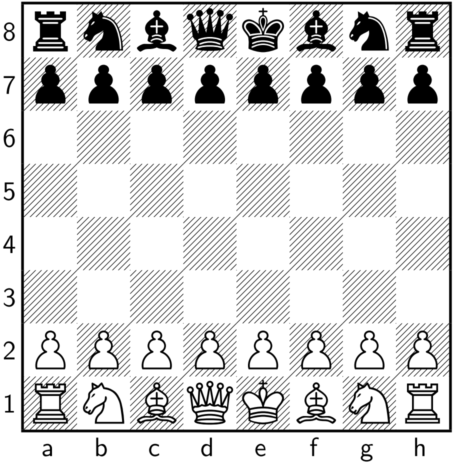
Games, optimization and phase transitions

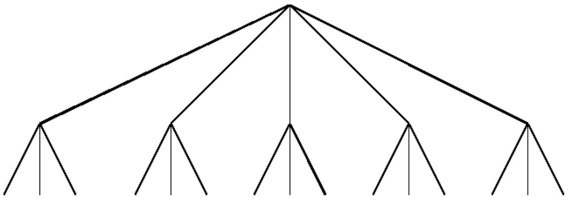
Johan Wästlund

Chalmers University of Technology

Two-person games

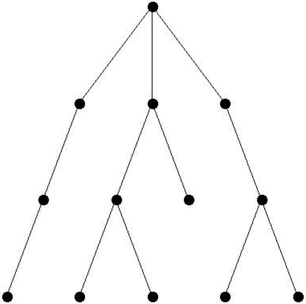


Computer's perception of the position



Random model: The Poisson Galton-Watson process

Random rooted tree. Each node has Po(λ)-distributed # children.



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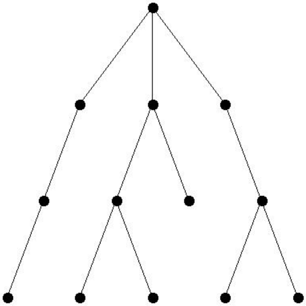

Convention: A player unable to move loses.

Replica Symmetric ansatz

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and

$$p = \frac{W(\lambda)}{\lambda}.$$

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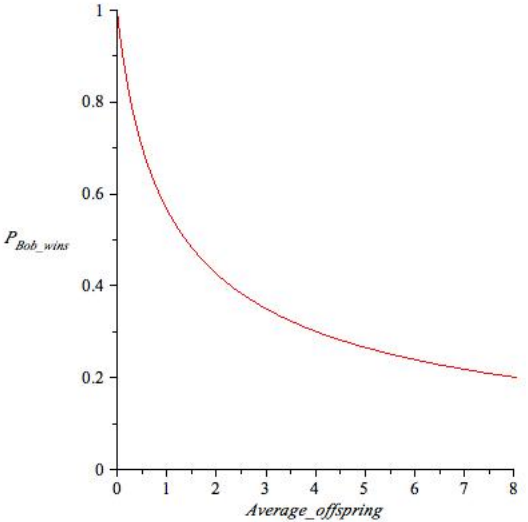
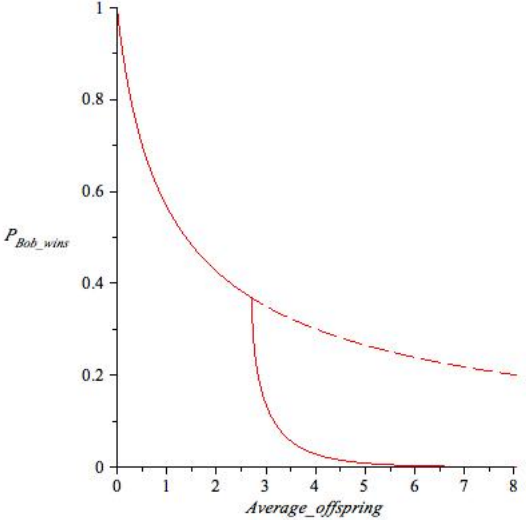


Truth



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What happened?

Truth

The "RS" solution $\lambda = \frac{-\log p}{p}$ is the *fixed-point* of the map

$$p \mapsto e^{-\lambda p}.$$

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But the truth about the game comes from *iterating* that map.

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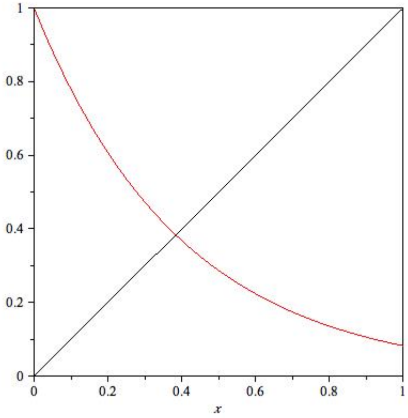
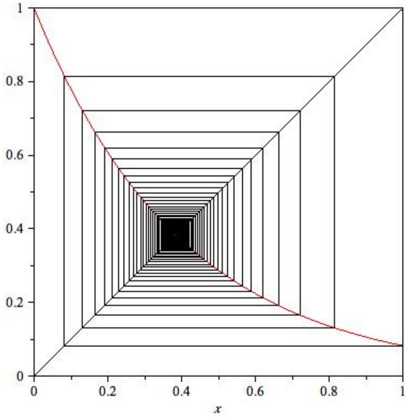


$$\lambda = 2.5.$$

Truth



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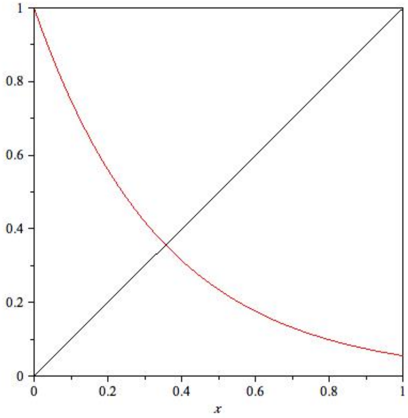

$$\lambda = 2.9.$$

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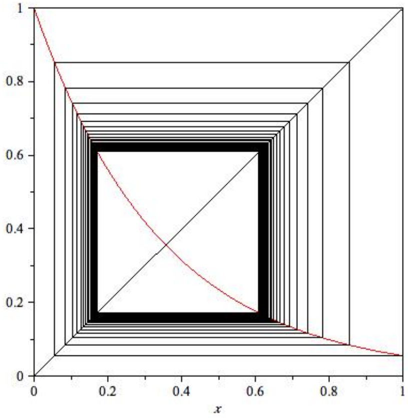

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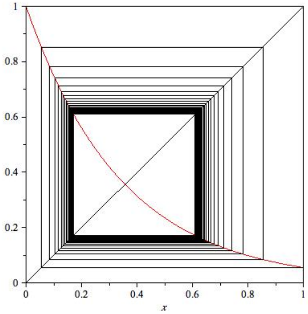
Truncated game

- The iterates show Bob's probability of winning if the tree is truncated after k moves.



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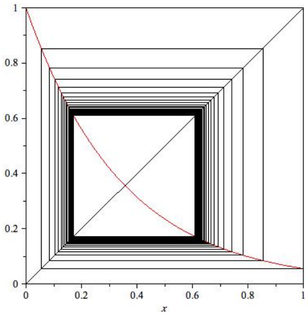

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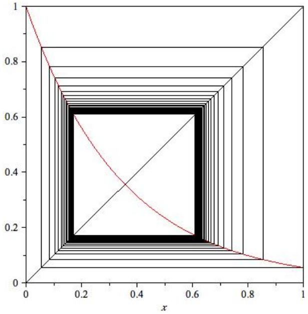
- Draw $\leftrightarrow$ influence of boundary conditions remains positive



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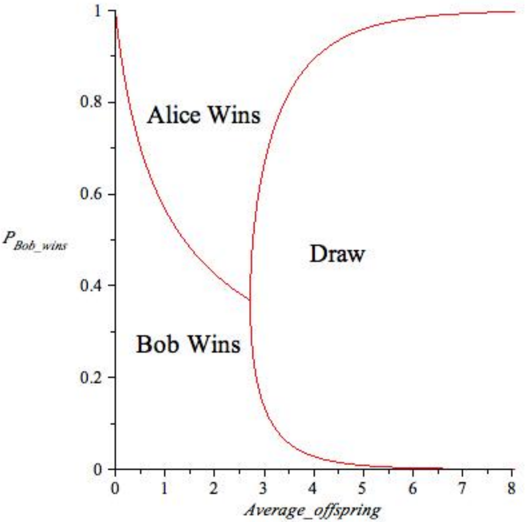
1
0.8
0.6
0.4
0.2
0
P_{Bob_wins}
Alice Wins
Draw
Bob Wins
0
1
2
3
4
5
6
7
8
Average_offspring

Geography

- Algorithmic Combinatorial Game Theory

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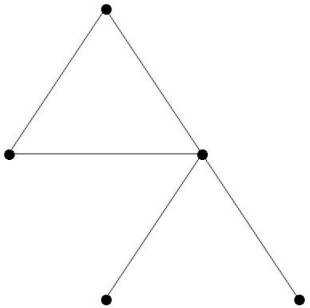
Undirected Vertex Geography

Theorem

On a finite graph, Alice wins if and only if every maximum size matching covers the starting point.



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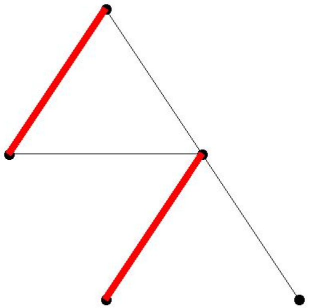

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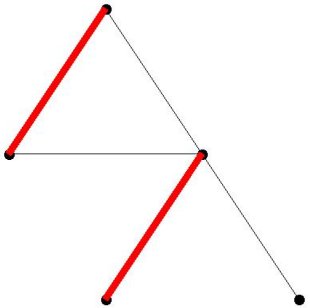

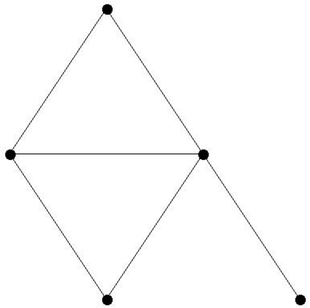

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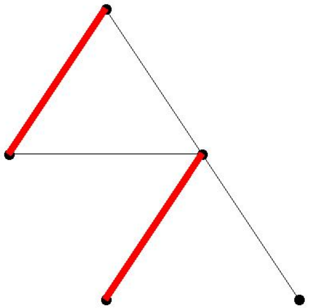

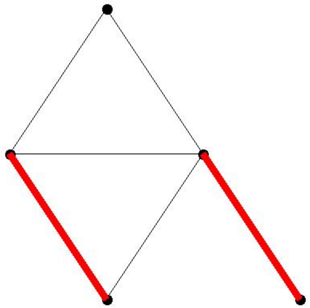

Erdös-Renyi random graph model

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- Each edge present with probability λ/N (average degree λ)
- Local weak limit: The Poisson Galton-Watson process (Poisson Bethe lattice)
- If $N >> \lambda^{2k}$, then the k-neighborhood of a random vertex is a tree (whp)

Max-size matchings

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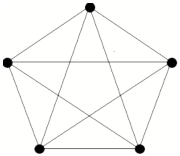
- Complete graph K_N with $\exp(N)$ edge-costs.



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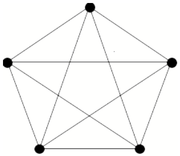

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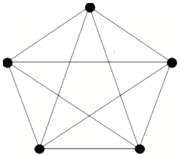
- Average cost per vertex $= \pi^2/12$ (Mézard-Parisi 1985, Aldous 2001)



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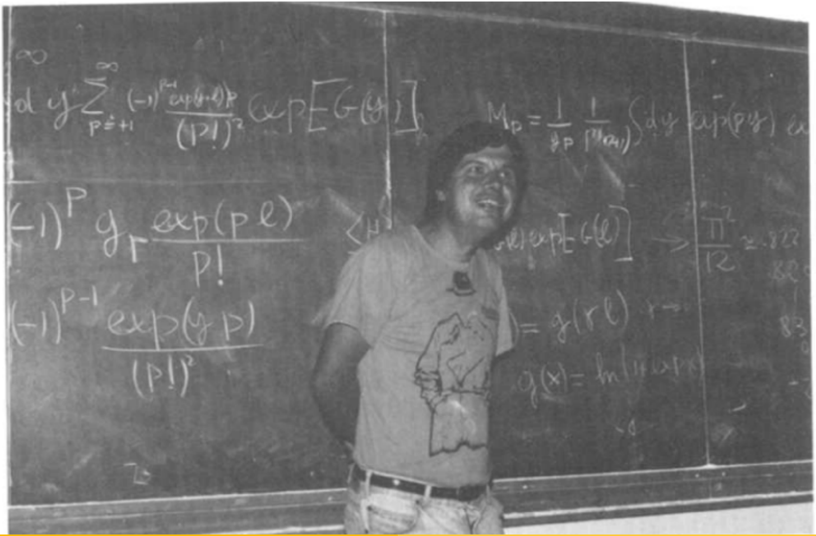
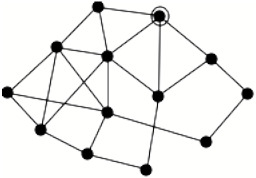


Graph Exploration

2-person zero-sum game:



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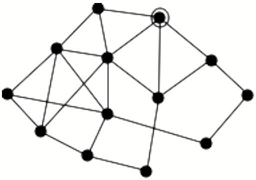

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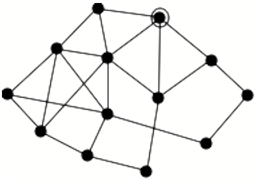

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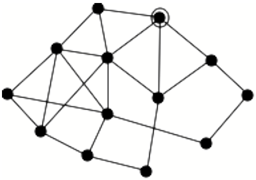

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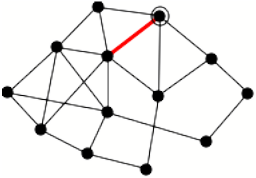

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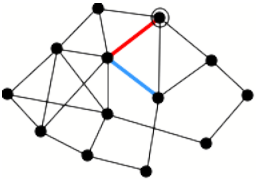

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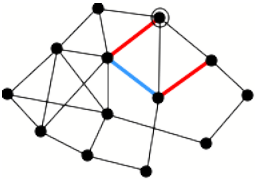

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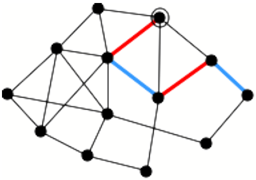

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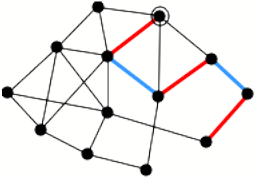

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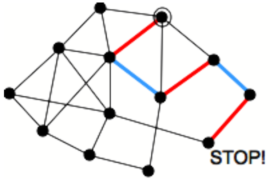

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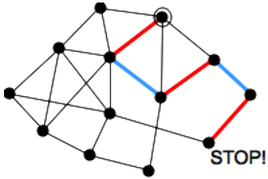
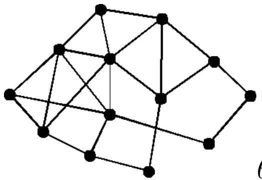
- Edges longer than λ are irrelevant!



Diluted Matching Problem

Optimization:



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Diluted Matching Problem

Optimization:

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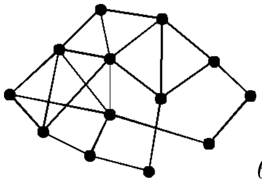

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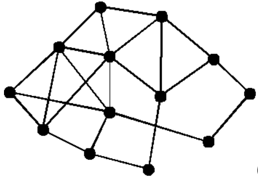

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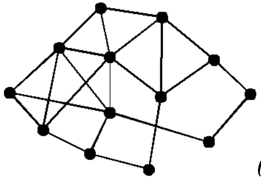
Diluted Matching Problem

Optimization:

- Partial matching
- Cost = total length of edges + $\lambda/2$ for each unmatched vertex
- Feasible solutions exist also for odd N



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Proof.

$$f(G, v) = \min(\lambda/2,\ l_i - f(G - v, v_i))$$

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$$M(G) - M(G - v) = \min(\lambda/2,\ l_i - (M(G - v) - M(G - v - v_i)))$$

Solution to Graph Exploration

Lemma

$$f(G, v) = M(G) - M(G - v)$$

Proof.

$$f(G, v) = \min(\lambda/2,\ l_i - f(G - v, v_i))$$

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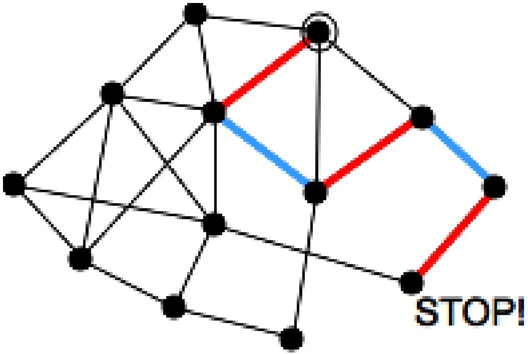
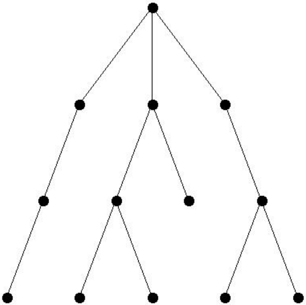
$f(G, v)$ and $M(G) - M(G - v)$ satisfy the same recursion. □

Solution to Graph Exploration



Tree approximation

Poisson-Bethe-Aldous-Galton-Watson-Erdös-Renyi-lattice/graph/process



Edge-costs from uniform distribution on $[0, \lambda]$

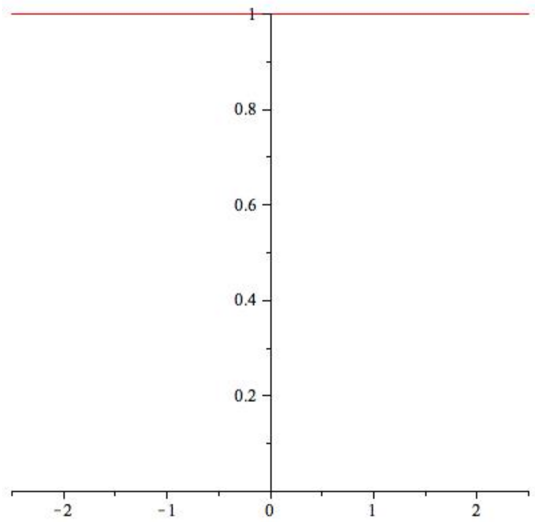
Payoff distribution

$F(x) = \mathrm{P}(\text{Bob's payoff} \geq x)$ in the truncated game

$$F \mapsto \exp\left(-\int_{-x}^{\lambda/2} F(t)\,dt\right).$$

Payoff distribution

P(Bob's payoff $\geq x$) in the truncated game ($\lambda = 5$)



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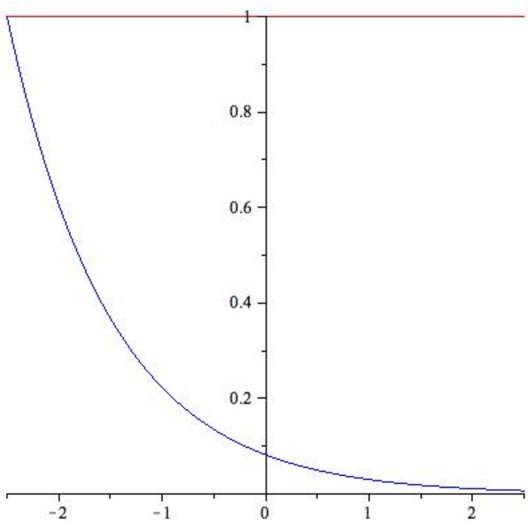

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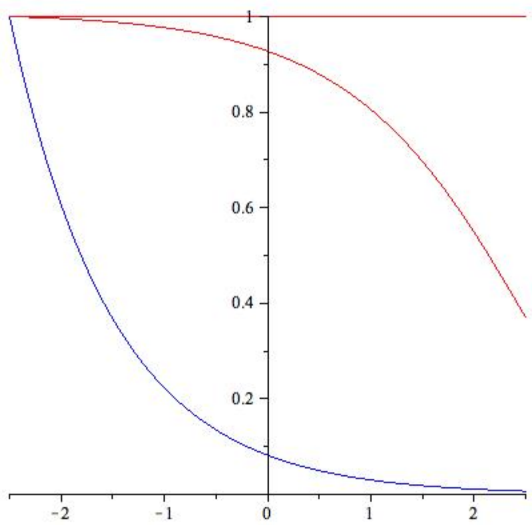

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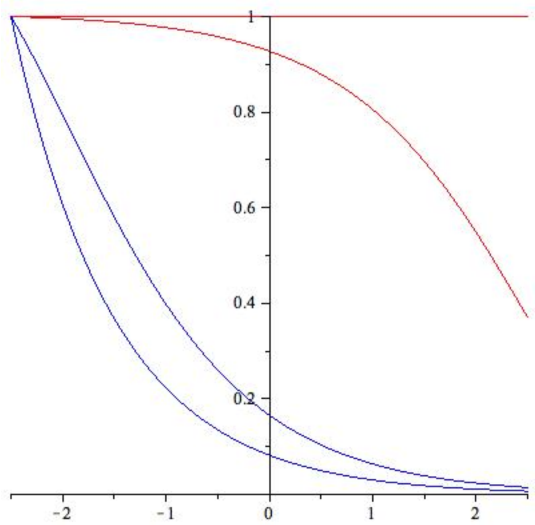

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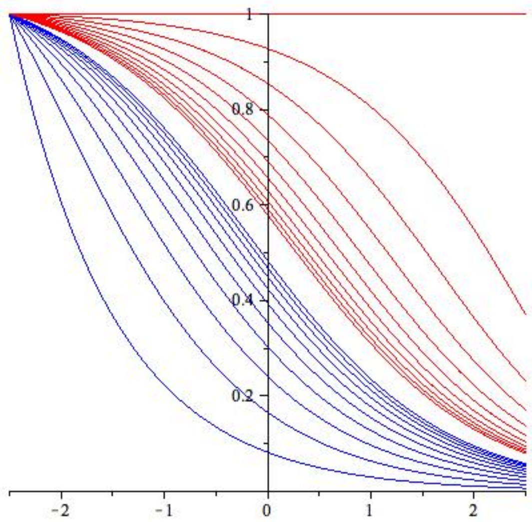

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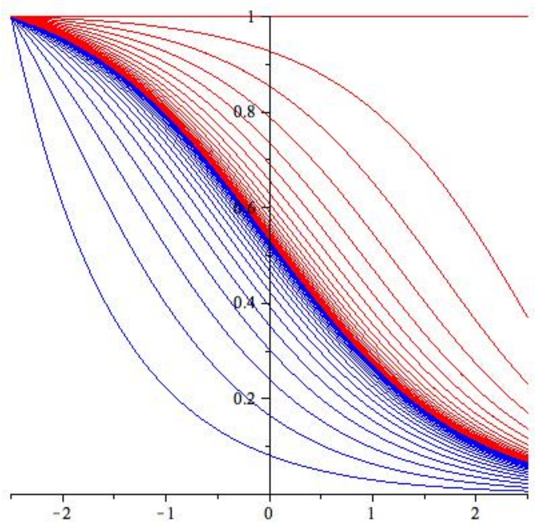
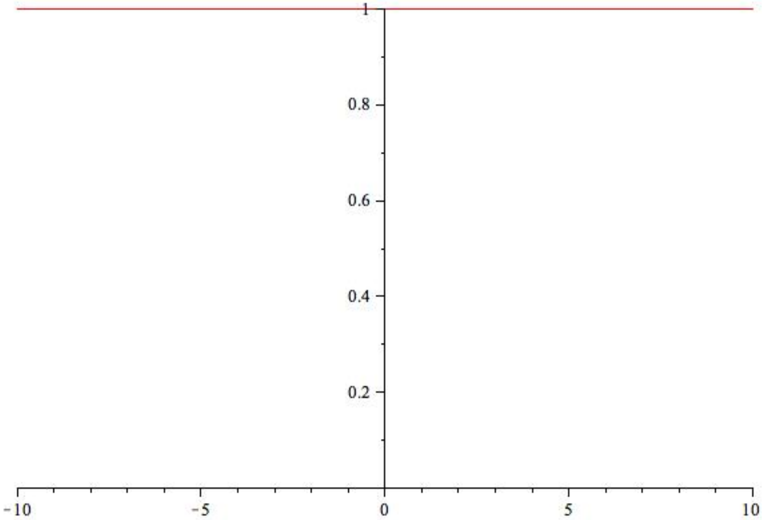


Payoff distribution

P(Bob's payoff $\geq x$), $\lambda = 20$.



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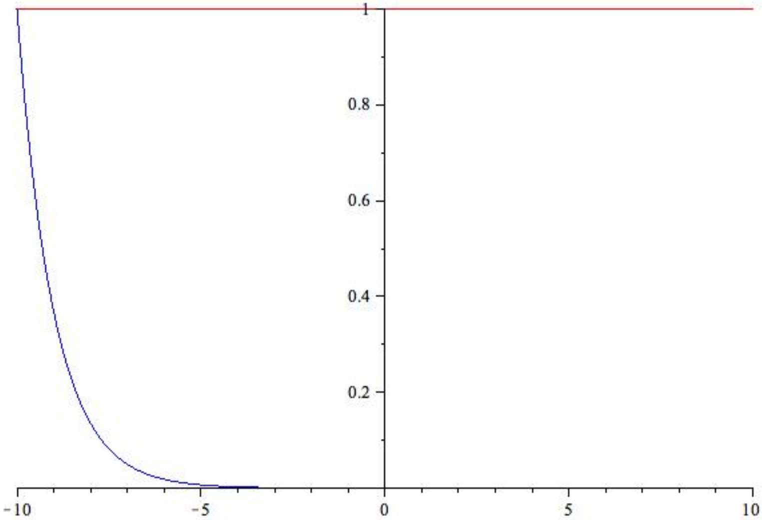

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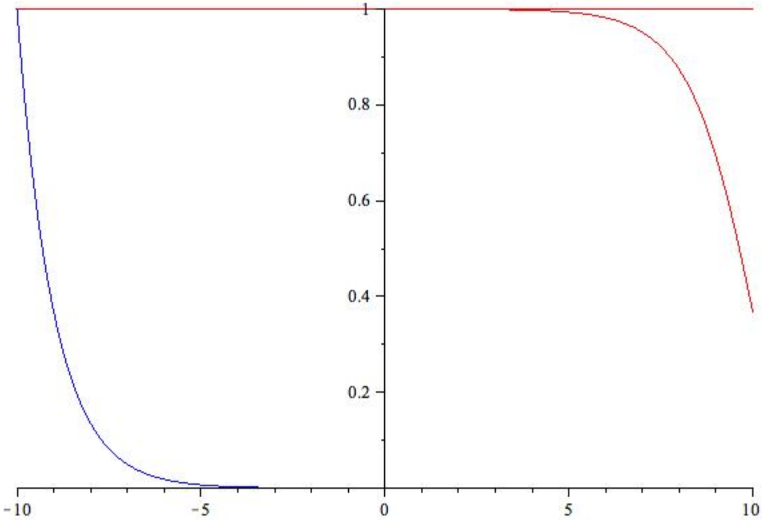

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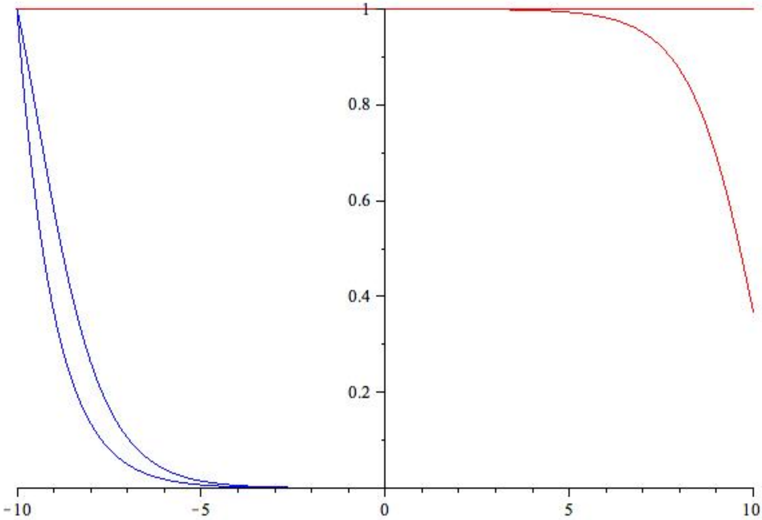

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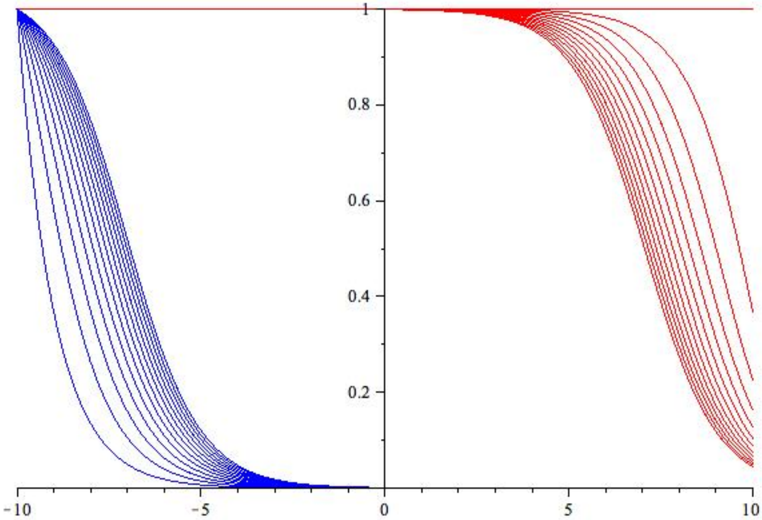

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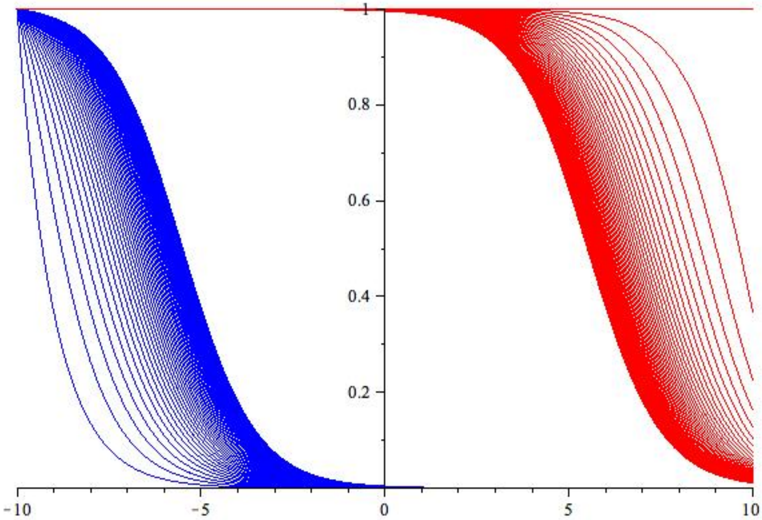

Convergence?

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Theorem

$$E\left|Payoff_{k+1} - Payoff_k\right| \leq \frac{\lambda e^{\lambda}}{k}.$$

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gives

$$F(x) = \frac{1+q}{1+e^{(1+q)x}},$$

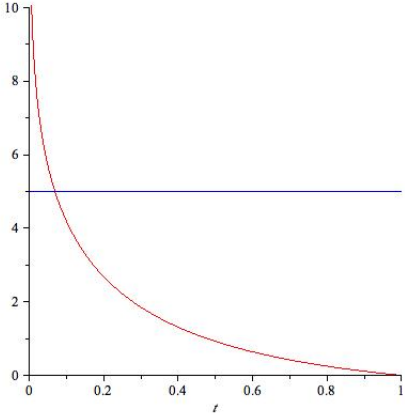
where

$$\lambda = \frac{-2\log q}{1+q}.$$

Cost of the diluted matching problem

- Average cost per vertex (from Alice's first move):

$$\int_0^1 \min\left(\lambda/2, \frac{-\log t}{1+t}\right)\, dt$$



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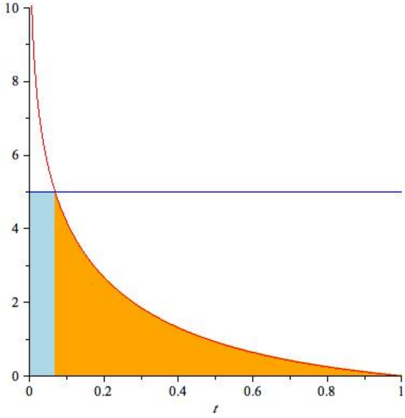


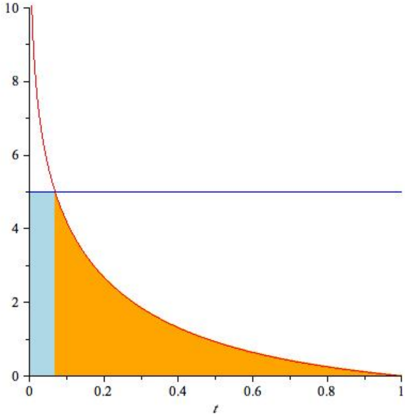
Cost of the diluted matching problem

- Average cost per vertex (from Alice's first move):

$$\int_0^1 \min\left(\lambda/2, \frac{-\log t}{1+t}\right)\, dt$$

- Limit cost as $\lambda \to \infty$:

$$\int_0^1 \frac{-\log t}{1+t}\, dt = \frac{\pi^2}{12}.$$



Proof of convergence, numerical values for limit costs

Problem	Limit cost	Pseudo-dim 2
Matching	$\pi^2/12 \approx 0.8224670336$	0.571759049 59888
TSP	2.04154818642	1.285153753372032
Edge Cover	$W(1) + \frac{1}{2}W(1)^2 \approx 0.7279690463$	0.55872