Random optimization problems

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1 Course description

This is a graduate course in mathematics intended for graduate students in mathematics and computer science. Random optimization problems is a field of research which is currently under rapid development. It lies in the intersection of probability theory, statistical physics, combinatorial optimization and computer science. Methods from all these fields are applied and often combined.

We will discuss topics as random k-SAT, (phase transitions, algorithms), and models of spin glasses, but we focus on the *mean field model of distance*, or equivalently, graphs with random costs associated to the edges.

To give an example of the sort of result that we are interested in, suppose that the $\binom{n}{2}$ edges of a complete graph on n vertices are assigned independent costs taken from uniform distribution on the interval $[0, 1]$. Consider the problem of finding a set of $k = \lfloor n/2 \rfloor$ edges of minimum total cost under the constraint that no two of them may have a vertex in common. Let M_n be the total cost of an optimal solution. Then as $n \to \infty$, M_n converges in probability to $\pi^2/12$.

This was conjectured by the physicists Marc Mézard and Giorgio Parisi in the mid-1980's based on the famous *replica method* of statistical mechanics, a method originating from the study of so-called *spin glasses.* It was settled by David Aldous in 2000. In the course, we develop tools that allow a relatively simple proof of this and related theorems.

While developing these tools, we encounter several topics of general interest which are often useful, especially in discrete mathematics and computer science:

- Properties of the exponential distribution and the Poisson process, and their use in problem-solving.
- Probabilistic methods in discrete mathematics.
- Expander properties of graphs.
- Algorithms and heuristics for combinatorial optimization.
- The Talagrand concentration inequality.
- Basic concepts of statistical mechanics and the replica-cavity ansatz.
- The Erdös-Rényi graph process and the giant component.

2 Organization

The course will begin with a lecture giving an overview of the field. This first lecture is intended to be of interest also for people who do not plan to participate in the course.

Then there will be meetings approximately once a week. At these meetings, the material is discussed and some literature is handed out. There will be home-work consisting of some exercises which are supposed to be doable in one week.

3 What the students are expected to do

The students attending the course are expected to

- Be present at a reasonable number of meetings.
- Give a short presentation of a paper/book-chapter/topic at one of the meetings.
- Solve a reasonable number of exercises.