The Music of the Primes

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The Riemann Hypothesis is probably the most celebrated open problem in mathematics. Traditionally three open problems used to be known to the general public. The four-color problem, Fermat's last theorem, and the Riemann hypothesis. As Norbert Wiener once remarked, every mathematician worth his salt, should try at least one of them. (He himself had tried all three). The first two are easily explained to the layman, while the third is more elusive[1]. Mathematicians have always considered the first two as rather frivolous, while the third is a real mathematical problem. Typically Hilbert did not even mention the first two in his famous list of twenty-three problems, while he did include the Riemann Hypothesis (as number eight), in fact giving it his blessings, and its elevated status. The Four-color theorem was the first to get cracked, first by the general case being reduced to a finite number of possibilities, than by each one of them painstakingly checked by a computer. The result was greeted by a mixture of disappointment and derision. A proof is not just verification, it is also illumination, and the solution to it provided none of the latter. It was so to speak an ill-posed problem, the search for it had generated very little mathematical development, and its solution made no one the wiser[2]. The Fermat theorem was different. It had spurred on much serious attempts which had indeed generated much new and exciting mathematics, and the solution to it was effected by its accidental connection to a central project in number theory and elliptic curves[3] Yet knowing that it is true does have no further ramifications. Both problems are but artifacts of the history of mathematics (although of course the Diophantine equation that Fermat's posits is rather simple and natural and would no doubt have popped up in alternative mathematical traditions). With the Riemann Hypothesis it is different, it holds such a central part in number theory, so many results hinge on it, that its eventual solution would significantly alter our conception and understanding of the mathematical landscape (to use a metaphor of which the author is very fond). It is the purpose of this book to explain to the layman, why the Riemann Hypothesis is so important, and more to the point, why it is fascinating to the mathematician, and hence ultimately to mankind as well. It is a tall order, the subject is of course a very subtle and technical one, and thus in principle impossible to convey without such distortions as to risk making it meaningless to layman and expert alike. The author tries nevertheless and the question is whether he succeeds, or rather to what extent

[1] Although I was first told of the Riemann Hypothesis by my mother, a teacher of elementary mathematics. Maybe the math teachers of the past were far more educated than those of today.

[2] I recall Mackey once expounding in the common room at Harvard, that the conjecture was so boring that if God had offered him success on the condition he spend two years on it, he would have declined with no regrets whatsoever

[3] This connection also provided that extra motivation to solve the more central problem. The desire for public glory is not one mathematicians in general are wholly immune to.

he does succeed.

The first question is to what kind of audience such a book should be addressed. Obviously it is not intended for the professional mathematician, although many mathematicians will of course read it and belong to its most appreciative readership. But who beyond the mathematician is game for mathematics? There is a wide hunger for scientific illumination among the public. After all science has in the 19th and 20th century profoundly altered the human condition and also our quotidian life. If ever there was a success story in the history of mankind, surely science would be the prime candidate. In the past the scientific curiosity was centered on the hard natural sciences, such as astronomy, physics and chemistry with geology and zoology and its combination - paleontology likewise provoking the imagination; later on in the 20th century biology and medicine should be added to the list. Where does mathematics fit in here?

The problem is of course that while the central concepts and problems of the sciences can be made intelligible to the general public, and their solutions successfully put across (through colorful and often misleading metaphors); it is far harder to convey even the most fundamental of mathematical objects, let alone explaining why they are interesting and important. And as to mathematical results and solutions, forget about it. Thus when it comes to explaining the importance of mathematics, too often only resource is made to the applications of mathematics as supposedly the only viable option. Although this might work with funding institutions (and mathematics is modest in its needs) it does not exactly fire the interest of the public, who is perfectly willing to be fascinated by science which has no practical applications at all, provided it addresses deep issues of human curiosity. It is nothing wrong with heralding the applications of mathematics, it certainly adds to its mystery, but it surely should never be the major reason, let alone the sole. A popular text of mathematics is successful first and foremost to the extent it can fire this fascination that should be inherent in any curious individual, and only secondly to the extent it provides instruction. However, it is not easy, nor desirable, to try and draw a sharp distinction between the two aspects, in fact it is is questionable whether such even exist at all. Yet, trying to make the first a priority does influence the way to present the story, and the obvious way is to appeal to the so called human interest, to present the players as people of flesh and blood, thus to convey the fascination indirectly, through the passion it provokes in people partial to it. Not everyone is a mathematician, and not everyone can really understand directly what makes mathematical questions so compelling; but most people with an open mind can to some extent be carried away by vicariously experiencing the excitement of others, and thus to take part in an enacted drama. And is this not the secret of our fascination of a dramatic enactment, to sympathize with emotions we recognize but the sources of which really do not concern us in any immediate sense. (Typically when we express our condolences to someone bereaved.) This can be very effective. I am told that Singh's book on Fermat not only sold a lot of copies but excited many people innocent of mathematics, without giving any real instruction. And in my own case, I decided to become a mathematician after reading Bells 'Men of Mathematics' when I was fourteen. This was also a book which conveyed very little actual mathematics, at least I cannot recall anything that widened my mathematical horizons, yet made it clear to me that the real giants were those with mathematical genius. What was a Napoleon to a Laplace? It made the great men of mathematics, Gauss foremost, into heroes larger than life. I believe that I am not alone in having been so affected by the book, still I doubt that it would have had a similar impression on most readers.

Still to make this drama intelligible for the reader, he or she must have some feeling for prime numbers. Everyone knows about the numbers, and their inexorable march, one step at a time, towards infinity. True, this can instill in many of us a sense of vertigo, an emotion that either can lead to a feeling of exaltation or one of nausea. Maybe the latter being the case for the majority of people, and such people would be very unlikely to pick up the book in the first place. So who is going to read the book? One very important group, maybe the most important is the impressible youth. In such cases a book like this could potentially change lives, thus it is important that a popularizer does not cheat. Complicated matters need to be simplified of course, but it is very important the way it is being done. The typical case is that of watering down. This is a top-down approach. You have the complicated story, which you cannot get across, it is too technical, depends on too many things the reader cannot possible have any clue of. There is no time, nor any patience to give a crash course, so what do you do? You take away at first the most egregious technicalities, then you start stripping things off, and in the end not much remains. But what can you do? The difficulty is that the final result may very well be as unintelligible to the expert as to the layman. This is an approach you expect from a journalist. The journalist is a professional, jack of all trades, king of none, as the hackneyed saying goes. Used to take any information and process it for the reader. It goes with an unsentimental attitude towards the material, no darlings exempted from being killed. The attitude is one of realism. Books should be understood, make sense, and not tax the reader in any serious way. The reader comes voluntarily, with some ambition to be instructed, but above all he or she expects to be entertained and reassured. Then there is another approach. Bottom-up. And it is here the expert steps in. A bottom-up approach means addressing the child in you, to see things from a primitive perspective and then to reconstruct it afresh. This can be very instructive for the author, enabling well-known matters to reappear fresh again. Of course you can get carried away, starting from literally nothing and then in some whirling pages of excitement and rediscovery leave the struggling reader in the lurch. Sometimes one may fault Penrose for such excesses, but I would call it forgivable, it is not cheating at least. The point being that an impressionable youth is not necessarily put-off by what he does not understand, on the contrary she may be fascinated by what goes beyond. When there is no cheating involved, such a reader will get hold of the end of some string liable to be unraveled. Maybe not in this particular book, but part of being a good book is to be a pointer to other books.

In addition to the impressible youth we have what I would like to call the intelligent layman. A discerning individual eager to enlarge its views. Ignorant, maybe, but not stupid, resentful of being dumbed-down to. A popular book on science, in particular mathematics, should not disappoint those two categories of readers, even if the majority of readers may be far less serious. In many ways this is a tall order, on the other hand anything worth doing is worth doing well. And it is here we are looking for the initiated author, who knows his subject inside out, i.e. being able to reconstruct it from bottom to top. When it comes to the skills of writing engagingly, they may not be universal in the mathematical community, but certainly there are enough such gifted individuals to go around when it comes to writing the relatively few books the so called market is expected to be able to absorb.

So the story is basically told chronologically, both in the sense of history and in the sense of going from the elementary to the more sophisticated; usually the two strands are reasonably synchronized. Thus the work is cut out for the author. One one hand we will follow the story of the main players, starting with Euclid, going through Euler and Gauss to Riemann, the latter in many ways the most central and enigmatic of all the characters in this particular story, and then concluding with the moderns. The great names in mathematics, surely are worthy of being generally known, and there are standard ways of trying to make them come out live, such as the regurgitation of the well-known anecdotes. (And the author certainly serves as many of them as he can reasonably impose.) On the other hand there are the opportunities for digressions and for minor characters to step forwards, as well as tidbits and occasions for some mathematical enlightenment on the side (you better take advantage of the attention you are temporarily holding). Yet, at times some digressions are felt like paste-ons. It is of course legitimate to introduce Julia Robinson, but when a big thing is made of her being female (a politically correct thing which is very hard to resist especially for male authors) and that consequently there is a quick review of other women in mathematics, be they rather unrelated to the main story (such as Emma Noether), one feels that the author is rambling. It is also somewhat annoying to encounter repetitions, which are not acknowledged as such, creating in the reader the impression that the author is distracted, not fully in control of the material. It is definitely not wrong to repeat oneself in a book, as long as a reader is reassured that it is a deliberate intention on the part of the author. On the contrary repetition is an invaluable didactic device, and in order to appreciate something it has to be shown in many different contexts.

Yet the crucial question is the one of priority. A book like this cannot be exhaustive, in fact it better not be, yet there are omissions that many professional readers might regret, and given the rather large amount of marginal mathematical tidbits served, surely some room could have been made at the expense of the irrelevant?

So what are the basic things that have to be presented? A definition of the primes, and a list of the first few. Furthermore that whether a number n is a prime or not can be straightforwardly checked by division by a finite number of candidates (obviously those less than n, and almost as obviously those less than $\sqrt{n}$). This is of course not a practical procedure for big n and thus individual examples of big primes are rather fascinating[4]. Then everyone needs to understand the sieve of Eratosthenes and do their own for primes say up to a hundred at least once in their lives[5] This will induce in a reader the notion that if the primes thin out too rapidly, becoming a prime becomes less exacting, and hence

[4] This ties up with the difficulty of factorization and the production of large primes with important business applications, which will play a big role later in the book.

[5] I recall doing such a sieve by hand for an interval above 10'000 in the late seventies while at Columbia. I used different color strokes and markings and thus got even more information. This rather inane occupation afforded me great pleasure at the time. I felt that the primes I unearthed were 'mine'. The satisfaction is close to that of bicycling, getting non-trivial distances on your own locomotion.

there will be a surplus. Thus the distribution of primes have to strike some balance, which however is very hard to make precise. The idea can actually be made to show that there must be an infinite number of primes, because if not, the quotient $\Pi_{n=1}^{N}(1-1/p_n)$ of numbers in an interval of length $\Pi_{n=1}^{N}p_n$ not struck out is strictly positive. Another way of putting it would be that if the primes are too sparse, there will not be enough around for every number to be a product of primes. The fact if the primes appear too often, there will be too many products of primes, so by the pigeon principle unique factorization will be violated. This however is harder to make precise. Now I would say that the proof above for the infinity of primes is superior to that of Euclid's proof. Why? It is in a sense more complicated, involves more steps, and lack the beauty and inevitability of Euclid. But contrary to myth, mathematicians do not always go for beauty, there is indeed space for ugliness in mathematics, pace Hardy. The problem with Euclid's proof is that it is too elegant, too slick, and must strike many people as a trick they would never have thought of themselves[6]. The above proof is more natural as it exploits a line of thought that must appear to everyone contemplating the sieve[7]. It also leads to the next guess that this proportion of unstruck numbers should go to zero, or what is equivalent, that the sum of the reciprocals of primes should diverge. But it is far from clear how to get this[8]. Then Euler steps in with a beautiful identity, using the unique factorization of primes. Namely that the infinite product of the infinite sums of inverted prime powers is equal to the infinite sum of inverted numbers. Or more precisely

$$\Pi_p(1+1/p+1/p^2+1/p^3+\ldots)=1+1/2+1/3+\ldots+1/n+\ldots$$

which can easily be simplified as

$$\Pi_p(p/p-1)=\sum_n 1/n$$

The so called Euler products. As the harmonic series diverges, this means that taking the inverted values, the product $\Pi_p(1-1/p)$ diverge indeed to zero. Now variations of Euler products were used by Dirichlet to prove that certain subsets of primes (namely those congruent to $a(b)$ with a,b relatively prime) similarly diverge, a fact also brought up by the author, although a bit peripheral, if instructive, to the main story. Now the dramatic climax of the story depends on two strands which are beautifully and tantalizingly combined.

First by doing the sieve the reader appreciates that the striking out of numbers becomes more and more chaotic the further on one goes. One needs to keep in mind the regular marching of a great (and growing) number of primes simultaneously. It is like a

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[6] There is also some subtlety to the proof of Euclid. It does not explicitly produce a new prime, which confused me when I as a child first encountered it. I wondered why it did not produce prime-twins, by considering subtracting one as well. Also, a trivial and incidental observation is that it does not even use unique factorization

[7] Or would that just be the prerogative of a mathematical mind?

[8] A clever and elementary proof of that fact is to be found in the first pages of Hardy & Wright

juggler, who has no problem keeping three balls in the air, but thousands and thousands prove far too much of a challenge. Thus for all intents and purposes the emergence of a prime is a chance event, the chance of it growing slimmer and slimmer the further along we go, as there are more and more primes to contend with. Thus we have the spectacle of a completely deterministic process that we nevertheless are experiencing as random and unpredictable. The primes exists 'out there' whether we want them or not, and their distribution is subtle beyond our understanding, not unlike a living entity with a mysterious will of its own. Just imagine instead that we had discovered the primes, not just 'invented' them, and we would after a long arduous process have discovered the simple algorithm that generate them. Would we then have decided that we had solved the riddle of primes by reducing it to a simple algorithm[9]? Would we not have seen it as a kind of materialistic explanation that somehow cheapened them? Just as we would be aghast at the discovery that our consciousness, not to say our soul, would be 'explained' by some simple algorithm involving configurations of material particles. Supposedly to 'God' the primes present to mystery, he has no problem keeping them all in mind; so the mystery of the primes is due to our own limited capacity. A problem of epistemology not ontology.

This of course ties up with the philosophical interpretation of mathematics. Mathematics is about objects that although mental exist independently of us, it concerns inescapable facts we have to live with. We are battling a hard unforgiving reality impervious to our wishes. On the other hand mathematics is not about facts, it is about understanding; and understanding is a human prerogative, not at all existing outside a human context. A mathematician is like a painter, painting a scenery. The scenery exists outside independent of the painter, the highly individual painting on the other hand, is that of the painter. Those two things are separate but of course intimately related, a fact that has sown much confusion in the philosophical discussion on the Platonic character of mathematics. It is with this in mind we should understand the child Gauss counting primes and coming up with an approximate formula for their density. Primes being almost physical objects, at least existing as numbers in a list, just a vanishingly tiny part of what we already understand to be an infinite list. Gauss came up with a nice approximation, namely that a number N has probability $1/\log(N)$ of being a prime. Thus one expects that the number of primes $\pi(N)$ below N should be approximated by the function $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}$, or equivalently that you expect the nth prime to be of the size $n\log(n)$. This is empirical experimental mathematics. In mathematical jargon it gives evidence. By treating the primes as actual numbers in a list, we make it all physical. Primes although generated by us (be it directly by hand, or by programming an artificial hand, i.e. a computer) exist outside us just like other physical phenomena, to be counted and measured. Now this approximation $\mathrm{Li}(x)$ is of course not exact, it cannot be, as $\pi(x)$ is a discontinuous integer valued function and the former is obviously not, but the contention is that the error percentage wise goes to zero, i.e. that the quotient $\pi(x)/\mathrm{Li}(x)$ goes to one. This is nowadays known as the Prime-number theorem, conjectured by Gauss, but not proved by him; that would take another century after it was first suggested.

The second strand is the next giant step, and in fact the pinnacle of the story, as well

[9] One is reminded of the Darwinian explanation of evolution, which in the words of Dawkins and Dennett reduces to an algorithm

as the dramatic climax of the book (or ought to be). It consists in the introduction of the zeta-function

$$\zeta(x) = \sum_n 1/n^x$$

by Riemann. Clearly this is an elaboration of the Euler approach, and by knowing this sum for every x encodes the knowledge of every prime, although it is of course exceedingly subtle how to extract that knowledge from the zeta-function. The stroke of genius is to consider this not only for real x but complex arguments z. The formula will only make sense for z with real part > 1 but it defines an analytic function that makes sense for a much larger domain. In technical language we can analytically extend the function to a meromorphic one with a pole at $z = 1$ and enjoying a symmetry around the line with real part equal to a half, the so called critical line.

Analytic functions were discovered by the early 19th century, or at least systematically explored and understood. Euler, of course, was already very familiar with such functions as e^z and the trigonometric ones with complex arguments in the 18th century. When a student encounters analytic functions he for the first time encounters pure magic in his mathematical education. The morale of analytic functions is that formalism makes much more sense than was ever intended. A trigonometric function defined in the usual way does not make sense for complex arguments. Of course we can always force some extension, but such an extension would be more in the nature of a social convention, a kind of whim. The miracle is that there is a natural extension, an extension that is not invented but so to speak forced upon us. To compute the values of a trigonometric function we employ infinite series, but those make as much sense for complex arguments as real, and the entity so defined throws much light on the behavior of ordinary trigonometric functions, as well as many other things. This is an example of a discovery in the Platonic realm, the kind of discovery that convinces the working mathematician that there is something 'out there' of which he is completely innocent and on which he just have stumbled. So in particular even if the formula for the zeta-function, its ostensible definition, only makes limited sense, the function it defines, makes sense beyond the formula. This is why it is meaningful to claim that suitably interpreted the sum of all positive integers is equal to $\frac{-1}{12}$ ([10]!) A fact proclaimed by Ramanujam and dismissed as the ravings of a madman by a less perspicuous mind.

So what Riemann did was to encode the prime-numbers into an analytic function, and through that very function derive an exact formula for $\pi(x)$ involving an infinite sum, each term corresponding to a non-trivial zero[11]. By knowing the zeroes of the function very precise information can be gotten as to the distribution of primes. The author likes to use a musical metaphor (hence the formulation of the title) of likening the zeta-function to an orchestra, each zero being a player, playing its particular tone at a particular amplitude. The essence of the Riemann Hypothesis in this parable, is that the orchestra is playing harmoniously, no player is allowed to play louder than the others, and thus throw the whole

[10] The interpretation is that of evaluating the zeta-function formally at -1

[11] The functional equation for the zeta-function involves a factor that introduces additional zeroes at the even negative integers

thing off. In musical terminology, the tone of each player is determined by the imaginary part of the zero, and the loudness (amplitude of the waves) by the closeness to the critical line. The further away, the louder. Only if all zeroes line up on the critical line, is the error from the Li-approximation given by the square root of N. Riemann's contention was that the zeroes did indeed line up, all their real parts were 1/2. A contention partly underscored by actual computations.

Now the author is forced to get all of this above to an ignorant audience. How does he do it? He does not allow himself a precise statement or an explicit formula. Such things the public would not understand, on the contrary would scare most potential readers away. On the other hand what damage could such an explicit formula do? It would not take much space, and it would not be deprived of precedents. The author does indeed present some formulas, such as the explicit formula for the number of partitions, a theme actually irrelevant to the main story. Even if people in general would not understand it in, they might be intrigued by it. So what he does is to water down the technicalities and replace them with some purple passages. An expert knows the formula anyway, a fellow mathematician who does not, might get a tantalizing suggestion of what is at stake, but be unable to make it very precise. (But such a reader could easily google anyway.) But the general reader what does he or she get out of it? This is of course impossible for a mathematician to guess, except that such a reader is bound to get less than the mathematician. If the letter does not understand, a good bet is that the layman understands even less. But maybe he or she does not understand even this fact, maybe they are willingly being bamboozled? The general reader may get the impression that mathematics is deep and is some kind of celestial music. Maybe this is not such a bad thing after all, although the mathematician may regret such a fanciful impression devoid of any mathematical understanding.

The year is 1859. Riemann has now set the stage, he has made a revolutionary connection between elementary number theory and complex analysis, two disciplines of mathematics which seem to have nothing to do with each other. He has put forward a hypothesis of what appears and should be true, namely that the (non-trivial) zeroes of the zeta-function should have real part $\frac{1}{2}$. A relatively simple statement that can be put across to the general public[12] without necessarily conveying its significance. What remains is to prove it.

What is meant by proof? The notion is used in extra-mathematical contexts, such as in courts. Faced in a court proceeding you may have the frustrating task of proving something you know is true, because you have experienced it yourself. But why should people believe your testimony? So you are challenged to provide objective arguments (such as documents) to convince outsiders who are deprived of your special insight. (Sometimes you may even doubt your own memory and crave something more tangible.) If you are convinced of your truth, you certainly find objective proof to be a pedantic appendage which has no bearing on the truth at all. Then there is the notion of truth in science. It is based on induction as opposed to deduction, as well as a belief in some general principles[13]. A scientific theory cannot be proved beyond all doubt, all we can do, according to Popper,

[12] such as my mother

[13] highlighted for the first time systematically by Hume who cast aspersions on such notions as cause and effect

is to test it, not to confirm but to give provisional reassurance. Theories can never be proved as opposed to disproved. But in mathematics surely things can be proved for all eternity? Mathematics being an ideal occupation, far above the uncertainties of everyday life on whose matters we can, according to Plato, only have opinions. The notion of a Platonic truth is a matter of faith for most working mathematicians, and the way this faith is grounded is usually through formalism. A proof is seen as a verification, a series of iron-clad arguments leading, often along a tortuous route from A (the hypothesis) to B (the desired result). However, this is usually a travesty of how mathematics is done. It is true, however, that much of proofs turn out to be in the nature of verifications, a higher kind of calculation, in which every operation is locally understood, but throws no light on the whole; and where the end result is what matters, just as in a calculation when after the successful completion the intermediate steps can be thrown away. This is why it is often very painful to listen to a proof in a lecture, unless you are extremely motivated. This is understood by most lecturers, who prefer to just 'get the ideas' across and wave their hands (and arms). The problem is of course that if there is a devil he is found in the details. It is quite another thing to string those arguments together in a chain. If your result hinges on a certain routine calculation, to perform it could be very exciting, akin to watching a game, with a vested stalk in its outcome. Now proof is often more than mere verification, it is also illumination. A good proof should be centered around a few beautiful ideas that fit together in a surprising yet as it will turn out inevitable way. Not all proofs are like that, but every mathematicians know a few, which once grasped can never be forgotten. In general proofs combine both features, new insights, new angles, as well as a tedious verifications of details. And it is the humbling experience of mathematicians that if there is an unbridgeable gap somewhere, it is for a very good reason. What you might wish to be true, might in a later more encompassing context turn out to be necessarily false. Such experiences, probably more than anything else, fix the working mathematician in his conviction of an outside Platonic reality of mathematics.

At this point it might be worthwhile to insert a digression. Proofs are not always afterthoughts, verifications of guesses, but proving is often a means of discovery (as the author actually acknowledges, likening them to travel reports, leaving the listeners to fill out the details). Thus you often prove something but it is not clear what you prove. The insight of your discovery can be formulated in many different ways. Thus knowing the statement of a theorem is not enough, a theorem is not a black box, or some new irreducible truth to be used in a more extensive edifice, although theorems are often thought of as stepping stones, to be used in chains of arguments. You do not become a mathematician by memorizing theorems. Mathematics is not a matter of knowledge. The most important things in mathematics are never written down, never even formulated, because in a sense they cannot be captured in words. (Although of course a clever aphorism can catch it, but only if the mind is already suitably prepared, otherwise it does not make sense.) Of course ignorance is usually a handicap in mathematics[14] but it is usually a minor one. So where does that secret knowledge reside? Somehow between the lines, and intelligence means exactly being able to read between lines. Paying attention to proofs, especially the details, may make you see between the cracks.

[14] although I think it was Mordell who claimed that it is often beneficial

Now as to the certainity of mathematical truths, there was great and disturbing progress in the first decades of the 20th century. The name of Gödel cannot be evaded, and the author brings him up trying to explain what it is all about. Now this is something that is almost metaphysical in nature and devoid of any real technicalities and thus very suitable, I believe, to get across to the general public, or at least to the intelligent layman. However, I think that the author fails here, or at least does not exploit fully the potential. Any discussion about technical presentations is bound to be quibbling, yet I feel that the authors omission of presenting the diagonal principle in some detail is a serious one, especially in view of the fact that the book contain so many other more frivolous digressions. It could hardly be that the author expects the reader to be already familiar with it, at least he gives no such indication. It is the diagonal principle which is the crucial ingredient both in Gödels incompleteness theorem and Turings undecidability. As the author decides to devote attention to them, it should be done well. Now usually Gödels result is split into two, one that states that you cannot prove the consistency of a formal system within itself, and that in every formal system there are true statements that are unprovable[15]. The underlying assumption beyond Gödels result is the metaphysical one that we can imagine infinite verifications[16]. An infinite verification is not a proof, it cannot be physically manifested, nor absorbed by a finite brain, but our belief is that it is in principle possible, thus leading to the possibility (indeed exemplified by Gödel) that something can be true, without we being able to find a snappy reason for it. Gödels result has in recent years attracted a lot of hype, it has also sent a mixed message to the mathematician. On one hand it goes against the optimism of Hilbert, whose program Gödel demolished, on the other hand it seems to indicate that the human mind transcends the materialism of mere algorithms and formalizations, that it has a special connection to the Platonic realm. This is the attitude taken most forcefully by Penrose.

What does this have to do with the Riemann hypothesis? Could it be that the task is ill-posed, that this Holy Grail of mathematics will forever be out of reach to us? If so it would be true, because would it be false, there would indeed be a disobedient zero, and its specification would constitute a finite proof of the falsity of the Hypothesis. (But maybe a proof so long that it cannot be contained in the universe, which would have disintegrated long before we even had gone halfway through. This scenario says something about the power of the Platonic assumption).

The bulk of the book is devoted to chartering the course in the 150 years that has followed upon the formulation of the Riemann Hypothesis[17]. It has generated much excitement and activity, probably more serious such than any other problem in mathematics, still in an absolute, as supposed to social sense, no progress has been made. The theorem will most likely still be open at its bicentennial, and when Hilbert will be reawaken by modern science five hundred years after his death, we will probably be forced to disap-

[15] The reason that one may want to conflate the first into the second is that the consistency of a formal system is surely a consequence of it, and that the example of an unprovable theorem in the second, is actually one that asserts its own unprovability.

[16] Thus in addition one needs to assume that the formal systems are powerful enough to encapsulate the integers, i.e. infinity, but anything else would surely have no interest to mathematics.

[17] This is literally true at the writing of the review, but not of the book itself.

point him and he might opt to go back to sleep for another five hundred years. It might even be the case that the proof will evade us for ever.

There has of course been some progress, more perhaps of a moral kind than a mathematical. The zeta-function has been useful, confirming the point of translating the theorem from the realms of the integers with very little structure (this is what makes it so hard) to that of complex functions, where there is an embarrassment of riches as far as structure is concerned. In 1896 Hadamard and de la Valléé Poussain managed to prove that no zeroes were to be found in the strip $0 \leq Re(z) \leq 1$ which implied an error estimate of a power of N strictly less than one, and hence implied the Prime Number theorem. (A fairly close approximation of the theorem was effected by Chebyshev using very elementary arguments which, however, did not seem to be powerful enough to go all the way, so when in the late 40's Selberg managed to find an elementary proof it made a stir[18] although it turned out to be something of a dead-end.)

Now in the beginning the resolution of the Riemann Hypothesis seemed a more or less forgone conclusion. Hilbert when he attached it to the list thought of it as one of the first problems to yield (provoking some sarcastic comments of Siegel). In fact it was considered by a British mathematician, somewhat out of touch with the cutting edge of mathematics as most of his colleges at the time, as a suitable thesis topic. It was given to Littlewood, who given little if any context, rediscovered its connection to prime number. With Hardy and Littlewood (irreverently compared to Laurel and Hardy by the author) the pursuit of the Riemann Hypothesis came into its earnest. First it was proved that an infinite number of zeroes were on the critical line, techniques and estimates improved by Selberg in the 40's (and presumably the real reason for awarding him the Fields medal?) so as to involve positive fractions, to be improved by Levinson in the 70's and finally Conrey in the 80's to about 40 percent. All of this is highly technical, and through the encoding via the Riemann zeta function, not based on any intimate acquaintance with primes at all. Thus someone like Andre Weil would not consider it to be number theory at all, but more a question of integral estimates. With the advent of the electronic computer (conveniently Turing was one of the first to dabble into this, giving the author a perfect excuse to bring him into the tale) the business of locating zeroes has taken off, and at the time of writing the first 10 trillion zeroes have been identified and verified to lie on the critical line[19] This seems to be overwhelming empirical evidence, but from a mathematical point of view it means nothing, we may not yet have entered the realm of typical zeroes, although the reach into the landscape of the Riemann zeta function that modern technology has allowed us is awesome compared to say what modern telescopes can penetrate into the Cosmos. (In fact the subject is notorious, the largest number that has appeared in a mathematical proof

[18] There ensued a controversy between Selberg and Erdös, who jumped on the bandwagon. Until recently the story has not surprisingly mainly been told from the perspective of the ambulatory Hungarian with his wide network, but recent unearthed material - Normat 56:1, has revealed the preeminence of Selberg.

[19] The author does not mention this, but because so far no multiple zero has turned up, the verification of lying on the critical line is simple, due to the symmetry of zeroes around it, but would there be a multiple zero, there would be, as I understand it, a serious computational problem, but maybe tractable because of lower estimates of how far from the critical line putative exceptions would have to be.

is that of Skewes, relating to when $\mathrm{Li}(x)$ first underestimates $\pi(x)$, a number beyond the intimate encounter by any physically conceivable computer in the universe as we know it.)

Circumstantial evidence, which really counts for mathematicians as opposed to mere numerical[20], is that during the 20th century more and more connections to number theory have been made, and thus the conjecture plays an even more central role, so many results having been proved modulo its truth. That the world is orderly and mathematical has been an element of faith without which progress in science, especially physical sciences, would not have been possible. But mathematics cannot rely on the subject being mathematical, our conceptions of beauty may be flawed (recall the Devil lurking in the details!). Furthermore there has been a proliferation of zeta functions and related complex functions based on discrete data in mathematics, for which suggestive analogues of the Riemann hypothesis can be stated and proved. The most striking example being the one suggested by Andre Weil and proved by Deligne using the machinery designed by Grothendieck. But of course no direct link between those results and the classical have so far been forged. The problem is once again that the integers have so little structure, everything blends into each other[21]. Another intriguing connection is given by the chaotic distribution of zeroes along the critical line being reminiscent of the distribution of eigenvalues for Hermitian matrices, and also occurring in physics. But it remains a tantalizing suggestion. It seems safe to say that a breakthrough would need more than refined technical mastery attacking the function as a complex function; this avenue seems to have been essentially exhausted[22]. What is indeed needed is to lift the problem onto a higher-level, to effect a second Riemannian revolution. Connes has stepped in and outlined a program involving non-commutative geometry. The author is too polite to say so, but clearly it seems to be a matter of a lot of hot air.

Now can this excitement be conveyed to the general public? As I noted earlier, probably only if the players of the game come across vividly enough. Now there are the old mathematicians long dead. We do not know very much about them, with the exception of Gauss no serious biographies have been written about them, and the few anecdotes we know tend to be used over and over again. Yet the task is of course not impossible. Then we have the contemporary mathematicians, and actually a majority of the people discussed are still alive and active today and thus approachable in the flesh[23]. Yet the way they come across is a bit too stereotyped, complying to the breezy manner you have come to expect from journalists. Maybe the author was encouraged to write so, and the many hackneyed phrases that peppers the text, may be due to the same sort of suggestion. This

[20] Although the psychological confidence of the latter should not be underestimated. A mathematician proving a theorem is rather pleased when he can get numerical confirmation whenever feasible

[21] The rings $F_p[t]$ are somewhat more complicated than the integers although very similar, with primes given by irreducible polynomials, but here it is rather easy to get formulas for the number of such, for each degree

[22] It is of course pretentious of an amateur to make such a pronouncement, yet the fact that Conreys estimate has not been improved in twenty years indicates a case of diminishing returns.

[23] I have actually met a fair proportion of them, once even renting Conreys house in Oklahoma for a year, the mathematical world being small. It is fair to assume that the author being a mathematician has the same experience.

is of course a matter of taste, but they grated on me for that very reason. Then as I have already remarked, there are many quibbles with technical details of the presentation, I do for example not believe that Gauss employed a formula that he proved in order to make his famous addition as a school-boy[24] it would be enough to observe the possibility of adding the first and last number, and then continuing, without thinking of a general formula. I also take exception to the story so often relayed without any independent thought that Gauss really believed that his small geodetic triangles might show up measurable deviations from the expected sum of their angles, unless of course he believed in local curvature. A unit the size of the radius of the Earth would in hyperbolic space result in dramatic parallaxes, of which Gauss surely was aware[25]. The story about Gauss and Hyperbolic geometry is of course too sweet to be rejected, but perhaps it ought to have been, irrelevant as it is to the main story. And here we come to a more serious objection, that of rambling. Admittedly rambling is a matter of taste, to digress, as any travel writer is not only invited to do but actually encouraged (and the book is in many ways written as a travel report, not only because of its frequent references to the mathematical landscape) and can often be charming, but too often I feel that the digressions of the author (such as when discussing the insanity of Grothendieck and Nash) are a bit too undisciplined, reflecting a desire to pack in everything he knows. On the topic of Grothendieck (if I will allow myself to ramble as well) I find his suggestion that Grothendieck laid new foundations on geometry in order to follow Gödels advice that new axioms had to be added to conquer the Riemann Hypothesis grossly misleading. Gödel was talking about new axioms, meaning new principles of inferring facts of the integers, Grothendieck was forging a new language. Besides I am rather doubtful that Grothendieck was seriously interested in the Riemann Hypothesis (as opposed to being gratified would it fortuitously drop out of his general project.).

This is the criticism, what about good things? I surely read the book with pleasure, not only picking up a thing or two I did not know before, but more to the point becoming fascinated enough to go to the real sources. This fascination may be due less to mathematics but the communal excitement depicted in the book. We are all social creatures being interested in what others are doing, and maybe even more concerned with others being interested in what we are doing ourselves, the two concerns intimately related. I also became moved to write a long review.

Finally any one solving the Riemann Hypothesis is liable to win a million dollars. This is paltry indeed, it is actually hard to think of more difficult ways of earning a million dollars. besides, given the connection with e-business, primes, and by extension the Riemann Hypothesis, are now commercially valuable, and although it might be hard to put a price on the hypothesis (but surely not as hard as proving it) those certainly would be worth much more than the prize money. But money apart, in this context who cares about money? More than any other mathematical achievement the proving of

[24] I believe that the task was somewhat more complicated than just adding the first hundred numbers but that might be pointless in view of Kehlmanns fictional biography in which he speculates that the story might be apocryphal.

[25] Lobachevsky certainly was, the lack of any measurable parallax at the time, indicated to him that the unit of length would have to be very large.

the Riemann hypothesis would have a tremendous social impact. Immortality would be ensured (at least for the foreseeable future) and compared to immortality, surely any heap of money dwindles to nothing.

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