letters to a young mathematician

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September 21-22, 2010

Hardy's 'A Mathematician's Apology' is a classic, and a gem at that. As far as being a statement on the role of mathematics in human culture, it is unparalleled, maybe because it is so easy to quote from. The edition I bought back in 1970 comes with a foreword by C.P.Snow. Not surprisingly, in view of Snows well-known cross-cultural ambitions, and indeed Snow takes his task seriously, his foreword is, as I recall, longer than the slim text of Hardy himself. Nevertheless, the book has been criticized, as being too haughty, and Hardy's avowed distaste for mathematical applications has been censured.

Ian Stewart, although paying homage to the brilliancy of the book, thinks it is out of fashion, and that there is a demand for an up-date. Maybe with a wink to Euler's letters to a German Princess, he presents his general views on mathematics and its practice, to an imaginary prospective mathematician, female of course, in the form of a sequence of letters. Does he succeed? And if so how does his book compare to that of Hardy's? To the first question one needs to take the intended audience into account. The tone of voice, especially in the beginning, is to address someone who is not only very ignorant of mathematics, but also very suspicious, and try to win her over. If that is to be taken literally, one wonders why one should try to entice somebody so reluctant into such a demanding field as mathematics. But just as in a novel, a logically flawed plot is irrelevant. Stewart may be politically correct in choosing a woman as his prospective mathematician, but he certainly is not politically correct when it comes to the role of innate talent in mathematics. This I think is one of the commendable features of the book, but one which he, for obvious reasons, only touches upon in the end, lest he scare away most of his potential readers.

Stewart dispenses with false modesty. He was always very good at mathematics, and he found out as the great majority of all our colleagues that he could do much better than his peers, and with a minimum of exertion too (if exertion is not too strong a word). In elementary mathematics, once you get the hang of the basic principles, everything follows more or less automatically[1]. Mathematicians, in general form their passions early, if needs be, with external promptings (often a parent or a teacher, and not seldom both)[2]. And being good at mathematics is a heady experience, it gives you a sense of intellectual power,

[1] A story I love to recall, is how I at the age of five or so, was piling hay onto a haystack together with my father. I asked him what 'trettio' plus 'trettio' was, and he replied 'sextio'. I knew that 'tre' plus 'tre' was 'sex', so in a flash I realized that it was all hanging together, and since then elementary arithmetic was a foregone conclusion. I retrospect I realize that what I did really discover was that just as you could count buttons and cows and other concrete objects, you could even count numbers themselves. And this is of course a proto-example of an important conceptualization that is prevalent in mathematics.

[2] As Stewart cautions. It is far easier to deflect a pupil with mathematical potential, than it is to empower the untalented. Thus it is more important that a teacher loves and understands mathematics, than possess more or less illusionary didactic skills.

which can lead to arrogance and (unrealistic) ambition. This ambition and the worry about your own ability can easily come in the way of the joy in mathematical discovery, during those early years when you discover an entirely new mathematical landscape[3]. Stewart does not have too much to say about this. Maybe his ambition never got in the way of his pursuit, testifying to the essential soundness of his approach, undeterred by neurosis. Still the competitive aspect of mathematical remains an essential ingredient throughout a career, as well as a forceful motivation. This the burgeoning mathematician knows in his bones, and thus needs no pointers as to the fascination of mathematics. When a mathematician loses this competitive urge, there is a real danger that she becomes complacent and degenerates into mediocrity, satisfied with churning out regular work repeatedly applying tools and techniques that has become a kind of route. Of course there may be real fascination and commitment that will allow surfing on the crest of curiosity for a lifetime, regardless of external reward, and this is of course how we all would like to look at our careers[4]. Yet obsession, except as to the very great, is no guarantee against being second-rate. Mathematics is cruel, on the other hand the society of mathematicians is made up by a very cheerful and tolerant group of people, and unlike other fields, your performance at say a talk is always met with politeness. And after all you should never forget, although it often is tempting to do so, that your colleagues, in spite of appearing dense and narrow, do possess mathematical talent to a degree unimaginable by the general population[5]. The book is meant to be up-beat, and thus it is hardly surprising that the author do not dwell on such depressing topics, although there are subtle hints dropped from time to time.

To change tack. What is the mathematical experience Really? One experience common to all mathematicians is the feeling that mathematics exists 'out there'. That it has an almost palpable existence, just as concrete and unavoidable as the chair you sit on, or the desk you work on. This notion is usually referred to as Platonism and considered philosophically naive. I believe that without that feeling, mathematicians would not bring up the emotional motivation to do mathematics, and incidentally Stewart is

[3] In my teens I was very competitive. Olympiads and taking advanced university exams while in high-school and all those things. I remember that after the mortification of failing an advanced exam, I started studying the subject matter again, and realizing how interesting it was for its own sake, regardless of my ability to master it. The whole thing did, however, repeat itself as graduate student at Harvard, when I spent the first three years brooding about my intrinsic ability, and only the final year, relaxing and writing my thesis.

[4] External rewards or not. There are of course always powerful internal ones. The beauty of mathematics, although not always present in talks of your esteemed colleagues, is a constant feature, as well as the psychological commitment you did in your early years. But beauty abstractedly enjoyed is too passive to be satisfied sustainably, you need, as Stewart rightly remarks, the repeated kicks you get when you suddenly understand something or crack an impasse in your problem-solving. Those adrenalin rushes are indeed as powerful as drugs, and possessing in addition the virtue of being deserved. But they need to be continually renewed, thus if you stop doing research, you get trapped into a vicious downwards spiral, into which it takes more and more effort to emerge from.

[5] Or is this just an illusion of mine. Could it be that some mathematicians without any distinct mathematical talent whatsoever achieve respectability in narrow fields, by dint of hard and disciplined work, and never being properly scrutinized? In later years I have started to wonder.

much impressed by the psychological finding that emotion is a prerequisite for rational thought. Philosophically naive or not, it is a testimony to 'common sense', and in my view a mathematician is just as justified in assuming the external existence of mathematics, as he is in assuming the independent existence of the physical world. In a sense, at least intellectually, his involvement with the former is more intimate than with the latter. (If I may make a concession to a stereo-typed caricature of a mathematician.). How many of us have not been frustrated by say an inequality going the wrong way, and praying for just one exception to save our argument. But mathematics is unrelenting, it admits of no such exceptions, it all hangs together like a seamless web, And indeed in the end you will be grateful that this inequality did not go your way, had it done so earlier, it would have spoiled and made impossible your final epiphany[6].

But of course 'common sense' provides little justification to a true philosopher (with the possible exemption of G.H.Moore), and Stewart in his digression touches upon some putative alternate philosophical explanations. One common one, is the Post-Modernistic one, namely that knowledge is but a social construct [7]. The most eloquent proponent of this is Reuben Hersh, who claims that mathematics is not essentially different from Law, Art and other human activities of the mind. That mathematics maybe objective from the point of view of the individual, but does not make sense without the context of human society. Money is a nice illustration of this. Money is based on a collective convention, which it is not in the power of the individual to flaunt. Similarly we can think of language. I would term this conception Jungian (in analogy with the 'collective unconsciousness'). Much as I agree and sympathize with Hersh, I still think that he addresses not the issue of mathematics per se, but the activity of doing mathematics. Clever as it is, it seems to ignore a deep mystery by simply postulating that it does not exist. I will return to this. As to formalism I would not say that it is opposed to Platonism, in a sense one can think of it as a complement, or even just an aspect. Hilbert is usually referred to as a formalist. It is true that he put formalism on a firm foundation, and by so doing made it vulnerable to a devastating attack. I suspect that Hilbert was in no way emotionally a formalist, his program had a very definite purpose, namely to once and for all do away with embarrassing contradictions, and move on to better things. Formalism is a way of making mathematics into something material (and indeed the computer is nothing but a way of turning the spirit into flesh), and I like to think of it as the analog of representing a picture by pixels. Great for various manipulation, but worthless for the human mind to understand[8]. Formalism reduces mathematics to a part of number theory, and hence as Gdel showed, incorporating reasoning of mathematics into mathematics itself. By allowing

[6] One is reminded of the theory that God is impeccable, the evil he apparently allows, turn out in the end to be for the larger good.

[7] Including 'Post-Modernism' itself. The author cannot resist making that jab against the naive self-referentiality of the pompous Post-Modernist. After all the statement 'There can be no truths' cannot be true, thus we conclude that there are truths, although we are unable to pin anyone down except the conclusion itself. This is an argument that incidentally goes back not only to Bolzano but even to St. Augustine.

[8] Just imagine given a typical digital image by a sequence of a few million numerical codings pixel by pixel. Does the picture 'pop' up in our minds?

the thought experiment of going through an infinite number of cases, it is no wonder that the human intuition trumps the finistic constraints of formal reasoning. Gdel was indeed a Platonist, and his most intriguing speculations were that we had not yet discovered the proper axioms and ways of reasoning when set-theory is concerned, but when we find them, we will recognize them immediately as natural, as if we had always known them. This does indeed tally well with Plato's suggestion that learning is just remembering what has been forgotten. An experience not uncommon to the mathematician.

Platonism comes in many flavors. The weakest form is simply that mathematics exists 'out there' and is in that sense not too far from formalism. Mathematical philosophers, who generally are rather ignorant of the variety of modern mathematics, tend to think of mathematics as giving a set of axioms, and mathematical activity a playing of the game. As with any invention or mechanical gadget, it comes with emergent features not designed. Even non-Platonists concede that[9]. A beautiful illustration of this is Evolution driven by Natural Selection. A stronger form of Platonism is that the historical development of mathematics is organic and natural and that the core concepts are not just human inventions, but part of the plan we are supposed to unfold. The notion of the natural integers is obviously such a core concept, and C.S.Peirce claimed that numbers are more basic than logic. And indeed, the attempts of Russell et all to derive mathematics from logic (just imagine the formal definition of say five initiated by Frege) seem to us as a glorious dead-end, especially when compared with Gdel basing logic on numbers. Popper on the other hand, claims that the integers and addition and multiplication may indeed be human inventions, but not the commutative and associative laws. Once again an example of a structure having unintended consequences for us to discover. The stronger form of Platonism is to a large extent boosted by the proverbial unreasonable effectiveness of mathematics to the physical world. Cosmologists tend to be unabashed Platonists, not to say strong Platonists, in their enthusiasm. In general one may argue that the applicability of mathematics is not so much a justification (such reasoning tend either to lead to circularity or infinite regress) as an illustration of its power.

One may think of mathematics as consisting of a central inhuman core, around which the fungus of mathematical activity is wrapped. Thus mathematical understanding is something that is not part of mathematics but of the mathematicians mind. Mathematical thoughts may be objective in the sense of being exportable from one mind to another, but as thoughts they do not exist outside humanity. Also most of the concepts a mathematician employs are inventions, or tools as Stewart would have it, and have thus no independent status. The inflexible core prevents us not from invention of tools and concepts, on the contrary, it provokes us to do so, but it sets restrictions on what we are allowed to do. As they say, facts kick back at you. It is this fungal growth that constitute mathematical activity, and from a social constructivist point of view, mathematics, and it is this growth that, in my opinion, Hersh refers to.

Philosophical speculations on mathematics in general run the risk of degenerating into harmless homilies. A more fruitful approach is to narrow the focus. One such is of course to discuss the nature of proof, to which the author devotes a letter. What is the purpose of proof? The naive attitude is of course that by proof we can determine what is right and

[9] Personal communication by Brian Davis

what is wrong. That proof is in principle infallible, and objective. As we all know, if two mathematicians argue, eventually one will concede the point of the other, and not take it personally. Unlike other disciplines winning an argument is not a matter of personality, the one that concedes does so because that is the way the world happens to be, and it is not up to the will of your opponent. Once you have understood your mistake, you internalize your understanding and by so doing you appropriate as yours, and thus the ownership of truth is shared. This is what is meant by rational understanding, you take possession of something, it becomes part of you[10]. Among mathematicians there is a remarkable consensus, and this consensus is not the result of a convention, it goes deeper than that. Once there are meta-issues about mathematics, what is useful, what is beautiful and what is important, not to mention matters such as intuitionism versus classical logic, there is the usual dissension and strife, mathematicians may be a cheerful and modest lot, but that might be a consequence of their common mathematical pursuit than a prerequisite for being a mathematician. Of course all of this ties up with the sense of palpable reality that mathematicians experience in their work.

Now how do you verify that a proof is correct? The standard idea is through formalism. You simply cut up the reasoning into its atomic parts, and check them one by one. Stewart refers to this as the structural approach. Sometimes, as he concedes, this can be useful, but this is not the way proofs are written and certainly not the way proofs are being understood. Understanding a proof is not a mechanical procedure, it has to do with assigning meaning. Thus if you want to explain something to someone, a not so bright student say, it simply does not help to add more and more arguments, taking smaller and smaller steps, on the contrary it is more apt to add additional confusion. A proof is a story, a narrative, and a proof well-understood is never forgotten, because there is a 'logic' to it, it hangs together not only on a local level, and that is why we compare it to narratives, which likewise have a global logic. Take an example such as Adam Smith 'The Theory of Moral Sentiments'. It contains lots of clever insights and striking arguments, but they are not structured into a narrative, they lead nowhere, and hence after reading it, you find yourself retaining almost nothing. A good piece of mathematics touches you and leaves an indelible mark. Proofs are there, ostensibly to justify and document, but their impeccable deductive nature is usually not enough to carry conviction. We speak about local understanding. Justifications are illuminating, if anything, and a proof is much more than the theorem it eventually leads up to. It is about tools, about other ideas, suggesting digressions and variations. A theorem does not encode a proof, in fact loosely speaking a theorem is just one out of many ways of articulating the insights provided by a proof. It is far more useful to a mathematician to master the methods employed than memorizing the results. And in fact it is in a sense useless to know that a certain thing is true if you do not understand why. Take Goldbachs conjecture. Imagine an infinite proof consisting of checking all the cases, this is indeed physically impossible[11] but possible to do as a thought experiment, making us believe that a result could be true without being provable. What

[10] This is why unintentional plagiarism sometimes occur among mathematicians. You are exposed to an argument, you take it in and in a sense it becomes yours, and you forget about its provenience, as being irrelevant.

[11] Although Stewart has a delightful little essay in which he imagines an infinite computer, making an

would such an infinite proof amount to? It would just be an oracle. It would satisfy the condition of verification, but hardly of illumination. And without the latter the former is of little interest. True, if on a less spectacular scale, large computations are also oracular in nature. An ordinary proof may rest on a large calculation, which similarly you have every reason to expect to be correct, but which provides no illumination whatsoever. One may suspect that such features will become more and more common in mathematics, and worry whether this will lead to an alienation, just as modern high technology alienates people, making them dependent on gadgets they do not understand at all[12].

And finally what makes a result convincing is how it fits with other results. The seamless web of mathematics is the ultimate source of conviction. It is of course a human conviction, and as such preliminary. Doing mathematics is not in principle different from doing other science, in the sense that theorems are true until falsified. If you want to check that a result is correct, it is not so efficient to check the details of the proof, as to go on forward reasoning and derive consequences. The experienced mathematician is here at an advantage, and his tacit conclusions can be thought of as acquired intuition. (Intuition is incidentally an intriguing subject in mathematics, which is however only fleetingly alluded to in the book.).

Otherwise the book is about giving advice. How to study mathematics, how to find your advisor, how to acquire a general mathematical culture, how to teach. Often sound advice, if not necessarily very original (which sound advice seldom is). As how to teach the author has not very much to say. This is natural, few people have. So apart from the homily of trying to put yourself in the shoes of the student, and the warning not to primarily amuse yourself when teaching (something I have found myself prey to) there is little to be said.

The book is written with a light touch, none of the bitterness and seriousness of a Hardy, and may as such be more accessible, but hardly outlive the latter, which for better or for worse has set much of the agenda of mathematics as a cultural phenomenon.

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infinite number of calculating step, still within a finite space and time, due to each component and each time interval decreasing geometrically.

[12] But that does not seem to bother people in our consumer oriented culture.