Prime Obsession

Bernhard Riemann and the Greatest Unsolved Problem in Mathematics

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This is a book that aspires to instruct the non-mathematical reader of what the Riemann Hypothesis is all about. To do so one has to assume a certain mathematical competence, and the author expects the reader to have had no particular problem with mathematics at school and maybe even having taken a course or two at college. Those readers who find the mathematics too advanced are to be warned that they will never find a more elementary treatment anywhere else which could impart any understanding. This is tough, but this is life, and you could as well face up to it. Nevertheless the author starts from the very bottom, reminding the reader about the most elementary mathematical facts, such as power-laws. He does it conscientiously and engagingly, yet in a text-book manner, save for the running asides. In fact it is so well-done that it can be read by a professional mathematician, without undue impatience, testifying to the pleasure we all mathematician find in recreating the subject from first principles. Slowly he climbs up the steep gradient of difficulty, engaging himself at the end in trying to make visual the actual graph of the Riemann zeta-function and discuss the infinite series, conditionally convergent, from which you can recapture the prime-counting function. The author, although billed as a mathematician, is no professional mathematician, although he obviously has had some serious mathematical training and did support himself at some stage in his life as a programmer on Wall Street. His treatment of elementary material shows that he is in command of the material, and his more advanced treatment is surely the fruit of his own study and ambition to improve himself, and is devoid of any (obvious) blunders. Thus he clearly knows what he is doing, and more importantly he is aware of his own limitations careful not to tread beyond [1]. His book is a piece of journalism. Good journalism, written as a popular book on mathematics should be. From bottom and up, sympathetically putting yourself in the shoes of an *ignoramus* and recreating the material, not just dumbing it down. Still he does of course know that an entirely mathematical text cannot hold the attention of a wider public. And that the kind of mathematical text he would produce would either be too elementary or too advanced. So interspersed with the mathematical instruction are chapters on more general human interest. (His original ambition was to make the odd numbered chapters of one sort and the even numbered of another, an ambition he was not able to pursue consistently to the very end). Thus we are treated to potted histories, not only of the major players, such as Riemann himself, but also historical reviews. What was the situation with Germany when Riemann was born? How many readers know that Hannover was actually in union with England, through common royalty, until 1838 when Victoria ascended the throne in Britain, and Hannover did not

[1] And one suspects in addition that the ms has been referred by the publisher

recognize female heads of states?[2]

Riemann himself was born and bred in Wendland, a sparsely populated area to the left of the Elbe, and grew up in the village of Quickborn, where his father was a Lutheran minister. He was a shy boy, closely attached to his family, the home of which was an object of nostalgia throughout his later student years in Gttingen and Berlin, and to which he would return as often as it was feasible. The author dips into the educational system in Germany, inspired by the visions of a Wilhelm von Humboldt. It was a system that had been thoroughly rehauled from bottom up and the elementary and secondary education available was probably the best there was at the time anywhere in the world. In particular much less emphasis was put on classical education. Riemanns talents were recognized and encouraged, as it is the duty of every working educational system. Not surprisingly Riemann was destined for a position in the Church, which was very reasonable given the times and the circumstances and the natural wish and expectation of his father. He went to Gttingen to study theology but was too enamoured with mathematics and asked his father for permission to change his subject. Hid father gave the permission, a very understanding father, the author remarks, as giving up a theological career was a risky thing, as the latter was a rather secure choice. Riemann, being a very dutiful son, eager to please, and of a shy and retiring temperament, and in addition a conscientious believer, would probably not have found the strength to oppose his father had that been necessary, and would have led a miserable life as a consequence [3]. Gttingen, however, turned out to be a kind of backwater, in spite of the presence of Gauss. But the great mathematician was aloof and not very active in giving lectures and interacting with students, not even with the very best ones. It was in Berlin the excitement was to be had, and the encounter with Dirichlet would prove to be crucial to the development of Riemann. Dirichlet was later called to become the successor of Gauss, but even before that Riemann had presented his Habilationsschrift at Gttingen, giving a lecture that provoked one of the rare opportunities when the great Gauss heaped praise. The reputation of Riemann was soon made, and in addition to becoming the successor of Dirichlet, who had died soon after becoming a professor, he became member of the Berlin Academy. This involved giving a paper, and it is this paper of 1859 in which Riemann presents the striking connection between the distribution of primes and complex functions, and which one finds the conjecture of what would later be known as the Riemann Hypothesis. It is fair to say that in the history of what we could call transiently as asymptotic number theory, this was the most important step that has so far ever been taken. Riemann single-handedly created the subject of analytic number theory, and set the context in which further research would be pursued. Now, Riemann did many things, and to most mathematicians his thoughts on number theory are not by any means the most interesting. Riemann revolutionized geometry, and made great

[2] Of course the personal relations between the British Royalties and Continental did not end at the time, the German Emperor William II was a grandson of Queen Victoria.

[3] It is easy to argue that it is good to be opposed, that this strengthens your resolve, and is part of growing up to become an independent adult. And that this provides some kind of automatic selection, only the most dedicated prevailing. However the case of Riemann illustrates that principles in education can never be applied universally.

advances in complex function theory, involving harmonic functions[4]. Riemann, who was such a shy and retiring individual, did in his chosen subject mathematics, show a daring and boldness of thought, seldom rivaled in the history of mathematics. Clearly he is one of the supreme giants of 19th century mathematics, and whose true significance only became apparent in the 20th century. It is a pity that he is not generally more known, he certainly could be placed next to Einstein, whose work would have been inconceivable without the foundations suggested by Riemann. Thus the Riemann hypothesis and his work in number theory was rather peripheral in his total *oeuvre*, although the extent of the work and devotion that he applied to it, only became apparent long after his death, when part of his *Nachlaß* became subjected to scrutiny by a mathematician capable of making sense of the cryptic jottings[5]. That Riemann is much more than his hypothesis, the author alludes to, but of course his aim, and that is a commendable one, is to focus on just one, if a truly spectacular feature of his mathematical work. Riemann died young from consumption, a very common fate at the time and probably the result of poverty on a less than robust frame. Had he been allowed to live another thirty years, one wonders as to what mathematical feats he could have achieved[6]. Now the problem with selling Riemann to a general public is that so little is known about him. I suspect that a large part of his correspondence is gone, and that the only witness to his life who bothered to write down his impressions for posterity, is his younger contemporary Dedekind, from whose accounts most of the more informal knowledge of his life derives, as well as being almost the only source of extant anecdotes. This presents problems to the biographer, as well as opportunities for the fictional account.

The author then proceeds with the other players, with a few digressions. However, unlike du Sautoy he does not ramble nor does he lose control, but keeps a tight rein on his fancy. He clearly announces when he makes a side-trip, and those are always kept short, to the point, and almost always have some relevance to the main theme[7]. Thus his cast of characters, unlike that of du Sautoy is limited, and many of the modern ones, such as Bombieri and Zagier, are not even mentioned. Instead the emphasis is on the older people, those who discovered the Riemann Hypothesis and made it the central part it now occupies in the sociology of mathematical research. There is of course the inseparable pair Hardy

[4] His contributions to the so called Dirichlet problem, i.e. that of finding a harmonic solution with given boundary conditions, play a central role in mathematical physics, and thus appreciated by more down-to-earth mathematicians.

[5] Not all of Riemanns *Nachlaß* remains. After his death, his housekeeper destroyed a large part of it in order to clear the house of debris, an unsentimental act which in most cases, with a few spectacular exceptions, is a laudable one. Surviving parts of his papers were naturally appropriated by his wife and descendants, and they were unwilling to part with it, because inevitably mixed in with it, were private jottings as well. In particular the fate of his Paris journal is unknown, and it is speculated that would it surface, it would reveal many other secrets he brought into his grave.

[6] Natural, and even more pressing questions in the case of Abel and Galois. It is not necessarily so that mathematicians do their best work when they are young, at least not among the very best, it could, however, be true for the second rate.

[7] The only possible exception, for which he apologizes, is the aside on Hadamard, whom he founds a most delightful character, worthy of a side visit.

and Littlewood, resisting the temptation to dwell on Ramanujam, whose relevance to the main story is low[8]. Crucial to the pair is of course also Hilbert and the Gttingen school, with Landau as the central figure in number theory. The later computational aspects, mostly associated with Odzlosky, are referred to, as well as the intriguing connection to physics via the distribution of the zeroes, which is not random, more like the self-repelling distributions of the eigenvalues of random Hermitian matrices. However, those ideas have so far not led to anything tangible[9].

The book is a model when it comes to serious popularization of mathematics. It is instructive, containing precise and correct statements, with no dumbing-down. It is also motivational, and thus also valuable to mathematicians not experts on analytic number theory, the latter maybe an indication that it might to some extent fail its popularizing ambitions. On the other hand it might then also help to refine and uphold those ambitions. In short it does combine the good features of a mathematical exposition, namely on one hand clarity and precision, on the other a willingness to go beyond the mere technical, to look at things anew, and to indicate what is important and what is not, as well as to provide crucial motivation through well-chosen metaphors. A good popular book on mathematics has the potential of being superior to any other book devoted to the popularization of science. In fact, if people did not have a thing against equations, mathematizing a subject is the quickest way of making it accessible to the mind of an outsider. But as it is, this might unfortunately, only be true for mathematical minds.

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[8] du Sautoy mentions that Ramanujam had to some extent rediscovered Riemanns way of recreating the prime counting function, but without the crucial error term. Did Ramanujam know more? In fact he did not, as Littlewood was able to prove, the standard estimate of the difference between the Li approximation and the true function cannot be improved

[9] The most flamboyant case being made by Connes.