Finding Moonshine

M. du Sautoy

August 12-14, 2009

What is it like, being a mathematician? What does a mathematician do? Perform long divisions every day? Have not computers put mathematicians out of business a long time ago?

It is questions like this that has motivated du Sautoy to write his book. It is a chatty book, in which he puts himself in the center, as if he would be hosting a TV-program. Here he is in the Sinai. It is 40^o (supposedly Celsius) and he is turning forty, a momentous transition for a mathematician, as it means that he will be too old to be awarded the Fields medal. He is in a sense over the hill as far as worldly mathematical progress is concerned. Most mathematicians do their best work while very young, very few do it in their maturity, although there are examples[1] . (Brauer might be the closest example he brings up, but he does so only in passing, because Brauer is being sidelined by a successful student of his.)

First the book has a mathematical focus and that is symmetry. This is an excellent choice, not only because the work of the author himself is connected with it, but that is only to become apparent to the knowledgeable reader, but because it is easy to explain. Symmetries are abundant, and the author takes us to a Alhambra which abounds in all kinds of symmetrical decorations. Symmetry is about patterns that repeat themselves indefinitely, either as in translations that go all the way to infinity, or rotations that eventually repeat themselves just as circles do. So on our trip to that Moorish palace we are challenged to see all the kinds of rotations and reflections (translations are to be tacitly understood) which leave the decorations invariant. It is not always so easy, as the professional du Sautoy has to admit, even if he has resort to hindsight. The morale is that it is not so much the actual decorations which are interesting as the kind of symmetries they exhibit. The idea that du Sautoy wants to get across to his readers is that behind symmetries there is something deeper lounging, something simple and yet profound that evaded mathematicians for a very long time. The professional mathematician knows of

[1] Of course this is much more pronounced in the arena of sports, where the window of excellence is much more narrow and often occurs early on in a career, which by necessity is cut short. This is not too surprising, performance in sports depend on physiological components such as muscular strength, lung capacity, components which can to some extent be developed but are nevertheless subject to inevitable decay as the body ages. And given the cut rate competition, even modest reductions in physical fitness, inevitably means the loss of that edge which makes the difference between winning and losing. In mathematics experience counts for more, and there are no simply identifiable neurological components of the human brain that translates into mathematical creativity. Furthermore in sports winning is all, while in mathematics work at different levels is still very useful in the collective enterprise, and unlike sports, different contributions cannot be directly be compared with each other. Thus to put it cynically, mathematical degeneration is masked, and shortcomings are not too obvious. This goes of course even more for other academic and scholarly pursuits.

course right away the right answer, namely the notion of a group, but du Sautoy is careful not to give the game away until far into the book when he introduces Galois, who made the concept explicit, and later on Cayley who formalized it. Thus just as behind every instance of three crows, three rats, three castles, you name it, there is the abstract notion of three, behind every manifestation of a symmetry there is a corresponding group. While we quickly spot the notion of cardinality, the notion of a group is far more elusive and as it not surprisingly turns out a far more subtle concept.

Now it takes a tour de force to prepare the reader for the introduction of the concept of a group and the question is whether the author has really honestly prepared the reader for it. It is not so easy once you know the concept to pretend that you do not. Thus he falls into the trap of repeatedly almost introducing it but always stopping short of it at the very last moment. Maybe this is also going to frustrate the innocent reader not only the professional. To do so convincingly, i.e. pretending not to know about a group, yet making it subconsciously manifest, would be a work of art only possible through some divine inspiration. His book is not a work of art, but few books are. So slips are noticeable. At one moment he talks about finite groups, as if he had ever dwelt earlier on infinite symmetries (true, translations, but those were quietly passed over), finite groups being what he has all the time having prepared the reader for, and even in his discussion of Lie he never makes explicitly the notion of a continuous transformation group, suddenly Lie has been transformed to certain finite groups of Lie-types, which it is not quite clear that Lie would ever have recognized as his.

Admittedly this is a book for the general public, not a text book, but a book to bring to the beach, thus there is no obligation to make definitions, let alone precise ones. The professional reader may be frustrated by the lack of them, which are the 17 groups of Alhambra really, what is the meaning of the Conway notation, how do you generate a group from it[2]?

Now the good ideas of a mathematician are never precisely articulated, in fact being ideas they cannot be pinned down without ceasing to be ideas. Ideas have be worked out, and this is the hallmark of a mathematician, to recognize an idea and to try to implement it. Now one may wonder whether it is actually possible to appreciate a mathematical idea without having any idea of what it means trying to implement it? Mathematical ideas, although vague are very potent, much more potent than well-articulated a precise statements (which only become graspable once you perceive the idea behind them), only make sense if you can try to bring them down to earth. This is a trap many popularizers fall into, believing that what is vaguely expressed will be as potent to the general innocent reader as to the educated professional. In the cases, unlike the present, when the authors themselves are not professionals, their vagueness may simply be a symptom of incomprehension and confusion, and then there is little potential for hidden potency, on the contrary they are apt to confuse expert and layman alike.

Ideas can only be fathomed and manipulated once they are given some rather concrete manifestation, concrete in the sense of being amenable to manipulation, i.e. to some kind of higher order calculation. Those manifestations are the analogues of the shadows projected

[2] This is a story that is well documented, and Conway has hit the lecture circuit himself giving vivid one-man shows on the subject. with a lot of bells and whistles (and whispers and wild shouts)

onto the material world of manipulation to use a Platonic metaphor. Any popular book on mathematics has to contain some kind of instruction, and this means exactly the formal manifestation of some simple mathematical ideas. There is some of this in the book, the author explains e.g. how to arithemitize geometry, how the geometry of a cube can be encoded in some short sequences of 0's and 1's (It would have been more elegant using ± 1) and that this readily generalizes, enabling us to manipulate things in several dimensions, beyond our visual powers. Furthermore there are explicit examples of composition tables, simple manipulations with permutations, and other such things, we as mathematicians cannot avoid informing the public about. Without some kind of anchorage, we are but vaporously suspended in mid-air.

All of this ties up with the authors unfortunate use of the notion of language. He speaks on one hand of mathematics as a language, on the other hand of the language of mathematics. Does he mean in the latter case supposedly languages created in order to pursue mathematics, as opposed to the former meaning of mathematics itself being a mere language, a formality whose value is but instrumental. More precisely, does he mean that mathematics is but a convenient tool to make sense of symmetries, that is that symmetries are part of the real world and not part of mathematics, and that there might possibly be other ways of understanding symmetry than mathematical? This is the problem of using metaphors too freely and mindlessly, metaphors should aid and stimulate thought, not be a substitute for it. Furthermore, to pursue the second case, are groups real or do they just make up a convenient language of mathematics, useful for the understanding of symmetry? In short are groups discovered or invented? This is supposed to be a crucial litmus test to decide the philosophical attitude of a mathematician. Yet invented or not, an invention inevitably carries aspects that are forced on it and not (consciously) designed, and thus those forced aspects acquire an objective, independent quality, examples of which in a sense are discovered. Most mathematician are Platonists whether naive or not being beside the point, and du Sautoy is no exception and consequently as the book will eventually disclose, groups are to be presented as real objects out there to be discovered.

Symmetry is manifest in nature, both true and unadulterated on the basic level, and approximate and deformed on a higher level. It is very important to clearly make the distinction and the author, although he makes some relevant hints, does not really do so. On the basic level symmetry is default, it is far harder to make something asymmetrical than something symmetrical, as the former requires more information. On the level of the symmetry of say human beings, the causes for symmetry are different. It is not so much, as du Sautoy try to make it out, that symmetry is something hard to achieve and something grown out of evolutionary pressures, similar to sexual selection, it is rather a kind of fossilized phenomenon. The 'Bauplan' of an organism, is something very basic, and as such once formed and rigidified not to be changed. Perfect symmetry does not, for a various of accidental reasons, occur in the macroscopic world, and although gross asymmetry may be repulsive to us, as the sight of a one-legged man, it is far from clear that the closer a face is to exhibiting symmetry, the more pleasing it is to us. This is getting leads to the intriguing topic of symmetry and the arts, especially visual art and music.

The book starts out with artwork, abstract non-figurative artwork designed and developed by Muslim artists (or rather artisans?) under the strictest obligation to be non-figurative (or more precisely not to depict the animate world). Is that art beautiful, and if so wherein lies its beauty? Is it in the contrast of colors or in shapes? Being non-figurative the issue of mimesis, successfully achieved or not , is not present. It is abstract art, although abstract art from the point of view of mimesis, is in fact as concrete as can be. A square is not a representation of a square, it is a square itself; and similarly an abstract design is its own representation. Yet the fascination of those designs, because after all there are more in the nature of designs and patterns than works of art, has a truly abstract component. Pleasing as they may appear to the eye, they inevitably would tend to bore the superficial viewer, and only by trying to figure out the precise nature of their symmetries, which in the case of a professional viewer means trying to compute their symmetry groups and see how they fit into the known classification, are they able to hold our sustained interest. And this is an appeal that goes beyond the visual, and in fact ultimately it has nothing at all to do with visual appreciation per se. In fact such art is mechanically generated, by choosing a fundamental domain for the group action, and making a random picture inside it, and then generating it by the action, the symmetric design is created. Incidentally the author does not speak about action of groups, although he is very close to so doing, by emphasizing the active and dynamic nature of symmetry, as being something that can 'be done' to the symmetric pattern, and yet preserve it. By not doing this he misses out another important aspects of those symmetric images, namely that not only are they dependent on the group action but also on the representation of the group, namely by orthogonal matrices preserving metric data. There are of course other kinds of representations, notably involving hyperbolic representations, as in prints by Escher, but of course this is not the kind of symmetry that cries out at us being ubiquitous. Escher is a mathematicians favorite, his standing in the art world is rather low, he is seen as exactly what I described above, as a mere clever technician exploiting mathematical tricks to generate intriguing images, ultimately hollow. Certainly Escher has done just that, taking an image and repeating it according to rules (i.e. finding a fundamental domain and repeating it according to a chosen symmetry-group[3]) but it would be nevertheless unfair to reduce his works to such cheap tricks (and after all there also takes a certain ingenuity in choosing the fundamental domain as well as choosing the picture inside it to be repeated) he has done not only that, although admittedly the fascination of his work hinges on the mathematical ideas depicted in them. Yet as art criticism considered, limited as it is, the ingenuity involved in mathematically disclosing symmetries in designs, is far more interesting and subtle, than much of the babble that qualify as such, looking for symbols as paintings were mere rebuses, and thereby trying to reveal the inner meaning of a visual work of art. There is of course mathematics in art, even if leaving aside the spurious case of harmony and the golden ratio, so often exploited in popular articles on mathematics and art intended to let some of the glory of visual art rub off on the former (as if this would be necessary to enhance it). I am of course thinking of perspective. Still perspective as used by artists is a mere technical device to achieve the illusion of mimesis. The connection between Music and mathematics is something entirely

[3] This is the idea behind the kaleidoscope, which in practice is restricted to hexagonal symmetries, but which was carried somewhat beyond that by Coxeter

different, something far more intimate. Or is it really?

Music is unconscious counting according to Leibniz, a well-known quote that du Sautoy introduces his section with. Mozart was supposed to be a mathematician at heart, only that he chose to articulate those insights musically. Thus when Mozart was able to recapitulate a long piece of music on hearing it once[4], it was not the case of having a mindless phonographic memory, but of understanding the 'inner logic' of music[5]. So far so good, but in what does this 'logic' consist? The author's claim that it resides in symmetry is such an egregious case of special pleading that he is made embarrassed by it himself. True some of the music of Bach exhibits striking symmetries, which become apparent when you represent the music visually by its score. Parts are played backwards, i.e. given by vertical reflections of the score, other corresponds to horizontal reflections of the same[6]. Furthermore by bringing in a musical friend of his to show the similarities of mathematics and music, he almost gives the impression that the composition of (at least some) music might be a case of solving sudoku-type problems. The discussion gives the impression that one may very well create music, by taking a so to speak fundamental segment of a more or less random music, then subject this to various symmetries, such as reflections and translations (suitably interpreted) and thus creating a musical kaleidoscope. In fact one gets almost the impression that one may generate Bach-typ music this way, and that the beauty of it resides entirely in those hidden mathematical symmetries[7]? Is Bach on the level of Alhambra designs? The author is backing out, he claims that he has never been able to appreciate fully the Goldberg Variations despite assurances from older colleagues that such an appreciation will come with age. Music is not perfect symmetry, that would be too boring. Almost symmetry maybe, but not full, it would be too predictable[8] There must be elements of surprise. By acknowledging this the whole explanatory approach comes tumbling down, and his foray into the elucidation of hidden symmetries reduces to an amusing aside of only peripheral interests, as it no doubt would be to most musicians. But the analogy points to an interesting aside, namely the relationship between a musical

[4] The piece of music was Gregori Allegri's *Misere* so much loved by the pope Urban VIII for whom it was written, that he issued a decree that on pain of excommunication it was not to be transcribed and carried beyond the Vatican. In fact its performance was to be confined in time as well to the Holy Week.

[5] But this is not analogous to Conways feat of memorizing thousands of decimal digits to π as du Sautoy claims. To do so is just silly. There are no hidden patterns to the successions of digits in π which the author obviously knows, if there were that would be news to us all. One should not confuse memorizing tricks with true patterns. The author is obviously carried away making a slip, unfortunately indicative of a certain slap-dash approach in his somewhat chatty and breathless writing, aiming not so much for permanence as momentary entertainment, admittedly with charms of its own.

[6] It would have been very helpful if the author had not only supplied illustrations but gone in somewhat more detail in explaining some musical notions, rather than assuming that those are well-known. But maybe the musical expertise of the average reader is beyond that of the intermittent mathematical reviewer?

[7] du Sautoy refers to a certain BBC competition in which you are supposed to distinguish between computer-generated pastiches of Bach and the real thing

[8] A similar point was made by Bernstein in his Eliot lectures at Harvard in the mid 70's, pointing out the near symmetries to be found in some of Mozart's music. I found that intriguing, but of course as far as having explanatory value it has next to none.

score and music itself.

A musical score is a visual encoding of music, and as such much more direct than the written encoding of human speech (not to talk about the linguistic encoding that words and combinations thereof make up), and thus should not the aural beauty of a piece of music be reflected in the visual beauty of the score? Musical symmetries are in fact even more manifest in the score than in the listening[9]. And if so would not the score faithfully substitute for the music itself? Now apart from the fact that a musical score is not a rigid recipe (unlike the encoding say on a CD disc) but leaves a lot to the discretion of a performer, it is quite remarkable that we need to have it translated into sound before we can enjoy it. Would the pleasure entirely reside in the abstract notion of its symmetry, the score itself would be enough. The actual timbre of the sounds do make a difference, just as actual colors are superior to merely imagined. Yet when we read a text we are all often able to vividly engender in our imagination the scenes the author wants us to imagine, but few of us are able to read music in the same direct way. That is by reading the score, the music simultaneously plays in our head. The great composers obviously had this ability, and in addition they could hear new music being played in their heads and effortlessly write it down, unlike more pedestrian composers who need to sit by the piano and painstakingly document their efforts[10]. But the question is whether the sensual imagination can substitute for the sensual impression itself. Consensus among philosophers, even among the idealistic ones such as Berkeley, is that the imagination is but a pale phantom of the real thing, so his deafness was to Beethoven a personal tragedy, even if it strictly speaking did not impair him from future compositions, and thus the tragedy did not extend to a public one. Our world of sensual perception is indeed a private one, and indeed if we naively share the same world, our qualia cannot be compared, let alone exported from one mind to the other. With more abstract notions it is different, although we cannot say that our perceptions of say colors are the same, in fact the notion of being the same presupposing some means of comparisons, we can say that they are (essentially[11]) isomorphic. Thus paradoxically the more abstract the easier it is to be shared between different minds, something explicitly pointed out already in the dialogues of Plato. In the interpretation of a score there are two things going on, on one hand its translation into sensuous sounds and on the other hand the (musical) emotions engendered in the listener. Of the last there is of course little control. In the interpretation of a text, one could detect three levels, in order to bring out analogies. At the first level the author has very little control, in a novel most of the perceptual detail is left to the readers imagination, and when not so done, it hampers it and becomes tedious[12]. In fact what the author exports

[9] As du Sautoy notes, music has to be heard linearly, and hidden symmetries can only become manifest after repeated exposures. That a piece of classical music 'grows on you' is certainly a common experience.

[10] My understanding of the actual process of musical composition is no doubt greatly aided by my ignorance enabling me to present simplistic explanations with great reassurance, a phenomenon of quite a general application. Yet it has little bearing on the discussion to follow.

[11] leaving aside the slight complication that is caused by colorblindness or the more general fact that different individuals differ in their way of being able to distinguish between nuances of color. In short a case of different equivalence relations.

[12] In a filmed version, very little of the perceptual world is left to the imagination, and that does not

are not exact perceptual representations, but something more abstract. And here it is of course important that we get it 'right'. This should then be compared to the actual performance of a score. Then finally the third level corresponds to the emotional reaction to this, over which which the author (or composer/performer) has only limited control.

This leads to a very interesting issue namely that of mathematical exposition, something that is central to du Sautoy's book, but which he does not really milk for all its worth. He is content to remark that what he aims for is unambiguity. If he wants to get a mathematical explanation across, he wants to get this very explanation across, and not something else. In other words he wants the reader to see the very same mountain that he has discovered himself. If he also would see and discover other mountains in his effort to do so, this is fine maybe even better, but nevertheless he should not miss his particular mountain. Mathematics is abstract, and the more abstract a thing is, the more portable across minds in principle. This of course underlies the Platonic image of mathematics and something hinted at in his dialogues on perception, mentioned above. Now the philosophical issue is complicated and confusing, as we on one hand are talking about facts of mathematics and on the other hand understandings of mathematics, and it is not entirely clear what du Sautoy is referring to, maybe both, or may be he thinks that it would be naive to make a clear cut distinction. Mathematical facts, in my opinion, can be seen as existing outside human minds, objective features of the universe, but that mathematics as a human enterprise is less concerned with finding facts than to achieve understanding in terms of 'patterns', something we will return to below. Furthermore how much detail should enter a mathematical exposition, or particularly a proof, being what the author mainly has in mind? Too much detail and the reader is bogged down, not seeing the forest for all the trees which are in the way, too little detail and the reader is saddled with the hard work of reconstruction. Unsurprisingly what may work for one does not work for the other, the novice needs a lot of details, the expert is annoyed by them. Also different things trigger understanding for different readers, thus the common experience of many mathematicians of having something come alive to them by some off-hand remark by a colleague in a cafeteria line. But a written down proof comes just in one version and has to do the work for many different readers. The purpose of a written down proof is to provide a documentation of the thought processes of an individual (a road map to the mountain, in the simile of the author) and not necessarily the last word on the subject. Understanding a proof involves much more than the local understanding that comes from following individual logical steps. In that sense a proof can be seen as a score, or, if you want, a digitalization of a picture into pixels, useful for transportation.

In his discussion of symmetry in the arts he fails to mention rhyme schemes in poetry. This is an obvious example of repetition (or translation to preserve the language of symmetry), a certain seemingly random pattern is to be repeated (a suitable number of times) . The scheme itself is a simple kaleidoscope. But of course the poetry that is made to comply to those rules cannot be generated in the same way, unlike the scheme

---

bother us, because we take it in directly, without the need to to having to decode it through the tedium of words. On the other hand, the more that is left out also in a visual presentation, the more effective it usually becomes. Acts of violence can e.g. be far more powerful when merely suggested than when shown in graphic detail, the same goes without saying for sex.

itself a small portion of it does not generate the rest. The constraint of such schemes is meant to act as a stimulation to the imagination by providing a challenge, and I believe that with most poetical compositions, the adherence to it is mostly subconscious, indeed a case of subconscious counting according to Leibniz, the poet not being reduced to a puzzle-solver trying to get the pieces to fit, even if occasional tinkering must intervene at times. But in principle this is not different from the convention that a picture should be rectangular[13]. As to bell-ringing it is in its modesty an excellent illustration. The ringing of bell in various combinations is not music but a kind of sudoku-like diversion that does engender some elementary but honest mathematical problems concerning permutations, which were raised and to a large extent solved, before it had become an articulated part of mathematics, testifying once again to the idea that mathematics exists before it has been discovered (just as colored natives).

So what has music really in common with mathematics? Is it true that a mathematician is really a musician at heart? As to the second question it can readily be disposed of. The notion that mathematicians should be musically gifted in any normal way that it is usually manifested in (i.e. that a musician would make an independent diagnosis of such) is a romantic notion and easily disproved by a multitude of examples. As to the second it leads to a discussion so vague as to make any denial impossible. I have referred to the 'inner logic' of music above. What is really meant by it? In the same way narratives have inner logics, which make it easier for us to remember them than randomly assorted facts, and more to the point, proofs have 'inner logics' above and distinct from that which guides every step. As du Sautoy points out a proof has to be read many times in order for you to pick up its themes. The 'inner logic' we are talking about cannot be made explicit, if so it would cease to be an 'inner logic'. In particular the 'inner logic' does not consist in those kind of symmetries the author has been explaining. This would be a far too concrete and manipulatable interpretation, just as the 'patterns' of mathematics we have referred to above, and which G.H.Hardy was so fond of referring to, are but abstract metaphors, and should never be taken too literally. The Music of mathematics is not the same as symmetries, but may arise when we contemplate them, and thereby making unexpected connections.

As to some other examples of symmetry which du Sautoy brings up, the less said about them the better. His discussion of coding and error-correcting codes has little to do with symmetry, at least in the way he presents them, its only justification being its reference to the 24 dimensional lattice now known to mathematicians as that of Leech, which will play an important role later on in the book. As to 'mirror neurons' this is downright embarrassing. The psychological fact that we often make unwarranted symmetrical assumptions is interesting, but can we really deduce so many far-reaching ramifications from it? The fact that we seem to have an uncanny ability to emphasize with people, as well as our tendencies to automatic emulation of other peoples behaviour, is of course extremely interesting, and it brings into focus many psycho-philosophical issues, such as the social formation and function of language, to what extent we are able to individuate in the sense of Jung, as well as pointing to other of his notions, such as collective unconsciousness and archetypes. But to reduce this to a case of mirroring neurons, is as puzzling as it is

[13] Circular and oval pictures only being frivolous variations on that basic scheme.

simplistic, and really a case of stretching the notion of symmetry, to suggest a physical neurological instance. (But of course in a book meant for the beach this might be perfectly legitimate.)

The over-riding purpose of the book is not so much its ostensible subject - symmetry, but to present to the public a credible account of what it means to be a mathematician, to describe what a mathematician does, what makes him tick, and in particular what his everyday life amounts to. This is done both by a self-biographical account of the author himself and giving thumbnail sketches of mathematicians of the past, as well as presenting some of the leading players of the classification of simple groups. As a self-portrait, the protagonist comes across as a likable and regular fellow, interested not only in music but in soccer as well. A guy who it should be fun to get to know and fun to be with, and who presents many a delightful vignette from his early life[14] . Now whether it is intentional or unintentional, the portraits of the other contemporary mathematicians, certainly play up to vulgar conception of mathematicians as unworldly, close to autistic individuals, singularly obsessed with narrow and esoteric pursuits. It may nevertheless to have been the personal experience of the author, and as such it certainly would be dishonest to play it down. Maybe his point is simply that mathematics is so great that it transcends its practitioners? Anyway the issue of autism and mathematical giftedness is an interesting one, and surely one which may grab the attention of the casual reader. But to that we will return.

So Marcus du Sautoy is a mathematician. A successful one, professor at Oxford, with a fair number of doctoral students and publications, with a few breakthroughs to his name, but not obviously of la creme de la creme. A fields medal will not come his way, as he ruefully confesses (yet tongue in cheek) in the introductory pages. Mathematics is about having ideas, of being to implement them and to get the rush of seeing that they work. Of making unexpected discoveries and glimpsing striking but elusive patterns. The high you get of solving a problem you have been struggling with transcends the social recognition such achievements may engender[15]. Now it is of course very hard to convey this elation to one who has not experienced it. Mathematicians all have, at least to the extent of having been seduced into the profession. Other people may at least have had some inkling of it, felt the satisfaction of things clicking into place, maybe when solving a sudoku? What does du Sautoy actually do? The non-professional reader will inevitable have but a very hazy idea, while the professional reader understands that the author is concerned with $p - groups$ counting the number of isomorphy classes of such depending on p and its power. This can of course easily be worked out by hand by any minimally competent mathematician, at least up to the case of p^4 but then it starts to become more subtle. Simply polynomial formulas exist for the next case, but pretty soon one knows nothing. So du Sautoy is up to

[14] He also presents a lot of personal material, that certainly adds to round off the picture, but also tends to stimulate in the reader a certain amount of titillation, and maybe prurient curiosity. So the IVF failed and twins were adopted from Guatemala, but what about his oldest son, is that his own, or his wife's (from a previous relation, if any)? Questions, and related ones, being of course totally irrelevant to the subject of the book, but nevertheless provoked.

[15] Borcherds experience of cracking the problem up in a stalled bus in the Kashmir, beats the glory of receiving the Fields medal for it in Berlin a few years later.

generalize this, finding some universal formula, maybe without having to do the intractable classification itself, but by making unexpected connections to other parts of mathematics. The author hints to the computation of rational points on elliptic curves, variants of zeta functions, motivic integration, all making it very intriguing at least to the professional reader, who would like to know some more, but this is obviously not the place to learn of it[16]. Now this is as much as we really need to know reading the book. Now mathematics is not all fun, especially not to the average mathematician. It is not just a tale of steady elations, but more often one of frustration and depression. No wonder that people drop out, or that mathematicians stop doing serious mathematics but resign themselves to do what they have always been doing, routinely churning out results by exploiting technical machines they have learned to master in their youths, in order to comply to the pressures of publishing and prove that they are active. The wonder is that not more people drop out. Mathematics involves both having ideas and doing calculations. By calculations I of course do not mean specifically long divisions and such things, but simply what belongs to mathematics as a craft, some of which can be automated and implemented on a computer. Calculations are hard work and no sane person is willing to engage in them indefinitely for their own sake. Calculations only make sense when they are in the service of a mathematical idea. Given en exciting idea, you are quite willing to go to great lengths in pursuing it, and calculations which otherwise would be too tedious to undertake can be obsessively pursued. Now ideas come not easily, and to pursue them might be fraught with endless frustration, but this is inevitable. If there is no frustration chances are that you are not doing mathematics really, only going through the motions of so doing. Now only hard work engenders ideas, but you need to have ideas to motivate hard work. Thus for the successful there is a positive spiral, the more you think, the more ideas you get; while it can as easily, and in fact more commonly, work the other way, putting you into a vicious circle. As ideas dry up, the motivation for hard work peters out, and without hard work the chances of getting new ideas drop dramatically. Sometimes the only solution is that you get a good suggestion from the outside. This is why conferences are useful, not so much to sit and learn from lectures, most people sleep through lectures, but to pick up something that might engage you. The chances of so happening are slim, especially if you are down and out already. There are some standard ways of protecting yourself. Collaboration is maybe the most common and effective. Humans are social creatures, even mathematicians are in general, and if you are lucky, a collaborative effort can be super-additive. Meaning not only that you add quantitatively but also qualitatively. If you are lucky you can add alternate rugs to a ladder that leads to the goal. The whole is more than the sum of its parts. I believe that most mathematicians would not survive and thrive without collaborations. Collaborations are different from merely meeting people and listening to what they have to say. Yet collaboration is also looked down upon. Individual contributions count for more, collaborative efforts are suspected, it is often assumed that the other one, has done the work. That in collaboration the sup-norm is the relevant one, not the integral norm (to quote a well-known Swedish mathematician). And mathematics is also a very competitive and individual pursuit. Unlike in science there are no research teams in mathematics, it is never really reduced to just a job, it behooves each individual mathematician to see to

[16] The author gives no further readings, but at least links to a few relevant web-sites

it that he or she not just goes through the motions. No one can go through the motions indefinitely, and when done for a long time, there usually is no way back. Even a young, bright, and energetic graduate student hungry for success typically spends several years before getting to the forefront of a field of research. An older jaded mathematician cannot expect to repeat such a feat, unless of course his previous expertise provides an unexpected link. To switch between different fields is a romantic antidote to drying up, available only to the very best, who usually do not need it. There are so many things that works against mathematics and its pursuit, but of course in a book like this, too much emphasis on those features, would defeat its ultimate purpose. So instead, apart from a few references to graduate student blues and midlife crises, there are vignettes from conferences at exotic locations, intra-cultural communication (mathematicians use a universal language and the subject being of such an abstract level, it is eminently communicable, although in practice very few people are in the position of taking advantage of it), regular visits to Oberwolfach, or to centers like Bonn and Paris. Most professional mathematicians recognize this and nod, to the general reader this might not seem so exciting, except if he or she is immobilized to a fixed location. Does he succeed in conveying? This is not for the professional reader to judge.

Mathematics contain some good stories of human interest. Maybe the two most classical being those of Abel and Galois, both of which fit into the general theme of the book. And in the historical sections the author obviously draws on secondary sources, repeating to a general audience distilled versions of what is already available and to which most professional readers would be privy. The real interest concerns the modern story, the story of the classification of simple groups, where the author can draw on first hand material, and which might be the most interesting to a professional readership, and I fear almost exclusively interested to such. But once again I am not in a position to judge, some of the descriptions of eccentric mathematicians may titillate.

The classification of simple groups stand out as somewhat of an anomaly in mathematics. It involved a clearly defined research project, which in its outlines, as opposed to its details, was readily understandable to any mathematician. (And in fact almost within the reach of an uneducated but curious audience[17]) Unlike most mathematical projects it was to some extent centrally planned, in fact under the auspices of Daniel Gorenstein, who used to travel around the colloquium circuit provocatively announcing that the problem was to such and such a percentage solved, when normally problems are solved or not, and even a graduate thesis may appear not to have gotten off ground until just before it is completed. It was a military pincher operation, one one hand trying to limit the possibilities of simple groups, on the other hand to try to find new ones, the latter activity to a large extent inspired by the gaps of the first. It is also known for the extreme

[17] Strictly speaking du Sautoy never makes entire clear what is a simple group, although he comes very close to. He does not define the notion of a normal subgroup, nor the notion of dividing a group by another. Although he makes the remark that there are subgroups with which one cannot divide. It would be petty to quarrel with him on this issue, after all he does make the analogy of building blocks as refers to them as some kind of generalized prime numbers. This is of course a natural one fitting in his project of thinking of groups as generalized numbers, concerned with symmetries not cardinalities, as noted initially in the review.

complicatedness of the proofs involved, that trend being set by the first breakthrough by Thompson and Feit taking 250 odd pages to show the Burnside conjecture to the effect that all groups of odd order are solvable. In toto the number of journal pages needed for the arguments constituted around ten thousand. Most of those were highly technical, in a sense maybe to be seen as higher order calculations, the pursuits of which would never have been undertaken had they not been part of this overarching goal. When the proof was eventually announced, a close to a thousand of pages were still missing, something that annoyed many outside mathematicians. Of course from a mathematical point of view this somewhat optimistic attitude of the group theorists only make sense if the project was low on ideas and high on calculation. Some of those calculations were indeed electronic, showing the existence of putative groups.

Many people were involved in the enterprise. The author mentions Thompson, Aschenbacher, Fischer, Grieß, Janko, and some Japanese. Totally there might have been two dozen people who made published contributions[18], but he concentrates on Conway and his associates, which makes sense, both because those are what he no doubt knows best, and because some of the most colorful characters are to be found in this gang.

John Horton Conway born in 1937 was something of a math prodigy from the very start, and soon became a long-haired fixture in the Cambridge Common Room, spending most of his time, in spite of his obvious mathematical talents, playing back-gammon. So much talent and so little direction? Given the first the second is not entirely rare. Conway has the reputation of being something of a recreational mathematician (maybe most known in wider circles for his 'Game of Life'), i.e. lot of dazzle but little serious activity or substance, a kind of mathematical clown. However, finite groups would turn out to be his mathematical metier, not surprisingly, although never considered part of combinatorics, finite group theory has a strong combinatorial flavour. Supposedly, according to du Sautoy, the turning point came in 1966, when John McKay suggested to him to look at the automorphism group of the Leech lattice. It gave him direction and he gave up back-gammon and started to work hard, reserving certain spots in the week to total immersion. At that time Thompson was at Cambridge, and Conway tried in vain to engage him as well, but the latter was evasive, finally coming straight out announcing that he would not think of it unless Conway gave him the size of the automorphism group of his group. Once that was computed, Thompson's interest perked up, and then things started to move very quickly. So not only did Conway have a new sporadic group, that group also contained many of the other sporadic groups as subgroups. Conway got engaged in earnest and gradually involving more and more collaborators he started the project of the Atlas, intended to give to every simple groups as exhaustive a description as possible. When du Sautoy came on the scene, the project was not yet finished, and he was, with certain provisos as to the spelling of his last name invited to add the latter to the title page, in which each author had six letters to their surnames, and their alphabetical order faithfully reflecting the chronology of their joining. For a variety of reasons, the author declined.

Of the twenty-six sporadic group, the largest [19] and the one most known to the general

---

[18] I am making a more or less informed guess, suspecting that many of them were actually anonymous, and that quite a few graduate students must have been involved.

[19] Its order is $2^{46}3^{20}5^9 7^6 11^2 13^3\ 17\ 19\ 23\ 29\ 31\ 41\ 47\ 59\ 71 \sim 10^{53}$

public of mathematicians is the so called 'Monster' [20] The fascinating aspect of course is that quite a lot is known about such a group before even its existence is confirmed. This, however, is not an unusual feature in mathematics, the point of a proof of contradiction is to establish so many properties of a putative object that they eventually contradict themselves and proves its non-existence. In particular its putative character table had been computed involving 194×194 entries with the smallest irreducible representation being of dimension $196883 = 47 \times 59 \times 71$. Finally it was exhibited as the automorphism group of a certain mathematical structure[21]. It was initially noted by McKay (known for his ability to find quirky details) that $1 + 196883 = 196884$ i.e. that the lowest dimensional representation is 'almost' the second coefficient in the q-expansion of the j-function. A case of mere numerology? The fact is that big concrete numbers are rare in mathematics, thus coincidences more likely than not are indicative of some hidden relations, i.e. invisible patterns. It turned out that this was not a freak, simple linear relations between the other dimensions of irreducible representations and the expansion of the j-function were discovered by Thompson and elaborated on by Conway and Norton, calling it Moonshine, the title of the present book. So this is exactly the kind of thing mathematicians dream about, finding totally unexpected connections between seemingly totally unrelated fields in mathematics testifying not only to the basic unity of the discipline but more importantly to its independent Platonic nature. Eventually the fact of Moonshine was established by Borcherd a former student of Conway, reinterpreting the Monster as a group of symmetries of a vertex operator algebra (incidentally containing the original algebra considered by Grieß). However, proving a fact and understanding it are different things, and in particular Conway is unhappy claiming that Moonshine has not yet been properly understood, testifying to the division between the Platonic nature of mathematics and our human understanding of it.

As the classification theorem was winding up, there was naturally a sense of dejection among the actors, especially Conway and his group. The Atlas was tidied up and published, but then what to do? Conway being in a nature of a collector would have been delighted had a 27th sporadic group turned up, because after all so many avenues of search had been closed off in the analysis of restrictions, and who knows, what oversights may be lurking? Such oversights are certainly not unheard of (the search for a few groups were close to being aborted due to faulty reasoning pertaining to their impossibility). Also the proof being so extensive no single mind had a real overview, let alone mastery of all the details. But barring this hope, which appears more and more desperate, what to do? In a sense one could liken the whole enterprise as to having the rules of chess discovered, now the time to start playing it would commence in earnest[22]. The next step of building up groups from their simple components is an extremely involved one, shown by the intractability of the very simplest case, that of studying p-groups. But maybe this is too daunting, once the universe has been mapped, to follow it in all its turns and twists, appear insufferably

[20] the alternative name 'the Friendly Giant' proposed by Griess, the one mathematician who together with Fischer was the one who had first been on its trail and who finally proved its existence, never caught on.

[21] Few but the very simplest groups are defined by exhibiting their composition tables.

[22] Feynman made such a comment as to the prospects after all the fundamental las of physics have been established.

tedious. Conway left Cambridge for Princeton, fell into a deep depression, made a half-hearted suicide attempt, and married for the third time. Are mathematicians strange? Autistic individuals, singularly inept to lead normal lives?

Autism is a tragic neurological condition, which inevitably cripples a victim, making a functional life more or less impossible. In recent decades the milder form of the so called Asperger symptom has received wide attention. Its diagnosis, not so much based on clinical neurology as on vaguer behavioral symptoms, has made it become the purpose of a somewhat popular social game of trying to identify its prevalence among friends and colleagues. Mathematics departments provide fertile grounds for such hunts. Mathematicians are very tolerant of socially deviate behaviour, and eccentric minds often find a sanctuary in their midst. What are the symptoms of the Asperger condition? A certain lack of empathy is often emphasized, but this is rather vague and often misleading, indicating coldness and cruelty, when it much more often entails mere puzzlement and incomprehension as to the lack of kindness that characterize much of social interaction. A tendency of getting obsessed with narrowly defined interests, a partiality to systematization, and a penchant for carrying logical arguments to absurd lengths. Especially the latter is extremely useful in mathematics but often socially disastrous. Ultimately the Asperger individual may seen as very clever, almost inhumanely so, but ultimately missing the point. The real great mathematicians surely are not effected by Asperger? True mathematics surely require much more than technical cleverness and logic rigor, there must also be some kind of vision and overall well-rounded maturity? Still the biographical anecdotes that has been handed down about Newton reveals many tell-tale symptoms, his passion for having forgers executed while he was Master of the Mint, belonging to the less sympathetic ones. Conway certainly is not an autistic personality. Extremely extraverted and gregarious he certainly does not fit the image of a shy retiring genius ill at ease in social situations. Still du Sautoy harbors serious misgivings about his humanity. Extremely self-centered he does not really allow a two-way communication, only speaking about things that interests him. Furthermore he is addicted to all kinds of quirks more normal mathematicians would disdain, such as the memorization of thousands of digits of π[23] or making a point of being able to name the day of every date given, typically a feat of idiot savants[24]. Somewhat embarrassing features that confirm the vulgar misconceptions of mathematicians by the general public, confusing the same as to what constitute real mathematical wizardy as opposed to flashy and shallow. Conway, by his own admission, has an uncanny ability to retain in his memory all kinds of quirky facts, but finds himself at the same time unable to commit to memory names of colleagues he has known for twenty years. A sign of close to total lack of interest in people who do not directly interact with his obsessions? As age increases, the powers of mathematicians slacken, as we have already noted, especially the kind of 'computational' powers that characterize a mind like Conway's. What is there to look forward to, du Sautoy rhetorically asks, desperate measures to keep mentally fit

[23] already referred to above in a previous footnote. And why only the decimal expansion, why not the binary? A case of being in the throes of social conventions?

[24] Every mathematician would be able to do this from scratch say within a minute, and more to the point they would be able to design some mnemonic algorithm to cut down the process to say twenty seconds, but Conway wants even to perfect the latter with his Doomsday procedure.

eventually will have to lose out against the great leveler of thermodynamic decay.

If Conway presents a complicated picture, the case of Simon Norton is far more clear cut. Even for a mathematician he appears strange, wearing tattered clothes, not out of extreme poverty but out of total indifference, cultivated or not. An overgrown child[25] whose main topic of extra-mathematical conversation concerns bus-tables and how to get from A to B in the most contorted way by public conveyance. Tables he carries around in plastic bags (reportedly mixed in with unpublished notes on the Monster and related topics). Or take the case of Borcherd, a far more accomplished mathematician, and a self-diagnosed Asperger, involved in a study made by the psychologist Baren-Cohen as to the prevalence of Asperger individuals among mathematicians. A study certainly intriguing to the author, and maybe to the average reader as well. Yet I believe that the emphasis on the mathematicians penchant and ability for abstract reasoning is somewhat exaggerated. Mathematics is an abstract science, but it does not require too much of abstract acrobatics among its practitioners, a certain facility with formal reasoning, surfing with tools already developed, seem to be sufficient for most academic careers. Most mathematicians could as well have been able to pursue other things, and among them you find your expected share of extro- and introverts. What is often misunderstood as an autistic streak, may be nothing more unusual than egomania. Competetive obsession and the reckless pursuit of narrow things, are quite common in many other fields of activity (sports, stock-trading, you name it).

Writing about mathematics to the public is a challenge, and a challenge du Sautoy loves to rise to. His ambition is not so much to instruct the reader, although as noted a certain amount of instruction is inevitable, as to convey to the innocent reader the excitement of doing mathematics. Whether he succeeds or not is beyond the ability of the professional reader to judge, sales figures of the books would be a more accurate gauge, but of course far from reliable as well, as consumerism adhere to elusive forces less based on real and concrete needs as imagined and abstract ones. One may though admit that the book is a definite improvement on his 'Music of the Primes', better structured, more focused, and maybe also more instructive on the mathematical level; the topic also being more accessible. He does keep his breezy, chatty style, which of course has become his trade mark as a mathematical popularizer. There are too many text-books in mathematics, there certainly is room for the book to bring to the beach, as he characterizes his efforts and ambition. The breezy style makes for fast reading, but this does not necessarily exclude reflection (as I hope this review has proved), although it certainly is conducive to slips, some of which a stern editor might have caught. But one should not get too uptight about those. The general reader will hardly be affected by them, and the somewhat autistic mathematical professional, may delight in them, confirming his sense of superiority.

---

[25] Born on February 28 1952, just one day short of the 'leap-day' birthday, which would conveniently cut his age by four. I met him when he was still officially a child, back in the Math Olympics in Moscow in 1968 (and then at the next event the following year in 1969). He was of course considered a math prodigy and with a glorious future predicted. I at the time was finally pitted against the spectre of encountering people more mathematically gifted than myself, and the strangeness of Norton certainly provided a consolation to that bitter pill, although its manifestation was not yet as striking as that du Sautoy encountered some fifteen years later.