biographical encyclopedia, and the unexplained references to P_{12} are incomprehensible except to experts on algebraic surfaces.

The book ends with a brief interview with Andrew Wiles, the mathematician next to Perelman, who is most well-known to the general public. Wiles ends with an admonishment. No one should ever opt for a mathematical career, unless they love mathematics so deeply that they cannot help themselves. Will such a collection of articles instill such a love? Or maybe more realistically help to support it? As Atiyah writes, close and extended co-operation with other mathematicians has been very crucial to him, as the hard abstruseness of mathematics is enlivened and mollified by human contact. Mathematics might be transhuman, but there is no denying that mathematical activity as such is driven by humans. It is important and inspiring to become acquainted with the people behind the names, and also to see how a few individuals have indeed shaped much of modern mathematics and to become aware of the various connections between them. It is true that most mathematicians, like academics in general, do not lead exciting lives. Of those discussed in the book maybe only the lives of Schwarz, Smale and von Neumann provide material, due to their political activity, that would lend themselves to narratives of more conventional extroverted interest. But many of the great mathematicians were highly idiosyncratic personalities and cannot fail to affect, especially when their lives were tragic as that of Nash, Gödel and Turing. The strangeness of Dirac and the diagnosis of him as typical

Asperger has been widely publicized, although someone like Freeman Dyson, who knew him personally for decades debunks such charges. Kolmogorov and Arnold were of the Russian mathematical tradition, characterized by wide learning and great mathematical appetite, and surely would provide material for fascinating biographies. When Bell wrote his biographies not much was available on the lives of mathematicians. This is different now. On Bourbaki countless articles have been written. At least when it comes to Hilbert, Dirac, Turing, Gödel and Nash there are full-lengths biographies available, Schwartz published his memoirs before he died, and by Russell there is an embarrassment of riches as to autobiographical writing, and even Grothendieck has (in character?) provided posterity with a wealth of reminiscences, the bulk of which so far is only available in French. And that of Hardy is, as I already indicated, required reading. But the project is too serious and important to be squandered. There is, I believe, a need for a real sequel to Bell, work done by a single author, and in the best of all worlds to be exhibited among other biographies in general bookstores and not stacked away in mathematics sections. It would be a real pity if works like this would forestall the appearance of a real serious attempt.

July 4, October 1, 2011 **Ulf Persson:** *Prof.em, Chalmers U.of Tech., Göteborg Sweden* ulfp@chalmers.se