Reprinted from Multivariate Behavioral Research, October, 1983, Vol. 18, pp. 361-390

Decomposable Models: A New Look at Interdependence and Dependence Structures in Psychological Research

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Decomposable models represent interdependence structures for observable variables. Each model is fully characterized by a set of conditional independence restrictions, and can be visualized with an undirected as well as a special type of a directed graph. As a consequence each decomposable model can be interpreted either in terms of interdependencies only or as a particular kind of dependence structure, as a recursive system or path analysis model. Under the assumption of normally distributed variables, decomposable models determine the structure of correlation matrices, and maximum-likelihood estimates of these can be calculated with the help of ordinary least squares estimation. Using several examples from psychological research, we discuss the interpretation of decomposable models. Furthermore, it is demonstrated how recursive dependence structures can be specified with the help of decomposable models in a hypothesis generating (exploratory) as well as in a hypothesis testing (confirmatory) manner.

Since correlations were first calculated, there have been attempts to integrate single relationships into structures or overall models based on theoretical considerations. It is appropriate to speak of interdependence structures if changes in one variable can lead to changes in a whole set of relationships, and of dependence structures if some of the investigated variables are thought of as being dependent or response variables. The problem of defining adequate structures for a set of variables has remained important until today and a great many approaches exist.

It is our aim to describe, for applications in psychological research, one particular class of models for normally distributed variables, the members of which have been called decomposable or multiplicative models (Wermuth, 1980). This class can be regarded as the intersecting class of models for interdependence structures named covariance selection by Dempster (1972), and of models for dependency structures introduced as linear recursive equations by Wold (1954).

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One essential feature of decomposable models is that they may be used in confirmatory and in exploratory types of analyses. A particular model can be regarded as a hypothesis on a structure of relationships so that this hypothesis can be submitted to a statistical test, or a search procedure (Wermuth, 1976b, 1980) can be used to find a well-fitting decomposable model for a given set of data. In both cases a likelihood ratio test statistic can be employed as criterion to judge the assertion that a hypothesized model is supported by the observations.

Other aspects of decomposable models that make them attractive for applications are the following: (1) each model can be represented by a particular kind of an undirected as well as a directed graph, and it is interpretable in terms of (conditional) independence statements; (2) each model leads to a condensed description of the interdependence structure in the sense that the whole covariance (or correlation) matrix is estimated by using only parts of the observed matrix; (3) for each model the maximum likelihood estimate of the covariance matrix is expressible in closed-form with the help of ordinary least squares estimates.

Decomposable models are distinct from factor-analytic approaches (Jöreskog, 1970; Lawley, 1940; Spearman, 1904), because they represent structures only for observable variables. Since decomposable models form a subclass of linear recursive equations, they represent a subclass of the models for linear structural relationships. Decomposable models, though, provide new possibilities for interpretation and data reduction. The actual analysis of any given decomposable model in terms of computing the maximum likelihood estimates and a test statistic for its goodness-of-fit could be done with the help of a computer program such as LISREL (Jöreskog & Sörbom, 1978), but because of certain properties of decomposable models, this is not necessary. In fact, it has been shown (Wermuth, 1980) that the paper-and-pencil methods described by Wright (1923, 1934) for path analysis do not only lead to the maximum likelihood estimates for equation parameters but also to the maximum likelihood estimate for the correlation matrix — if the investigated structure is a decomposable model.

Before we present an overview on theory relating to covariance selection, linear recursive equations, and decomposable models, we first identify the models with the help of their graphical representations.

Graphical Representations of Special Interdependence and Dependence Structures

Description of Graphs for Unrestricted Models

In the following, we consider graphs with p points and, at most, one connecting line for each pair of points. Each point represents a variable, and a connecting line an interdependence or a dependence relationship. The p variables are assumed to follow a joint normal distribution so that interdependencies as well as dependencies can be expressed with the help of correlation coefficients.

A graph with p points is called *complete* if it has exactly $\binom{p}{2}$ connecting lines, and *incomplete* if not. It is an *undirected* graph, if all connecting lines have no arrow, or equivalently all are two-headed arrows. It is a *directed* graph if at least one connecting line is a one-headed arrow.

Given these definitions, we can say that a *complete undirected* graph corresponds to an unrestricted interdependence structure of the p variables with a connecting line between points i and j representing the partial correlation of variables i and j given all the remaining p - 2 variables: $\rho_{ij \cdot \{1, \ldots, p\} \setminus \{i,j\}}$. It represents an unrestricted covariance selection model (Dempster, 1972).

A subgraph of $s < p$ points is obtained by deleting all $p - s$ points as well as all connecting lines to these points. Thus, a subgraph of s points can have at most $\binom{s}{2}$ connecting lines, in which case it is a complete graph.

Of the possible directed graphs, we consider only particular kinds of *complete directed* graphs which satisfy the following two conditions:

(1) there are $k < p$ points at which one-headed arrows are directed and these points can be numbered such that exactly $p - i$ arrows beginning at points $i + 1$ to p point at each $i \in \{1, \ldots, k\}$;

(2) the subgraph of the $p - k$ remaining points is undirected.

Such a graph corresponds to an unrestricted recursive dependence structure in k response variables, where each response i depends on all of the variables $i + 1$ to p but on none of the variables 1 to $i - 1$, and it represents uniquely what is known in econometrics (Goldberger, 1964) as a complete system of k linear recursive equations with uncorrelated errors. A one-headed arrow pointing from j to i denotes the partial dependence of variable i on j given all other variables that influence response i. This dependence may be measured by the partial correlation coefficient: $\rho_{ij \cdot \{i+1, \ldots, p\} \setminus \{j\}}$. A two-headed arrow between any two points s and t from the last $p - k$ points represents the partial correlation of variables s and t, given all of the remaining $p - k$ variables: $\rho_{st \cdot \{k+1, \ldots, p\} \setminus \{s,t\}}$.

It is known (e.g., Wermuth, 1980) that an unrestricted interdependence structure is equivalent to several of complete recursive systems that differ only in the number of the response variables. For four variables, Figure 1 shows all possible equivalent complete recursive systems that can be defined for a fixed ordering of the variables.

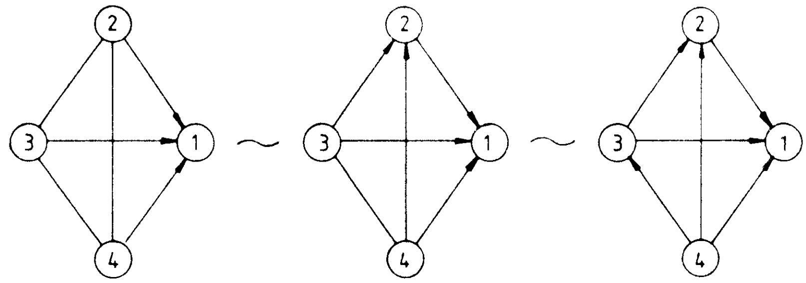

Figure 1.
Graphs of equivalent complete recursive systems.

Graphs for Covariance Selection Models

Members of the class of covariance selection models (Dempster, 1972) can be thought of as fully characterized by a complete undirected graph for normally distributed variables together with a set of restrictions on variable pairs: $I^A \subseteq \{ (i,j) \mid 1 \leq i < j \leq p\}$ such that $\rho_{ij \cdot \{1,\ldots,p\} \setminus \{i,j\}} = 0$ for each $(i,j) \in I^A$. The graph of a restricted model differs from he complete graph by missing connecting lines for all pairs $(i,j) \in I^A$.

The set I^A of restricted variable pairs in a covariance selection model is equivalent to a set of unrestricted subsets of variables, $\{N_t\} = \{N_1, \ldots, N_T\}$, that has been called the generating class of the model. The elements of $\{N_t\}$ separated by dashes have been used as (shortcut) notation for the model (Wermuth, 1980). It has been noted by Darroch, Lauritzen, and Speed (1980), in the context of equivalent models for qualitative variables, that the generating class can be read from the graph as the set of maximal complete subsets. A subset of points in an undirected graph is called *maximal complete* if the subgraph of these points is complete, and, if by including one more point, an incomplete subgraph results. The equivalence to I^A is then defined by $(r,s) \notin I^A \Longleftrightarrow$ there exists a $N_i \in \{N_t\}$ such that $\{r,s\} \subseteq N_i$. To give n example we use the graph of Figure 2.

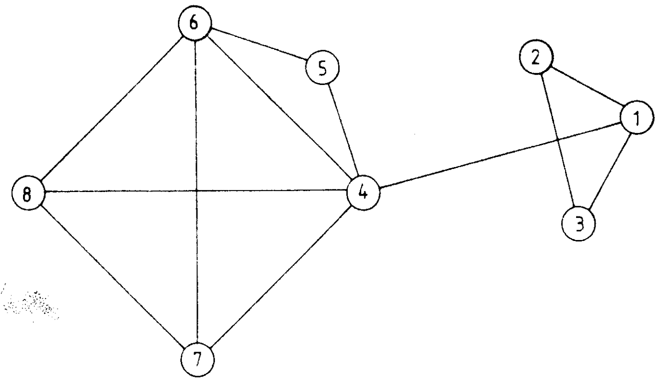

Figure 2.
Graph for a special covariance selection model.

This graph represents a covariance selection model with $I^A = \{(1,5), (1,6), (1,7), (1,8), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (5,7), (5,8)\}$, with $\{N_t\} = \{\{1,4\}, \{1,2,3\}, \{4,5,6\}, \{4,6,7,8\}\}$, and model notation 14/123/456/4678.

The interpretation of a covariance selection model is facilitated with the following result by Darroch, Lauritzen, and Speed (1980):

> If in the undirected graph for p variables two disjoint subsets of points A and B are separated by a subset D in the sense that all paths from A to B go through D, then the variables in A are conditionally independent from those in B given the variables in D.

For the model 14/123/456/4678 one obtains, for instance, that variables 2,3 are conditionally independent of variables 5,6,7,8 given variables 1 and 4; that variable 2 is conditionally independent of variable 4 given variable 1; that variable 5 is conditionally independent of 7,8 given variables 4 and 6, and so on.

Graphs for Incomplete Linear Recursive Equations

Systems of k incomplete linear recursive equations with uncorrelated errors (Goldberger, 1964) can be thought of as fully characterized by a complete recursive system in $k < p$ response variables (which gives an ordering of the responses such that variables $i \leq k$ may depend only on variables $j \in \{i+1, \ldots, p\}$ but not on variables $h \in \{1, \ldots, i-1\}$) and a set of restrictions $I^D = \{(i,j) \mid 1 \leq i < j \leq p$ and $i \leq k\}$ such that $\rho_{ij \cdot \{i+1, \ldots, p\} \setminus \{j\}} = 0$ for each $(i,j) \in I^D$. The graph of an incomplete recursive system results from the graph for a complete recursive system by leaving out the one-headed arrows for all pairs $(i,j) \in I^D$.

The set I^D of restricted variable pairs in a recursive system defines for each response variable i two subsets, A_i and B_i, of its potential influencing variables $\{i+1, \ldots, p\}$. The set A_i lists the variables on which response i actually depends: $A_i = \{j \mid j>i \text{ and } (i,j) \in I^D\}$, and B_i lists those on which it does not depend: $B_i = \{j \mid j>i \text{ and } (i,j) \in I^D\}$. Figure 3 gives an example of a recursive system with responses 1,2,3,4, and 5.

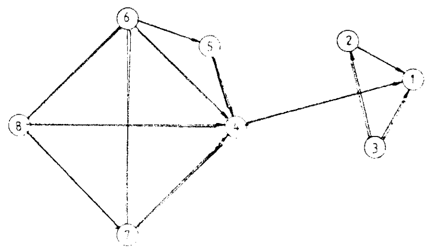

Figure 3.
Graph for a special recursive system.

The model represented by Figure 3 has the set of restricted variable pairs $I^D = \{ (1,5), (1,6), (1,7), (1,8), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (5,7), (5,8) \}$, and as sets of influencing variables, $A_1 = \{2,3,4\}$, $A_2 = \{3\}$, $A_3 = \emptyset$, $A_4 = \{5,6,7,8\}$, and $A_5 = \{6\}$. Though th set I^D is identical to I^A of the covariance selection model: 14/123/456/4678, represented by Figure 2, these two models are not equivalent. Such an equivalence can occur, however, for decomposable or multiplicative models, which are described in the next section.

Recursive systems can be interpreted in terms of conditional independencies as follows:

> Each response variable i is conditionally independent of the variables in B_i given the variables in A_i.

Thus, for the system in Figure 3, variable 2, for instance, is conditionally independent of variables 4,5,6,7,8 given variable 3.

Graphs for Decomposable Models

A method for recognizing a decomposable model from the incomplete undirected graph is to detect the so-called closed loops (Bishop, Fienberg, & Holland, 1975):

> An undirected graph characterizes a decomposable model if and only if this graph does not contain a subset of $r \geq 4$ points such that the graph of these points has exactly r connecting lines and each arbitrary starting point is reached with r lines.

With this result the graph of model 14/123/456/4678 in Figure 2 is recognized as one of a decomposable model, since it does not contain any subgraphs like those drawn in Figure 4.

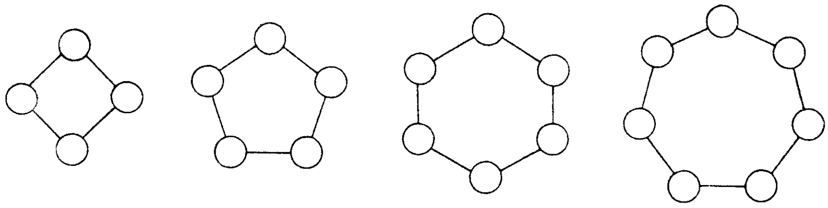

Figure 4.
Subgraphs that characterize non-decomposable models.

For each decomposable model—and only for these—the variables can be ordered so that the model can equivalently be formulated as an incomplete system of recursive equations (Wermuth, 1980).

If, for instance, the variables in Figure 2 (model 14/123/456/4678) are renumbered as for Figure 5, then the generating class becomes $\{N_i\} = \{\{3,6\}, \{1,2,3\}, \{4,6,7\}, \{5,6,7,8\}\}$, and the model notation becomes 123/36/467/5678. It is equivalent to the recursive system in Figure 6 with response variables 1,2,3,4 that have as sets of influencing variables $A_1 = \{2,3\}$, $A_2 = \{3\}$, $A_3 = \{6\}$, and $A_4 = \{6,7\}$.

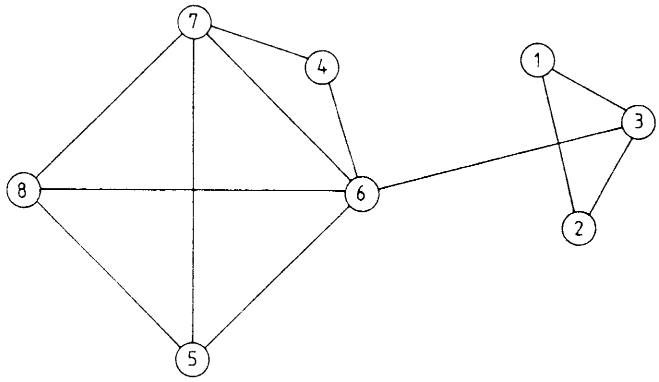

Figure 5.
Graph for the covariance selection model in Figure 2 with renumbered variables.

To determine from the graph of a recursive model whether it is equivalent to a decomposable model, the following result is useful.

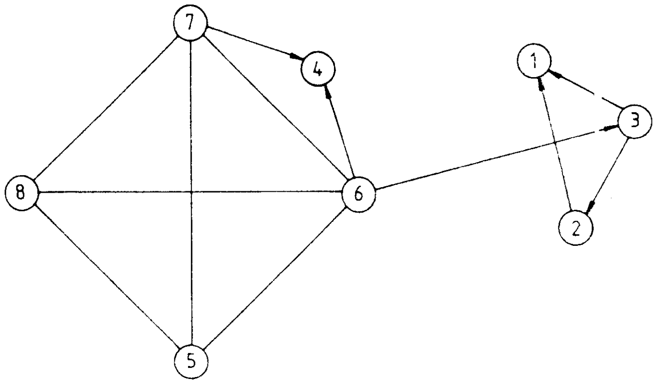

Figure 6.
Graph for the recursive system corresponding to the covariance selection model in Figure 5.

An incomplete recursive system in k responses is equivalent to a decomposable covariance selection model, if and only if in its graph the subgraph of each set of influencing variables, A_i, is complete.

This is just a reformulation of a necessary and sufficient condition given by Wermuth (1980) in terms of the set of restricted variable pairs. Another equivalent formulation is due to Kiiveri and Speed (1982): if the origin points of two one-headed arrows have no connecting line, then there is a restricted set of influencing variables in the system.

Using these results it is seen that the recursive system in Figure 6 is equivalent to a decomposable model, but the one in Figure 3 is not. This distinction does not only imply different possible interpretations of the model, but it also has consequences for test statistics and maximum likelihood estimates, to be discussed in the next sections.

Overview of the Theory for Decomposable Models

Covariance Selection

Covariance selection (Dempster, 1972) provides the theory for obtaining the maximum-likelihood estimate of the covariance matrix in a multivariate normal distribution, where for each $(i,j) \in I^A \subseteq \{(i,j) \mid 1 \leq i < j \leq p\}$ the concentration σ^{ij}, which is the element in position (i,j) of the inverse covariance matrix, is restricted to be zero. For positive definite matrices Σ, the restriction $\sigma^{ij} = 0$ is equivalent to the restriction $\rho_{ij \cdot \{1, \ldots, p\} \setminus \{i,j\}} = 0$ (Wermuth, 1976a), since

[1] $$\rho_{ij \cdot \{1, \ldots, p\} \setminus \{i,j\}} = - \frac{\sigma^{ij}}{(\sigma^{ii}\sigma^{jj})^{1/2}}$$

and since all diagonal elements (σ^{ii}) of the inverse of a positive definite matrix are nonzero.

As Dempster has shown, the estimate $\hat{\Sigma}$ always exists and is determined uniquely from certain elements s_{ij} of the observed covariance matrix S (with $s_{ij} = \Sigma_l (x_{li} - \bar{x}_i)(x_{lj} - \bar{x}_j) / n$).

$$[2] \quad \begin{aligned} \hat{\sigma}_{ij} &= s_{ij} \text{ for } (i,j) \notin I^A \text{ and for } i = j \\ \hat{\sigma}^{ij} &= 0 \quad \text{for } (i,j) \in I^A. \end{aligned}$$

This indicates that the submatrices of $\hat{\Sigma}$ having only unrestricted variable pairs (these are just the variable groups listed in the generating class $\{N_t\}$) match the corresponding submatrices of the observed covariance matrix. The estimated covariance $\hat{\sigma}_{ij}$ of each restricted variable pair $(i,j) \in I^A$ instead, may differ from the observed covariance s_{ij}. Its value is implied by the particular pattern of restrictions and by the observed covariances of all unrestricted pairs.

Since only parts of the observed covariance matrix are needed to compute $\hat{\Sigma}$, each covariance selection model can be said to give, with $\hat{\Sigma}$, a condensed description of the interdependence structure. It will differ little in positions $(i,j) \in I^A$ from the uncondensed description, the observed covariance matrix S, if the model assumptions are correct. Thus, each well-fitting covariance selection model provides the researcher with a good data reduction in the sense that the observed covariances of all restricted variable pairs are closely reproduced from only knowing the covariance matrices of the unrestricted subgroups of variables.

The likelihood ratio test statistic, LR-χ^2, represents a measure of the deviation between $\hat{\Sigma}$ and S. It can also be shown to measure the deviation between $\hat{P}$, the maximum-likelihood estimate of the correlation structure of a given covariance selection model, and R, the observed correlation matrix. The determinants of S and R relate as $|S| = |R| (\Pi_{i=1}^{p} s_{ii})$, and those of $\hat{\Sigma}$ and $\hat{P}$ similarly, since from Equation 2, $\hat{\sigma}_{ii} = s_{ii}$ for all i. The statistic is defined as

$$[3] \quad \begin{aligned} \text{LR-}\chi^2 &= n \ln (|\hat{\Sigma}| / |S|) \\ &= n \ln (|\hat{P}| / |R|). \end{aligned}$$

For large numbers of observations, n, this statistic follows a chi-square distribution with degrees of freedom equal to the number of restricted variable pairs for Σ, the number of pairs listed in I^A.

The actual computation of $\hat{\Sigma}$ or $\hat{P}$ has to be done by iterative algorithms like those described by Dempster (1972) or by Wermuth and Scheidt (1977)—unless the model is a decomposable one. In the latter case the equivalence to a recursive system leads to the simple closed form estimate in Equation 12 below.

Systems of Linear Recursive Equations

Linear recursive equations represent a subclass of the linear structural equation models studied by econometricians (e.g., Goldberger, 1964). Some of the p variables, $X_1, \ldots, X_k$ $(k < p)$, are considered to be response (or endogeneous) variables. In recursive systems these can be ordered such that each response X_i may depend on variables X_j with $j \in \{i+1, \ldots, p\}$ but not on any one of the variables X_k with $k \in \{1, \ldots, i-1\}$. The remaining variables, $X_{k+1}, \ldots, X_p$, that are not thought of as responses are called exogeneous.

The dependence relationships in an unrestricted or complete recursive system take on the following triangular form:

$$[4] \quad \begin{aligned} X_1 &= \alpha_{12}X_2 + \alpha_{13}X_3 + \ldots + \alpha_{1k}X_k + \alpha_{1,k+1}X_{k+1} + \ldots + \alpha_{1p}X_p + U_1 \\ X_2 &= \qquad\quad \alpha_{23}X_3 + \ldots + \alpha_{2k}X_k + \alpha_{2,k+1}X_{k+1} + \ldots + \alpha_{2p}X_p + U_2 \\ &\;\;\vdots \\ X_k &= \qquad\qquad\qquad\qquad\qquad\quad \alpha_{k,k+1}X_{k+1} + \ldots + \alpha_{kp}X_p + U_k \end{aligned}$$

In systems with uncorrelated errors the residual U_i is not only regarded as independent from the influencing variables or regressors $X_{i+1}, \ldots, X_p$, but also from residuals in other equations, that is, from U_j with $j \neq i$. Recursive models (that have also been called incomplete systems) result from a complete system by requiring the regression coefficients α_{ij} to be zero for each $(i,j) \in I^D \subseteq \{(i,j) \mid 1 \leq i < j \leq p$ and $i \leq k\}$.

For positive definite covariance matrices the zero restriction on a regression coefficient α_{ij} is equivalent to a zero restriction on a correlation coefficient, since.

$$[5] \qquad \rho_{ij \cdot \{i+1, \ldots, p\} \setminus \{j\}} = \alpha_{ij} \sqrt{\frac{\sigma_{jj \cdot \{i+1, \ldots, p\} \setminus \{j\}}}{\sigma_{ii \cdot \{i+1, \ldots, p\} \setminus \{j\}}}},$$

and, since in positive definite covariance matrices, not only are all variances σ_{ii} nonzero, but also are the residual variances nonzero,

$\sigma_{ii \cdot K_i} = \sigma_{ii} - \Sigma_{k \in K_i} \alpha_{ik} \sigma_{ki}$, which result from regressing variable i on an arbitrary set of influencing variables $K_i \subseteq \{1, \ldots, p\} \setminus \{i\}$.

In complete and incomplete recursive systems with normally distributed and uncorrelated errors, the maximum likelihood estimates of all regression coefficients in the system can be obtained by least squares estimation applied to the equations separately. This follows from a result by Wold (1954) or from the factorisation property of recursive systems. Let X_{A_i} with $A_i = \{j \mid j>i \text{ and } (i,j) \in I^D\}$ denote the influencing variables for each response i, then the joint distribution factorises as

$$[6] \qquad f(X_1, \ldots, X_p) = (\Pi_{i=1}^{k} f(X_i \mid X_{A_i}))\, f(X_{k+1}, \ldots, X_p),$$

and the maximization of the likelihood function can be split up into $k + 1$ independent maximizations.

For each fixed i one can obtain the (negative values of) α_{ik} as the solution of the familiar normal equations:

$$[7] \qquad s_{ij} = \Sigma_l \hat{\alpha}_{il} s_{lj} \text{ with } j \in A_i,\ l \in A_i,$$

where s_{ij} are observed covariances again.

Equation 7 shows that incomplete recursive systems do not necessarily lead to data reductions. If for instance, the first variable is unrestricted, so that $A_1 = \{2, \ldots, p\}$, then all observations are needed to compute the estimates $\hat{\alpha}_{1k}$, no matter how many other restrictions there are in the system.

The maximum likelihood estimate of the covariance matrix can be derived as shown in an example by Wermuth (1980). Equation 6 shows that $\hat{\sigma}_{lt} = s_{lt}$ for all $l > k$ and $t > k$. The remaining parts of $\hat{\Sigma}$ can be computed in the order $i = k, k\text{-}1, \ldots, 1$, as

$$[8] \qquad \begin{aligned} \hat{\sigma}_{ii} &= s_{ii} + \Sigma_{l \in A_i} \Sigma_{t \in A_i} \hat{\alpha}_{il} \hat{\alpha}_{it} (\hat{\sigma}_{lt} - s_{lt}), \\ \hat{\sigma}_{ij} &= \Sigma_{l=i+1}^{p} \hat{\alpha}_{il} \hat{\sigma}_{lj}, \end{aligned}$$

with $\hat{\alpha}_{ij} = 0$ for $j \notin A_i$. The first line of Equation 8 indicates that $\hat{\sigma}_{ii} = s_{ii}$ if the estimated covariances and variances $\hat{\sigma}_{lt}$ of the influencing variables A_i match the corresponding observed covariances and variances s_{lt}. This is the case for those recursive models which are decomposable models.

In any case, though, the likelihood ratio test statistic for incomplete recursive systems takes on a simple form that can be derived from the triangular decompositions of $\hat{\Sigma}$ and S:

$$[9] \quad \begin{aligned} \text{LR-}\chi^2 &= n \ln (| \hat{\Sigma} | / | S |) \\ &= n \ln (\Pi_{i=1}^{k} s_{ii \cdot A_i} / s_{ii \cdot A_i \cup B_i}), \end{aligned}$$

where $s_{ii \cdot A_i}$ is the estimated residual variance (s_{ii} - $\Sigma_{l \in A_i} \hat{\alpha}_{il} s_{li}$) from regressing variable i on all influencing variables A_i in the incomplete system and $s_{ii \cdot A_i \cup B_i}$ is the corresponding residual variance in the complete system (with $B_i = \{j \mid j > i$ and $(i,j) \in I^D\}$ and $A_i \cup B_i = \{i+1, \ldots, p\}$). Whenever an equation is unrestricted, it does not contribute to the test statistic, since then $A_i = A_i \cup B_i$. The degrees of freedom are again equal to the number of restricted variable pairs, to the number of elements in I^D. Since $\hat{\sigma}_{ii}$ need not coincide with s_{ii}, this test statistic may not, in general, be computed from knowing only the correlation matrices P and $\hat{P}$.

As has been noted by Tukey (1954) the (negative values) of $\hat{\alpha}_{il}$ may alternatively be computed as

$$[10] \quad \hat{\alpha}_{il} = \hat{\alpha}_{il}^{*} \sqrt{\frac{s_{ii}}{s_{ll}}},$$

where $\hat{\alpha}_{il}^{*}$ are obtained as the solutions of the normal equations for standardized variables, $Z_i = (X_i - \bar{X}) / \sqrt{s_{ii}}$:

$$[11] \quad r_{ij} = \Sigma_l \hat{\alpha}_{il}^{*} r_{lj}, \text{ with } j \in A_i \text{ and } l \in A_i,$$

and r_{ij}, the observed correlation coefficient for variables X_i and X_j.

Equation 11 is also known as Wright's (1923, 1934) rule for calculating path coefficients ($\hat{\alpha}_{il}^{*}$) for a system of causal relations that is represented by a path diagram. In the case of a recursive model with independent errors, this rule yields maximum likelihood estimates of the equation parameters, and a path diagram is equivalent to our graphical representation of the model. For recursive systems with correlated errors, however, or for non-recursive systems, Equation 11 does not, in general, lead to consistent estimates of the equation parameters.

Wright's second rule, the one for computing implied correlations, is restated in our terminology in Equation 12 below. It has been shown (Wermuth, 1980) that—even for incomplete recursive systems—this rule does not yield the maximum likelihood estimate of the correlation matrix, but that it always defines the maximum likelihood estimate, if the recursive model is a decomposable model, too.

Decomposable or Multiplicative Models

Decomposable models form a subclass of covariance selection as well as of recursive models. Members of this subclass possess advantages of both approaches, but avoid disadvantages. Unlike covariance selection models in general, the maximum likelihood estimate of a decomposable covariance matrix can be expressed in closed-form: no iterative algorithms are needed for the computations. And unlike general recursive models, a well-fitting decomposable recursive model provides the researcher with a good data reduction: only parts of the observations are needed to compute the maximum likelihood estimates of all equation parameters and of the covariance or correlation matrix of all t variables.

A decomposable model may be characterized not only by its previously discussed graphic representation but also by a property of its set of restrictions. A set of restricted variable pairs has been called "reducible" by Wermuth (1980) if one can reduce the dimensions of a p-dimensional normal distribution over variables $1, \ldots, i\text{-}1$, and for variables $i, \ldots, p$, the set of restrictions in the marginal distribution remains the same as in the joint distribution of all p variables. Formally we write:

Definition

A set $I \subseteq \{(i,j) \mid 1 \leq i < j \leq p\}$ is reducible if, for each pair (i,j) contained in I, and for all $h = 1, \ldots, i\text{-}1$, the pairs (h,i) or (h,j), or both, are also contained in I. Two results from Wermuth (1980) are then useful:

> A covariance selection model is a decomposable model if and only if the variables may be ordered (renumbered) such that I^A becomes reducible;

and

> A recursive model is equivalent to a decomposable model if and only if the set I^D is reducible.

Each model with a reducible set of restrictions can therefore be interpreted as a recursive system and the maximum likelihood estimate of the decomposable correlation matrix $\hat{P}$ can be computed (Wermuth,

1980) in the sequence i=p-1, . . . , 1 as

$$[12] \quad \rho_{ij} = r_{ij} \quad \text{for } i>k \text{ (two-headed arrows), and } j \epsilon A_i \text{ (existing one-headed arrows)}$$

$$= \Sigma_l \hat{\alpha}_{il} \, \rho_{lj} \quad \text{for } j \epsilon B_i \text{ and } l \epsilon A_i \text{ (missing one-headed arrows).}$$

This is just Wright's rule for computing implied correlations. The maximum likelihood estimates of the equation parameters are given by Equations 7 or 10 and 11, and the test statistic can be computed from the covariance *or* correlation matrix as

$$[13] \quad \text{LR-}\chi^2 = n \ln (|\hat{\Sigma}| \,/\, |S|\,),$$
$$= n \ln [\Pi_{i=1}^{k} (1 - R_{i\,.\,A_i}^2) \,/\, (1 - R_{i\,.\,A_i \cup B_i}^2)\,],$$

with $R_{i\,.\,A_i} = \sqrt{1 - s_{ii\,.\,A_i} \,/\, s_{ii}}$ being the multiple correlation coefficient. All these are, in principle, paper-and-pencil methods which are, of course, facilitated by computer programs for least squares regression.

Applications in Psychological Research

In what follows, some examples of applications in psychological research will be described (see also Hodapp, 1982; Wermuth, 1978; Weyer & Hodapp, 1979). The first example, taken from developmental psychology, is based on a study by Rubin (1973), dealing with relationships between egocentrism measures and age. The purpose here is to interpret the correlation structure of these data in terms of a decomposable model.

The later examples demonstrate the two different strategies of data analysis which can be pursued with the help of decomposable models: testing or searching for a structure. In an example from social psychology, we present an analysis of marital and parental variables influencing marital satisfaction, which were investigated by Miller (1976). This latter author stated, as an hypothesis of a causal structure, a path diagram. In the corresponding system of incomplete linear recursive equations, a subsystem takes on the form of a decomposable model. We test the goodness-of-fit of this model and, after finding it to be incompatible with the observations, suggest a better-fitting model.

In a separate section dealing with the exploratory approach to decomposable models, a search procedure is discussed. We use it to re-analyze the data of Zeiner and Schell (1971) from experimental learning psychology. This re-analysis combines model search, interpretation of a covariance selection model, and reformulation of a covariance selection model in terms of a system of linear recursive equations.

An Example from Developmental Psychology

To describe a well-fitting decomposable covariance selection model, we have taken an example from a study in developmental psychology by Rubin (1973), who stated:

> Egocentrism, a central concept in Piaget's theory (Piaget 1950), has been examined in terms of the young child's communicative, cognitive (problem-solving), role-taking, and perceptual activities . . . . Previous theory and research have suggested that a single factor subsumes the various forms of egocentrism. Indeed, some (Feffer, 1959, 1970; Piaget 1950) have identified this factor as "the inability to decenter," that is, the child's inability to shift his attention from one aspect of an object or situation to another. However, few studies exist in which investigators attempt to relate the different types of egocentrism. Those that do exist present inconclusive results as to the exact nature of the correlations . . . Thus, one purpose of this study was to examine the nature of correlations among tasks purporting to measure communicative, cognitive, role-taking, and spatial egocentrism in childhood. (pp. 102-103).

Investigating this question, Rubin additionally collected data on a few "marker variables," which he postulated should be related to the egocentrism measures. Of these we include only the two variables, Age and Conservation. In the classical analysis of Piaget, conservation skills are crucial for the ability of decentration.

Table 1
Intercorrelations of Childhood Egocentrism Variables According to Rubin (1973)

	Cogn. Egoc.	Spat. Egoc.	Cons.	Comm. Egoc.	Role Tak.	Age
Cognitive Egocentrism	1.00					
Spatial Egocentrism	.28	1.00				
Conservation	.25	.63	1.00			
Communicative Egocentrism	.37	.65	.25	1.00		
Role Taking Egocentrism	.32	.73	.65	.72	1.00	
Age	.47	.66	.73	.73	.78	1.00

Note. n = 60.

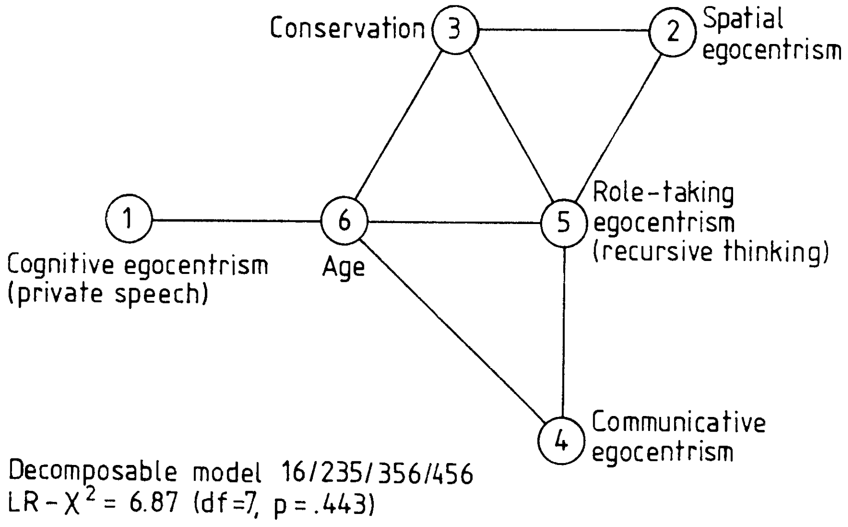

Figure 7.
Decomposable model for Rubin's childhood egocentrism variables.

In Table 1, the correlation matrix from the study of Rubin (1973) is presented. Figure 7 illustrates a well-fitting interdependence structure for these variables. The question of how this model has been obtained is deferred for the time being. Instead, we show how such a model can be interpreted.

In Figure 7, the variable Cognitive Egocentrism shows no direct relationship with the other egocentrism variables. There exists only an indirect relationship between Cognitive Egocentrism and the variables Conservation, Role-Taking Egocentrism, and Communicative Egocentrism, where this indirect relationship is linked to age. All variables, except for Spatial Egocentrism, are directly linked with the variable Age. This indicates that at every stage of the developmental sequence, it is individual experience and maturation that determines performance. Up to this point, this interpretation coincides with the interpretation of Rubin (1973), who wrote:

> A correlation analysis provided initial confirmation of the "centration" hypothesis. Measures of spatial, role-taking, and communicative egocentrism, as well as conservation seemed to form a cluster defined by their significant interrelationships. . . . It is probable that the age variables account for the significant relationships found to exist between the cognitive measure of egocentrism and conservation. With either mental or chronological age partialled out, cognitive egocentrism was no longer related to the latter variables. (p. 108)

In addition to this we are able to make some more differentiated statements. Communication, the most complex of the analyzed variables, is linked with the variables Conservation and Spatial Egocentrism through recursive thinking (Role-Taking Egocentrism). According to a proposition of Feffer (1970), there exists an isomorphy of cognitive structures which the individual develops towards *physical objects* and also in *interpersonal relations*. According to Piaget (1950), conservation skills and the overcoming of spatial egocentrism are the criteria indicating that a person has reached the concrete-operational stage. At this stage, cognitive operations are available to the child which have been described by Piaget with the concepts of decentration and reversibility. Role taking in a metaphoric sense—the ability to predict cognitions, motives, and feelings of other persons—is represented by the variables Recursive Thinking and Communication. The structure model shows a stable relationship between these social abilities and the cognitive operations, confirming an essential assumption of cognitive developmental theory (Flavell, Botkin, Fry, Wright & Jarvin, 1968; Selman, 1976). These relationships are maintained even when the age variables[1] are held constant. Furthermore, recursive thinking seems to form an intermediate function. According to Miller, Kessel, and Flavell (1970), recursive thinking means that a subject is able to comprehend the cognitions of another person with regard to his own or other person's cognitions ("thinking about thinking," "thinking about thinking about thinking"). Although recursive thinking is characterized by the distinction between, and subjectivity of, personal and extraneous viewpoints, real communication goes beyond this; it requires that the other person's role is understood and that one's own actions are adjusted accordingly.

With the rather elaborate interpretation of this example, we hope to have shown how the approach described here is different from the analysis using latent variables or factors. Such latent variable analyses are based on an abstract centration factor which is supposed to underly all egocentrism variables (see Rubin, 1973). For the above example, the structural analysis of the directly-observed variables has led to a more differentiated model of variable relationships.

An Example from Social Psychology

The next example illustrates how an hypothesis about a dependence structure can be tested and modified with the help of decompos-

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[1]To avoid multicollinearity, only one of the highly correlated age variables ("chronological age" and "mental age") can be included in the analysis. Our interpretation remains unaffected regardless of which of the age variables is incorporated in the analysis.

able models. The data are taken from a study by Miller (1976) who proposed a multivariate developmental model of marital satisfaction:

> Conjugal companionship and communication have sometimes served alone as criterion variables in studies of the marital relationship, and they also have been combined with other components in measures of overall "adjustment." However, it seems likely that objective marital interaction variables, such as the amount of communication and frequency of companionate activities, intervene between the child-related variables noted above and reported satisfaction with marriage. Consequently, the number and spacing of children was theoretically linked to how often spouses engaged in various activities, and this frequency measure was then related to the repondents' attitudinal reports of marital satisfaction. This was an attempt to examine the hypothesis that one of the key reasons children appear to depress marital satisfaction lies in the reduced frequency of companionate activities that husbands and wives typically engage in with the arrival and rearing of children. If children generally reduce marital companionships, then child spacing could be expected to positively influence companionate activities (longer intervals between births would allow more companionship). (p. 645)

The duration of marriage and the family socio-economic status were introduced by Miller as exogenous variables because empirical studies have supported a decrease of satisfaction in the course of marriage, and also have pointed to relationships between child-related variables and socio-economic status. A subsystem of the postulated dependence relations is shown in Figure 8 (cf. Miller, 1976).

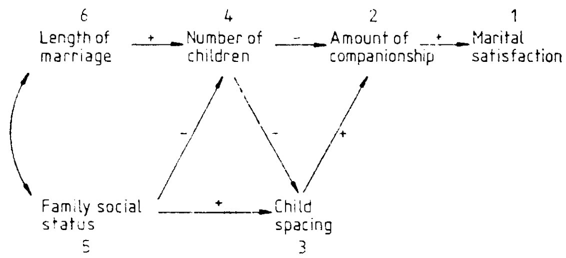

Figure 8.
Miller's theoretical model of variables antecedent to marital satisfaction.

This subsystem can be translated into the following recursive model that is equivalent to a decomposable model:

$$
\begin{aligned}
Z_1 &= \alpha_{12}Z_2 + U_1 \\
Z_2 &= \alpha_{23}Z_3 + \alpha_{24}Z_4 + U_2 \\
Z_3 &= \alpha_{34}Z_4 + \alpha_{35}Z_5 + U_3 \\
Z_4 &= \alpha_{45}Z_5 + \alpha_{46}Z_6 + U_4.
\end{aligned}
\tag{14}
$$

The coefficients of the equation system 14 represent the path coefficients and can be estimated using the observed correlations in Table 2 (below the diagonal). Further, the residuals, i.e., the discrepancies between the observed and the implied correlations, are presented in Table 2 (above the diagonal).

Table 2

Intercorrelations of Variables of Marital Satisfaction According to Miller (1976)

	Marit. Satisf.	Amount Comp.	Child Spac.	Number Childr.	Fam. Soc.St.	Length Marr.
Marital Satisfaction		.000[a]	-.056	.125	.115	.170
Amount of Companionship	.370		.000	.000	.244	.016
Child Spacing	-.062	-.016		.000	.000	.244
Number of Children	.047	-.211	-.041		.000	.000
Family Social Economic Status	.132	.289	.048	-.217		.000
Length of Marriage	.127	-.100	.216	.552	-.240	

Note. *n* = 140.

[a]Above diagonal residuals for the model postulated by Miller.

The implied correlations were calculated according to Equation 12. The correlation matrix with the implied correlations represents the estimate of the theoretical correlation matrix under the assumptions of the model represented by Equations 14.

From the test statistic for this model given in Figure 9 and from the residuals given in Table 2, it can be seen that this model postulated by Miller (1976) has only a weak correspondence with the data. To obtain new hypotheses about the variable relationships, we applied the model search procedure—described in detail in the next section—and obtained the modified model illustrated in Figure 10.

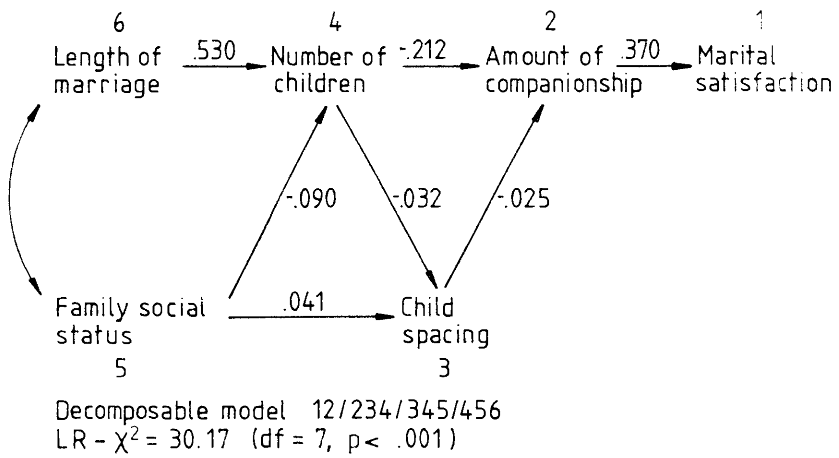

Figure 9.
Path diagram of Miller's theoretical model of variables antecedent to marital satisfaction.

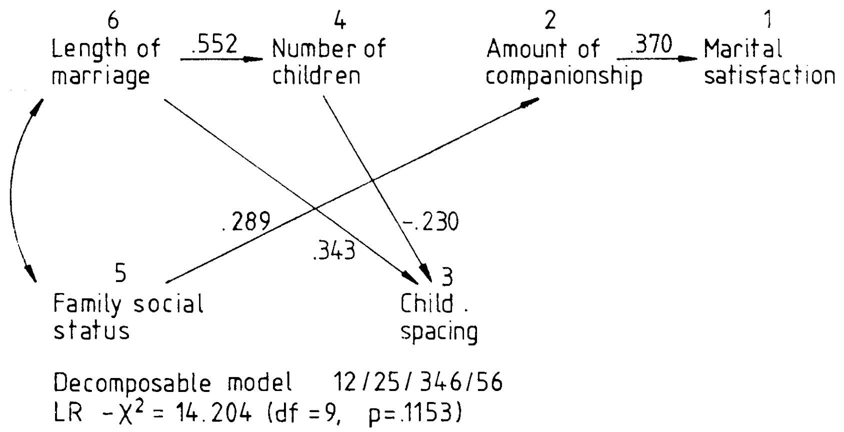

Figure 10.
Modified model of variables antecedent to marital satisfaction.

The models represented in Figures 9 and 10 are clearly different. The most obvious difference lies in the absence of any direct relationship between the variables Number of Children, Child-spacing, and Marital Satisfaction in the new model. Furthermore, two new direct relations were lacking in the former model: relations between Duration of Marriage and Child-spacing, and between Socio-economic Status and Amount of Companionship. Common to both models is a direct dependence of Marital Satisfaction on the Amount of Companionship. The influence of the socio-economic status on the latter was already recognized by Miller (1976):

> By analyzing these data more carefully and looking at the average companionship scores for five categories of family social status, it appeared that this relationship was strongest in the lower categories of social class. That is, those who were in the lowest category of family social status reported extremely low companionship scores relative to those in the upper four categories. This is understandable because some financial resources would be necessary to participate in several of the companionate activities referred to in the items. (p. 654)

A different explanation of the discrepancies between Miller's hypothesis and the modified model may be that satisfaction or dissatisfaction are representing only one possible way of responding to the challenges within the family. Orden and Bradburn (1968) and Hodapp and Weyer (1980) have pointed out that responses such as tensions, strain, and emotional disturbances may also be means of marital adjustment, independent from any satisfaction/dissatisfaction dimension. In agreement with other authors, Miller mentioned role strain as a possible source of conflict. Often these tensions may be caused by children, as the spouses have to fill their new roles as parents. According to Hodapp and Weyer (1980), a significant relationship has been found to exist between the number of children and the subjective pressure experienced by housewives, although not between the number of children and dissatisfaction.

Model Search

Description of the Procedure

The systematic search for partial zero-correlations or zero-concentrations aims at the simplification of an interdependence structure. The so-called "Simon-Blalock approach" (Blalock, 1964; Simon, 1952) was an earlier attempt in this direction. This technique, however, is characterized by two basic difficulties. First, there is the problem of selecting appropriate models from the large number of all possible models (see, for example, Blalock, 1962, who described a variety of possible causal models for four variables). Second, there is the problem of evaluating an overall model with the aid of test statistics. The two problems can be solved if a model search procedure based on the theory of covariance selection is applied.

A search procedure among decomposable models can be characterized in the following way (Wermuth, 1976b, 1980): The starting point is a situation where no zero-concentrations are required; this is followed by a step-wise examination of the variable pairs to determine how many and which variable pairs can be considered to have a zero-concentration. In the first step, that variable pair is selected for which the assumption of a zero-concentration is most consistent with the data. This is done by calculating the likelihood ratio test statistic for every variable pair under the hypothesis of a zero-concentration and by selecting the variable pair with the smallest statistic. In a second step, a second variable pair is selected whose statistic has the smallest value for the additional zero-concentration. If $\hat{P}_1$ represents the estimated correlation matrix with one additional zero-concentration in selection step $n+1$, compared with the correlation matrix $\hat{P}_2$ in selection step n, then Equation 15 below yields the likelihood-ratio test statistic of an additional zero-concentration, which is approximately chi-square distributed with one degree of freedom:

$$[15] \qquad \text{LR-}\chi^2 = n \ln (|\hat{P}_2| / |\hat{P}_1|).$$

Again, only for decomposable models, can the statistic in Equation 15 be calculated directly from determinants of parts of the observed correlation matrix without explicitly determining the estimates $\hat{P}_1$ and $\hat{P}_2$ first. When the test statistics of successive selection steps are summed, one obtains the test statistic in Equation 13 for the goodness-of-fit of the decomposable model (with $\hat{P}_1$) to the original matrix R without zero-concentrations. For a decision about whether a given model is suitable, one should take into account both the test statistics of the selection step (Equation 15) and the test statistic in Equation 13 for the overall model. Neither should correspond to a small fractile value p of the appropriate chi-square distribution. The p-value stands for the probability of obtaining the observed or an even larger chi-square value if the model assumptions are correct.

In order to use an objective criterion, we speak of a well-fitting model as long as none of the p-values for the test statistics in Equations 13 and 15 falls below .05. Clearly, this criterion is a very crude one, but it at least excludes all poorly-fitting models from the list of plausible models for the interdependence structure. In particular, the use of p values introduces a dependency on sample size.

An Example from Experimental Psychology

Zeiner and Schell (1971) reported the results of a conditioning experiment in which discrimination performance was investigated as a function of the orienting reaction (OR):

> Sokolov (1960, 1963) holds that OR's elicitated by innocuous stimuli, either produce or are correlated with the individual's heightened sensitivity to environmental stimulation which in turn leads to increased information intake and the facilitation of learning. On the other hand, the defensive reflex elicitated by

> intense or noxious stimuli attenuates these effects. Sokolov's theory predicts that Ss [Subjects] giving large responses to an innocuous stimulus should condition better than Ss giving small responses to an innocuous stimulus. Whether or not similar predictions can be made for groups differing on defensive reflex (DR) magnitude to a noxious stimulus is unclear .... It was the purpose of the present experiment to determine the relationship between individual differences in response magnitude to both innocuous and noxious stimuli and first and second interval SRR [skin resistance response] magnitudes in discrimination conditioning. (pp. 613-614)

The experimental setup of Zeiner and Schells (1971) study followed the pattern of classical conditioning and consisted of repeated, combined presentation of light stimuli (CS) of different colors and electrical shocks (UCS). In each instance, only light stimuli of *one* specified color were reinforced by a shock 5 sec. later. From the mean difference of the skin resistance responses to the reinforced and non-reinforced light stimuli, two measures of discrimination learning were derived. The measures related to the 5-sec. interval after the light stimuli were called first interval response discrimination (FDIS), whereas the measures taken 6 to 10 sec. after the light stimuli were called second interval response discrimination (SDIS). In addition to these two measures of discrimination performance, Zeiner and Schell defined two simple response measures which comprised nothing but the strongest skin resistance responses to the light stimuli within the first (1 - 5 sec.) and second (6 - 10 sec.) interval. These responses were called first interval responses (FIR) and second interval responses (SIR). Furthermore, the main skin resistance responses to the noxious and innocuous stimulus before the conditioning procedure were recorded; these responses were called orienting reactions, UCS-OR[2] and CS-OR.

[2]The term "defensive reflex" would be more appropriate, but we follow the variable notation of Zeiner and Schell.

Table 3

Correlation Matrix for the Variables of the Conditioning Experiment of Zeiner and Schell (1971)

	SDIS	FDIS	SIR	FIR	CS-OR	UCS-OR
1 SDIS	1.00					
2 FDIS	.72**	1.00				
3 SIR	.32*	.45**	1.00			
4 FIR	.30*	.54**	.60**	1.00		
5 CS-OR	.19	.43**	.31*	.71**	1.00	
6 UCS-OR	.25	.29*	.09	.31*	.26	1.00

Note. n = 48.

*$p < .05$.

**$p < .01$.

The correlations between the orienting reactions and the four response measures are shown in Table 3 (see Zeiner & Schell, 1971). In this example, there is a clear distinction between independent (exogenous) and dependent (endogenous) variables, as the authors used the orienting reaction to the noxious and innocuous stimulus as the independent variable in this experimental setup. As to the structure of the relationships between the response measures, Zeiner and Schell pointed out that all four response measures show significant intercorrelations. For this reason, it is not possible to give a definitive interpretation of the relationships between the response measures and the independent variables. As appropriate conceptions are missing about the relationships between these measures, it seems appropriate to apply the model search procedure, with the orienting reactions as exogenous variables.

Table 4

Results of Model Search Procedure for the Data of Zeiner and Schell (1971)

l=Number of Conditionally Independent Variables	Set I^A of Variable Pairs with Zero-Concentrations	Model Notation	Test Statistic with 1 df for One Additional Concentration	p	Test Statistic with l df for the Total Model	p
1	(2,6)	12345/13456	.15292	.69576	.15292	.6958
2	(2,6) (2,3)	1245/13456	.66090	.41624	.81383	.6657
.	. . .	.	.	.	.	.
10	(2,6) (2,3) (2,5) (1,5) (3,5) (3,6) (1,6) (1,3) (1,4) (4,6)	12/24/34/45/56	1.66086	.19749	10.94781	.3616
11	(2,6) (2,3) (2,5) (1,5) (3,5) (3,6) (1,6) (1,3) (1,4) (4,6) (2,4)	12/34/45/56	16.54783	.00005	27.49563	.0039
.	. . . .	.	.	.	.	.
14	(2,6) (2,3) (2,5) (1,5) (3,5) (3,6) (1,6) (1,3) (1,4) (4,6) (2,4) (3,4) (4,5) (1,2)	1/2/3/4/56	35.07079	.00000	117.65449	.0000

The results of the model search procedure are presented in Table 4. There exists a computer program which calculates the information set out in this table (Wermuth, Wehner, & Gönner, 1976). For every selection step, certain characteristics of each decomposable model are given: the set I^A of restrictions on variable pairs with zero-concentrations and the model notation as a shorthand characteristic of the variable groups $\{N_i\}$ with variables belonging together. Additionally, the statistic is shown for an additional zero-concentration as well as for the respective overall model. We chose model 12/24/34/45/56, because the model of the next step would lead to a weak correspondence with the data, not only for the single relationships [variable pair (2,4), $p = .00005$] but also for the overall model (model 12/34/45/56, $p = .0039$).

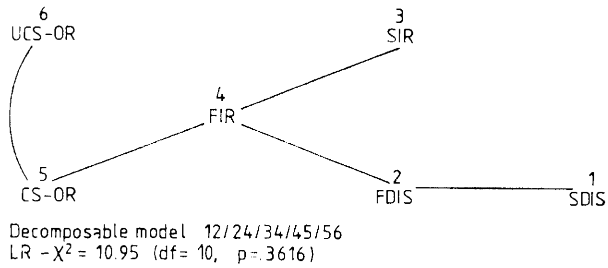

Figure 11.
Decomposable model for the data of Zeiner and Schell (1971).

This decomposable model is presented in Figure 11. The model shows a very simple variable structure, as in each case only two variables are directly connected with each other. In particular, the discrimination performance is determined only by the orienting reaction to the innocuous light stimulus, where the first interval response represents the crucial intermediating variable. Also, no direct relationship between orienting reaction and the second interval measures can be shown.

Although the decomposable model permits no statements about directed relationships at this point, the interpretation of this interdependence structure as a dependence structure is highly suggestive in view of the experimental setup. When the discrimination performance (Z_1,Z_2) is taken as dependent on the direct response to the light stimuli (Z_3,Z_4) and the orienting reaction (Z_5,Z_6), and the second interval measures are taken as dependent on the first interval measures, then the decomposable model may be transformed into the following recursive equation system:

$$[16]\quad \begin{array}{lll} \text{SDIS:} & Z_1 = & \alpha_{12}Z_2 + 0 + 0 + 0 + 0 + U_1 \\ \text{FDIS:} & Z_2 = & 0 + \alpha_{24}Z_4 + 0 + 0 + U_2 \\ \text{SIR:} & Z_3 = & \alpha_{34}Z_4 + 0 + 0 + U_3 \\ \text{FIR:} & Z_4 = & \alpha_{45}Z_5 + 0 + U_4 \end{array}$$

The coefficients of this equation system can be read directly from the correlation matrix because the regression equations can be traced back to the most simple case where only one variable is used as independent variable. Figure 12 shows the path diagram with the likelihood ratio test statistic for the overall model (cf. Equation 3).

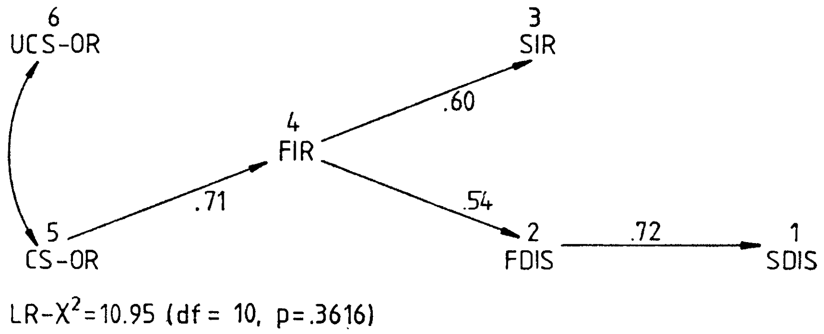

Figure 12.
Path model for the variables of the conditioning experiment according to Zeiner and Schell (1971).

The correlation matrix is estimated by using Equation 12. The implied correlations for this example are:

$$\hat{\rho}_{46} = \hat{\alpha}_{45}\,\hat{\rho}_{56} = r_{45}r_{56},\ \hat{\rho}_{35} = \hat{\alpha}_{34}\hat{\rho}_{45} = r_{34}r_{45},\ \hat{\rho}_{36} = \hat{\alpha}_{34}\hat{\rho}_{46} = r_{34}(r_{45}r_{56}),\ \text{etc.}$$

The path model confirms the Zeiner and Schell (1971) hypothesis that persons who show stronger orienting reactions to innocuous stimuli demonstrate better discrimination performance in the conditioning experiment. From the responses to a noxious stimulus, no predictions about discrimination performance can be made. As a direct determinant of first interval discrimination (FDIS), the response to the innocuous stimulus *during* the conditioning procedure—the first interval response—could be identified. The first interval response in turn depends on the orienting reaction *before* the conditioning procedure, the CS-OR. Sokolov (1963) and Maltzman (1967) supported these results, theorizing that the orienting reaction is a valid index of attention, and that differences in the intensity of attention are responsible for differences in discrimination learning between individuals.

Discussion

As examples from different areas of psychological research have shown, decomposable models are well-suited to order multivariate relationships and to obtain a unified overview of complex variable relationships. They represent structures for systems of observable variables which are relatively easy to interpret and which can be tested as to their goodness of fit. For normally distributed variables, models of interdependencies as well as dependencies are based on measures of linear associations, as for example, correlations or regression coefficients. This means that such a model should only be interpreted if there are no non-homogeneous subgroups or nonlinear relationships. Given these restrictions, decomposable models can help the researcher to pinpoint direct interdependencies between variables and to identify interrelationships that exist only indirectly via the covariation with other variables.

Presently, probably the best known models of dependencies are linear structural equation models. One of the most serious problems encountered in the application of linear structural equation models is that the investigator may not be sure if the model chosen is identified. If the models are restricted to only observable variables, we get the linear structural equations, of which decomposable models are a subclass. In this case, necessary and sufficient conditions for parameter identification are known.

For LISREL models, in general, no sufficient conditions, and no necessary *and* sufficient conditions are known for parameter identification. Instead, the program checks only a necessary condition and continues to compute estimates if this condition is satisfied. As a result, there are situations (all those in which the necessary condition is satisfied although the parameters are in fact unidentified) in which estimators are computed though "it does not make sense to talk about an estimator" (Jöreskog & Sörbom, 1978, p. 10). We regard this as a serious problem indeed. Researchers using LISREL cannot be certain that it makes sense to interpret the resulting computations.

Decomposable models have important properties which they do not share with other dependence structures. The most important of these, stated in technical terms, is the following: covariance matrices of subgroups of variables form the set of minimal sufficient statistics for the parameters of the joint model for all variables. The practical implications of this can be viewed as a generalization of the notion of a spurious correlation to variable sets. The models offer a true reduction of dimensionality. It is necessary only to know the associations of subgroups of variables to be able to reproduce associations of all variables. Put somewhat differently, a well-fitting decomposable model provides the justification to classify an association as less important because it can be closely reproduced from knowing only the association in the set of sufficient statistics. What we have here is Sir Ronald Fisher's important notion of sufficient statistics as it applies to covariance matrices.

The second important property of decomposable models is that such models have interpretations in terms of dependence structures (directed graphs), as well as in terms of interdependence structures (undirected graphs), and that we can read off from the graphs interpretation of these structures in terms of independencies.

Decomposable models can be used in confirmatory as well as exploratory types of analyses. Although testing of a priori hypotheses is generally preferred, a model search among decomposable models can be a valuable tool for gaining a better understanding of the data if,

a. the researcher's knowledge about the interrelations is too weak to formulate hypotheses,
b. an hypothesized dependence structure is not supported by the data, or
c. the intention is to find a condensed description of the data.

In all these instances, the results of the model search permit the formulation of hypotheses on dependence structures. These should be judged in terms of subject-matter considerations and can be tested as hypotheses on a new set of data.

Ordering of decomposable variable structures according to their independence restrictions illuminates the implications of adding or deleting certain connecting lines in the graphs of models. This is an advantage of decomposable models, since for more complex models, like simultaneous equation models or latent variable models, the implications of adding or deleting connecting lines are not well understood. Within the class of decomposable models, each model has well-defined interpretations. Each model represents a proposal for data reduction, and there is hierarchical ordering of the models: the simplest model states the complete independence of all variables; the most complex model states that all variables are interrelated so that no reduction in dimensionality is possible (except possibly in terms of an underlying unobservable factor).

In summary, application of decomposable models is an attempt to get away from overly complex models and to use—whenever this is possible and feasible—simple descriptions of the data.

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