Introduction to the Use of Graphical Chain Models

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Abstract

A description of possibilities and limitations in analysing data with the help of graphical chain models is attempted. For several sets of data the research hypotheses and their evaluation in the light of the empirical evidence are discussed. Thereby, it is pointed out, which further developments regarding statistical theory, strategies in data analysis and computational aspects appear desirable.

1 Introduction

I appreciate the opportunity to give this tutorial on graphical chain models at the 8th Symposium on Computational Statistics in Copenhagen.

Graphical chain models were introduced by Lauritzen and Wermuth (1984) as tools for the analysis of relationships among variables, some of which are qualitative and some quantitative. The investigated relations are symmetric or directed associations taking into account that in some research situations the variables are considered to be on equal footing and in others some variables are regarded as responses, some as potential influencing variables and some as intermediate variables playing the role of both influence and response. As the models apply to data from cross-sectional studies, causal analyses are neither intended nor possible.

*This is the revised version of a tutorial given in Copenhagen

By now, the main aspects of mathematical and statistical theory are well developed (Lauritzen and Wermuth, 1989), Frydenberg and Lauritzen, 1988). First important results for computational procedures to perform maximum likelihood estimation (Frydenberg and Edwards, 1988), as well as corresponding computer programs (Edwards, 1988) are available, and a survey (Lauritzen, 1988) as well as an expository paper have been written (Wermuth and Lauritzen, 1988). Thus, the way has been paved to address those who might actually want to use the models for data analyses and also those who can develop computational procedures and software, still necessary to turn the available methodology into a flexible, user-friendly tool for analysing data.

In view of this, I give only a very brief outline of the theoretical basis (Section 3 and Appendix), while the strategies employed in analysing the available data, as well as underlying principles and intentions are described in some detail (Section 2). In the actual presentation of the rather heterogeneous research questions and analyses (Sections 4 and 6) the same type of strategy has been applied throughout.

2 Employed strategy

Each strategy of evaluating empirical evidence for research purposes comprises the following elements

- formulation of the research purpose,
- judgements on whether the data rather support or rather contradict the hypotheses, or whether the evidence is inconclusive,
- summary of results.

2.1 Formulating the research purpose

The substantive research questions, considered here, concern relations between variables. We call a total set of such relations an *association structure.*

The research purpose can be presented in a number of different ways. Either a mere verbal description of subject-matter theories and their expected implications can be given, or one can attempt to characterise expected structures with the help of graphs or with the help of parameters in statistical models, or both. In each instance, it is necessary to contemplate the implications of alternative formulations in order to be able to decide whether the intended research purpose lends itself at all to an empirical examination and whether some discrepancy has to be expected between what can actually be observed and analysed and what should be investigated.

In this context not only measurement problems regarding the variables are important, but also possible difficulties in adequately representing effects or consequences of a subject-matter formulation in terms of parameters in statistical models. It is my conviction that a judgement on the last aspect requires an intensive communication between statisticians and subject-matter researchers.

One example can illustrate this. In the social sciences it is common to speak of direct as opposed to indirect relations between variables. There are clearly different ways in which these notions may be understood and be translated to have parameter equivalents in statistical models. It is the task of the statistician to state most explicitly the precise meaning of the parameters as well as the restrictions on the parameters, not only in a formal way, but in a language which permits to reach a common understanding between statistician and subject-matter researcher on the appropriateness of the parameters in the given context.

In graphical chain models quantitative properties of observational units correspond to continuous random variables, while qualitative -sometimes also called categorical- variables correspond to discrete random variables. Within that framework substantive research questions of the following two broad types can be studied: (i) a variable pair has an only indirect relation, (ii) a variable pair has a strong direct relation. We use the term *substantive research hypothesis* to remind that both of (i) and (ii) are postulated, while only (i) is assumed in a corresponding statistical model.

In order to obtain a precise meaning of such hypotheses a conscious decision has to be made on which variables are to be considered simultaneously. We call the variables belonging to one such set the *concurrent* variables. Within graphical chain models any indirect and direct relation of a variable pair means: *given all of its remaining concurrent variables.*

If certain distributional assumptions are satisfied, an indirect relation does not only show up in certain parameters being zero, but can also be interpreted as conditional independence given all of the remaining concurrent variables.

By using the notion of conditional independence it becomes possible to see quite distinct parametric situations as analogous ones. This is formalised by using so-called *conditional independence graphs.*

In such graphs discrete variables are drawn as dots and continuous variables as circles. An example with eight variables and three sets of concurrent variables is given in Figure 1. The sets of concurrent variables are fixed with a *dependence chain* which is an ordered partitioning of the set of all variables. The dependence chain in Figure 1 is $\mathcal{C} = (a, b, c)$ where, for instance, the chain element a contains variables A, X, Y. The chain elements are drawn as boxes. The dependence chain in Figure 1 defines three sets of concurrent variables: $a \cup b \cup c$, $b \cup c$ and c. Set a contains responses with potential influences in $b \cup c$, b contains intermediate variables with potential influences in c, and in set c no variable is considered to be a response. Within boxes there is a line connecting two points if a strong direct relation is hypothesised. A line indicates that the variables are thought of as being on equal footing so that their relation is symmetric. If a strong direct relation is hypothesised for variables in different boxes then there is an arrow pointing from the influence to the response. There is at most one connection permitted for each variable pair. Such a connection is called an *edge.*

Two graphs may coincide (i) in the number of variables, (ii) in the number of edges, and (iii) in the type of edge for each variable pair, but differ in the type of involved variables. Then, these graphs have quite different parametric implications, but an association structure with the same interpretation regarding direct and indirect relations. Which parameters are attached to any given conditional independence graph depends mainly on the type of distributions of the involved variables. Examples are given in Sections 4 to 6 and in the Appendix.

2.2 Judgement of empirical evidence

When judgement of empirical evidence is treated here, I concentrate on those aspects that help to decide which relations may be regarded as direct and which as only indirect. This implies in no way that standard checks of plausibility of observed values, of deviations from distributional assumptions or of residual plots are to be neglected. The view is rather that these should supplement the analyses, discussed here. Whenever I had access to the raw

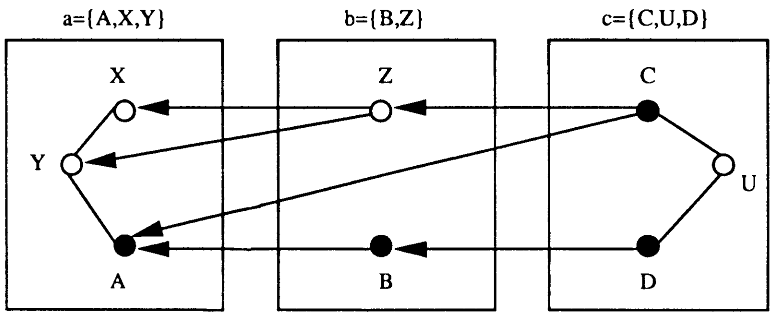

Figure 1: Example with four qualitative variables (A,B,C,D), four quantitative variables (X,Y,Z,U). The dependence chain is $\mathcal{C} = (a, b, c)$. It implies the following three sets of concurrent variables

set 1: all variables $(a \cup b \cup c)$,

set 2: $b \cup c = \{B, Z, C, U, D\}$,

set 3: c.

data, such checks were, in fact, performed.

In order to get to a well-founded decision on those aspects of an association structure, which are defined by complementary sets of direct and of only indirect relations, a fairly standardised strategy was chosen:

- First, a test statistic and a corresponding p-value was computed for each variable pair under the hypothesis that the pair is conditionally independent given its remaining concurrent variables. Small p-values were judged as evidence against an only indirect relation of this pair, high p-values as an indication that the relation might be indirect, and an intermediate value, like $0.01 < p < 0.2$, to permit none of these two conclusions.

- Second, the individual test results were judged in relation to the hypothesised structure. The simplest (and rarest) case occurred when all assumed direct relations corresponded to very small p-values and all postulated indirect relations to very high ones. Then only two tasks remained. It was checked (i) whether a good fit to the observations was preserved after all independencies were assumed simultaneously, and (ii) whether more detailed analyses of any apparently independent pair gave no counterindications. My method of choice for (ii) was to look at studentised interactions, i.e. estimates of interactions divided by their estimated asymptotic standard deviation, computed after utilising results by Dempster (1973).

  If only the postulated direct relations were unsupported by the observations the substantive research hypothesis had to be modified by postulating additional indirect relations.

  On the other hand, if the assumption of any only indirect relation was contradicted by what had been observed, then the hypothesised structure was rejected altogether and a search for a sensible revised hypothesis was performed.

- Third, alternative structures which appeared to fit equally well and structures suggested just by looking at the data, were summarised with the help of corresponding graphs. Their meaning is to be contemplated in discussions with the researchers in the subject-matter areas.

2.3 Summarising results

Summaries of the statistical results concerning association structures should not stop with reporting values of test statistics or corresponding well-fitting graphs. What is actually needed are simple, interpretable data summaries. However, in general, a decision on which measures of association represent the best way to summarise results is not easy. A clear advantage of standardised, sample-size, and unit-independent measures of associations, such as correlations or path coefficients, is that they facilitate a judgement on the relative importance of different variables and on whether the smoothing of data has led to large departures from the observations or not. Such standardised measures are not yet available for pure qualitative or for mixed data, and more experience is necessary regarding the ease in interpreting and communicating different types of summaries.

3 Theoretical basis

The basic distributional assumption used for deriving the theoretical results on graphical chain models is a particular joint distribution of discrete and continuous variables named CG-distribution. It is defined by a conditional joint Gaussian distribution of continuous variables given the discrete variables, and by positive probabilities for each level combination of the discrete variables. As explained in more detail below a graphical chain model is specified by distributional assumptions based on CG-distributions together with a set of conditional independence restrictions.

The use of this distribution for deriving the theoretical results does not imply that graphical chain models should only be used if observable variables follow a CG-distribution, but only that the described techniques are best in a certain sense if this assumption is satisfied. For graphical chain models with only continuous variables the same parameters and estimates can be obtained by just assuming linear relations but not a joint normal distribution (Wermuth 1988).

Graphical chain models may be viewed as a formalisation of common processes in actual data analyses, of looking at levels and differences in means, standard deviations, correlations and counts. The reason is that these familiar measures appear as parameters in a CG-distribution and are estimated

by corresponding sample equivalents.

If the joint distribution of a set of q continuous variables, denoted by $\Gamma = \{X, Y, Z, U, \ldots\}$ and a set of p discrete variables denoted by $\Delta = \{A, B, C, D, \ldots\}$ is a CG-distribution, then the joint density of all variables ($V = \Gamma \cup \Delta$) can be written with the help of moment characteristics. These are the probabilities Π_l, the means $\boldsymbol{\mu}_l = (\mu_l^x, \mu_l^y, \ldots)^T$, and the conditional covariance matrices $\boldsymbol{\Sigma}_l$. The joint density is then a product of conditional Gaussian densities $g_{\Gamma|\Delta}$, and of the marginal probability function $g_\Delta = \Pi_l$:

$$g_V = g_{\Gamma|\Delta} g_\Delta = \left[\left(\frac{1}{\sqrt{2\pi}} \right)^q |\boldsymbol{\Sigma}_l|^{-\frac{1}{2}} exp\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_l)^T \boldsymbol{\Sigma}_l^{-1} (\mathbf{x} - \boldsymbol{\mu}_l)\} \right] \Pi_l$$

where $l = 1, \ldots, L$ denote the level combinations of the discrete variables, $\mathbf{x}^T = (x, y, z, u, \ldots)$ are the realisations of the continuous variables. Occasionally, $\mathbf{K}$ is referred to as concentration matrix having concentrations as off-diagonal elements and precisions along the diagonal.

Equivalently, the logarithm of the density may be written in terms of canonical characteristics as

$$log\, g_V = d_l + \mathbf{h}_l^T \mathbf{x} - \frac{1}{2} \mathbf{x}^T \mathbf{K}_l \mathbf{x},$$

where the discrete, linear and quadratic canonical characteristics are denoted by $d_l, \mathbf{h}_l^T, \mathbf{K}_l$, respectively.

The relations between the two sets of characteristics $(d_l, \mathbf{h}_l, \mathbf{K}_l)$ and $(\Pi_l, \mu_l, \boldsymbol{\Sigma}_l)$ are

$$\begin{aligned} d_l &= \log \Pi_l - \frac{1}{2}\{q \log(2\pi) + \log|\boldsymbol{\Sigma}_l| + \boldsymbol{\mu}_l^T \boldsymbol{\Sigma}_l^{-1} \boldsymbol{\mu}_l\}, \\ \mathbf{h}_l &= \boldsymbol{\Sigma}_l^{-1} \boldsymbol{\mu}_l, \\ \mathbf{K}_l &= \boldsymbol{\Sigma}_l^{-1}. \end{aligned}$$

A CG-regression is a conditional distribution derived from a CG-distribution by conditioning an arbitrary subset of variables on all of the remaining variables. In graphical chain models CG-regressions are used as building blocks.

Let, as an example, $\mathcal{C} = (a, b, c)$ be an ordered partitioning of the variable set $V = a \cup b \cup c$. The assumed joint density f_V of all variables in a graphical chain model can then be written as

$$f_V = g_{a|bc}\; g_{b|c}\; g_c.$$

Here $g_{a|bc}$ and $g_{b|c}$ denote densities of CG-regressions, and g_c the density of a CG-distribution. The dependence chain $\mathcal{C} = (a, b, c)$ defines three sets of concurrent variables: $a \cup b \cup c$, $b \cup c$ and c.

Graphical chain models for f_V result by requiring selected variable pairs to be conditionally independent given the remaining concurrent variables. Such restrictions on a variable pair correspond to zero two-factor and zero higher-order-factor interactions involving this variable pair in a parametrisation of the associated CG-distribution in terms of interactions. An example for this parametrisation in the case of $V = \{A, B, X, Y\}$ and $\mathcal{C} = (V)$ is

$$\begin{aligned} ln\ g(i,j,x,y) &= \lambda + \left(\lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB}\right) \\ &+ \left(\eta^X + \eta_i^{AX} + \eta_j^{BX} + \eta_{ij}^{ABX}\right) x \\ &+ \left(\eta^Y + \eta_i^{AY} + \eta_j^{BY} + \eta_{ij}^{ABY}\right) y \\ &- \frac{1}{2}\left(\psi^X + \psi_i^{AX} + \psi_j^{BX} + \psi_{ij}^{ABX}\right) x^2 \\ &- \frac{1}{2}\left(\psi^Y + \psi_i^{AY} + \psi_j^{BY} + \psi_{ij}^{ABY}\right) y^2 \\ &- \left(\psi^{XY} + \psi_i^{AXY} + \psi_j^{BXY} + \psi_{ij}^{ABXY}\right) xy \end{aligned}$$

Whenever there are no continuous concurrent variables the λ-interactions are identical to interactions in a log-linear model (compare, e.g., Bishop et al., 1975). Whenever there are no discrete concurrent variables the two-factor ψ-interactions are identical to the concentrations in covariance selection models (Dempster, 1972). Mixed interactions, like η_i^{AY}, ψ_i^{AY}, involving both discrete and continuous variables, have not seem to have appeared in other statistical models. Interactions in a CG-distribution are related to, but distinct from interactions in analyses of variance models. For a discussion of this notion in the latter context see Cox (1984).

4 Case studies with only quantitative variables

The examples with quantitative variables only contain

- a simple one where the hypothesised structure fitted well and the results on the association structure could be confirmed in several samples
- another one where the observations contradicted one of the assumed independencies leading to a search for a sensible revised hypothesis.

4.1 Personality characteristics

4.1.1 The substantive research hypothesis

The variables anxiety and anger are of central importance in trying to understand effects of stress and of coping with stressful situations. Spielberger et al. (1970, 1983) have designed questionnaires which are to measure the variables viewed as personality characteristics on the one hand (called trait) and as capturing behaviour considered to be specific to particular situations (called state) on the other hand. A discussion, as to whether the distinction between the two constructs may be adequately measured with questionnaires is ongoing.

If state is measured in a fairly neutral setting then a plausible hypothesis concerning these variables is derived from the following. Associations of a linear type are considered to be appropriate descriptions of pairwise relations between the variables. Expectations regarding the correlation structure of the four variables are : (i) all marginal correlations are positive and of moderate size, while no partial correlation is negative, (ii) emotions in particular situations (states) are influenced by the dispositions (traits) of a person and not conversely, (iii) if either state variable is predicted in terms of the other three variables then there is no direct dependence on the other trait variable.

A reformulation of (i) and (ii) in terms of a graphical chain model is displayed in Figure 2. Another, equivalent formulation (compare Appendix) in terms of the graph of so-called block recursive linear regression equations (Wermuth, 1988) is shown in Figure 3. The parametric implications of this hypothesised structure are expressed with the help of a matrix of linear regression coefficients in Table 1.

Figure 4 and Figure 5 illustrate that the same independencies can be specified in a model with no responses, and also in another one in which trait variables are predicted in terms of state variables. The corresponding statistical models are, in fact, equivalent. This means that parameters and estimates for the one model may be obtained by one-to-one transformations from the other. The equivalence of statistical models in which the role of responses and influences are reversed, explain why causal analyses are not possible.

4.1.2 The empirical evidence and summaries

The data in Table 2 indicate a good agreement between hypothesis and observations, since the two relevant partial correlations are close to zero.

In parametric terms the good fit can be seen either in the covariance matrix, in the concentration matrix, or in the matrix of path coefficients (Table 3), defined as linear regression coefficients of standardised variables. Path coefficients are obtained by computing least squares estimates from the (estimated) correlation matrix.

It has been a traditon in univariate recursive path analysis (Wright, 1934) to draw a graph of the structure (actually a chain graph) and to attach the estimated path coefficients and standardised residual variances to them, as in Figure 6. As shown here, this practice may be retained, when the models are not univariate recursive, but block-recursive.

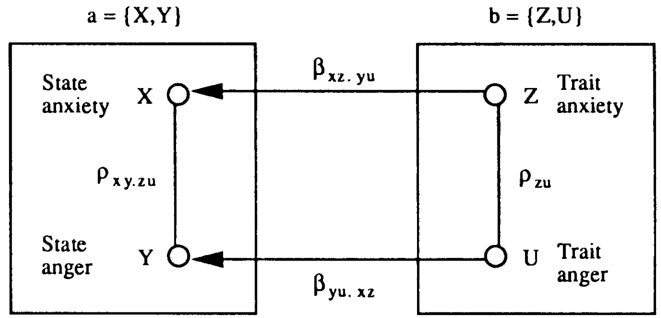

Figure 2: A research hypothesis concerning personality characteristics pertaining to special situations (state) and to typical attributes of the person (trait) expressed as chain graph having chain $\mathcal{C} = (a, b)$ and $X \perp\!\!\!\perp U \mid (Z, Y)$, $Y \perp\!\!\!\perp Z \mid (U, X)$ or $\beta_{xu.yz} = \beta_{yz.xu} = 0$

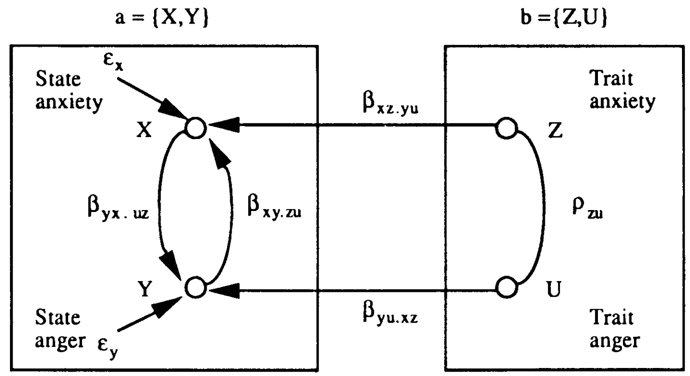

Figure 3: The same research hypothesis but a different parametrisation than in Figure 2 expressed with a graph for block-recursive linear regression equations and $\beta_{xu.yz} = \beta_{yz.xu} = 0$

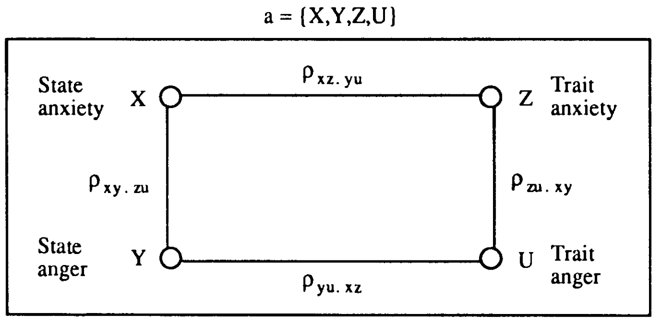

Figure 4: An equivalent statistical model as in Figure 2, but a different research hypothesis, since there are no response variables: $\mathcal{C} = (a)$ and $\rho_{xu.yz} = \rho_{yz.xy} = 0$

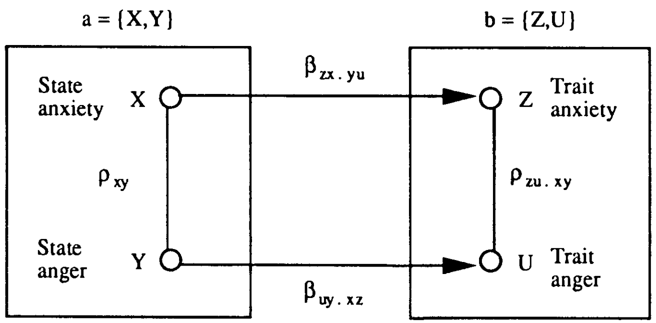

Figure 5: An equivalent statistical model as in Figure 2, but a different research hypothesis, since there is a different set of response variables: $\mathcal{C} = (b, a)$ and $\beta_{zy.ux} = \beta_{ux.zy} = 0$

Table 1: An equivalent hypothesis as in Figure 2 expressed with a corresponding matrix of block-recursive regression coefficients (compare Appendix)

Variable	X State anx	Y State ang	Z Trait anx	U Trait ang
X:= State anxiety	1	$-\beta_{xy.zu}$	$-\beta_{xz.yu}$	0
Y:= State anger	$-\beta_{yx.zu}$	1	0	$-\beta_{yu.xz}$
Z:= Trait anxiety	0	0	1	$-\beta_{zu}$
U:= Trait anger	0	0	$-\beta_{uz}$	1
Variable's chain element	a	a	b	b

Table 2: Observed marginal correlations (lower half) and observed partial correlations given all remaining variables (upper half), n=88 females; Data from Hodapp, 1988

Variable	X State anx	Y State ang	Z Trait anx	U Trait ang
X:= State anxiety	1	.57	.40	**−.11**
Y:= State anger	.71	1	**−.03**	.28
Z:= Trait anxiety	.63	.54	1	.58
U:= Trait anger	.48	.54	.70	1

Table 3: Path coefficients (off-diagonal) and one minus coefficients of determination (diagonal), as observed (first row), and as estimated under the hypothesis of Figure 2 (second row); n=88

Variable	X State anx	Y State ang	Z Trait anx	U Trait ang
X:= State anxiety	.40	.54	.41	−.11
	.41	**.52**	**.35**	**.00**
Y:= State anger	.60	.44	−.03	.27
	.59	**.45**	**.00**	**.23**
Z:= Trait anxiety			.50	.71
			.50	**.71**
U:= Trait anger				.50
				.50

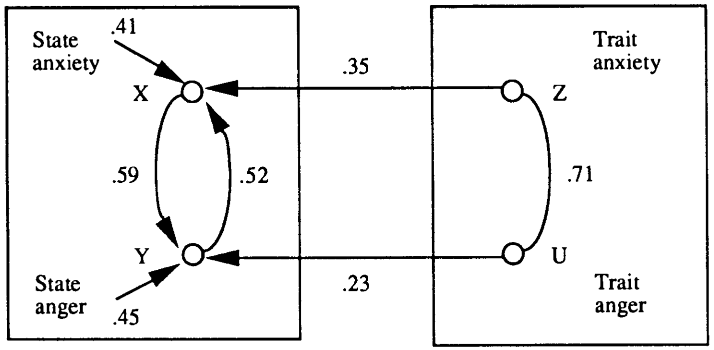

Figure 6: Presentation of results with path coefficients estimated under the hypothesis (Figure 1) (n=88)

4.1.3 An alternative, disjoint hypothesis treatable within the framework of linear structural equations

The hypothesis, treated above, cannot be studied within the context of traditional structural equations (Goldberger, 1964). If one wants to use this latter framework one first has to remove overparametrisation in the equations, leading, for instance, to Figure 7, and one may then impose additional restrictions, to get to a hypothesis as in Figure 8.

Though, in general, in pictures of traditional structural equation models no simple interpretations are possible of missing arrows, they correspond to conditional independencies in this particular example. As these independencies concern two of the marginal distribution and hence do not imply independencies in the joint distribution of all four variables, the specification has been called incomplete, elsewhere (Wermuth, 1988). The test result shows a very poor agreement of this hypothesis with the observations: the value of the likelihood-ratio chi-square statistic on 2 degrees of freedom is 88.8 for $\sigma_{14.3} = \sigma_{23.4} = 0$.

4.1.4 Confirmation of results in further samples

The remaining part in this section presents analogous data and tests for the same variables, but for much larger samples of female (Table 4, Table 5) and male (Table 6, Table 7) college students. It is reassuring to see exactly the same type of structure in both samples. The natural follow-up question is whether sex of the respondents has a moderating effect (compare Wermuth, 1989) on the structure. This leads, for instance, to the hypothesis concerning mixed variables in Figure 9.

The corresponding test results (Table 8) appear to indicate a good fit. The tests have all more than one degree of freedom; thus more detailed analyses become necessary. A look at relevant studentised interactions in Table 9, which should, roughly, behave like standard normal deviates, provides reassuring evidence. Since there is not a single large value (say larger than two), there is no indication for a poor fit.

The results of this analysis are displayed in terms of observed correlations as well as correlations estimated under the hypothesis of Figure 9 in Table 10. It would be preferable to report path coefficients, instead. However, no software is available yet to permit easy calculations.

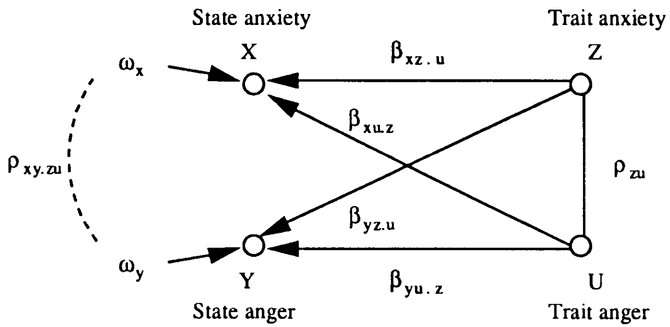

Figure 7: A saturated (exactly identified) structural equation model for the personality characteristics, which is identical to a multivariate regression model

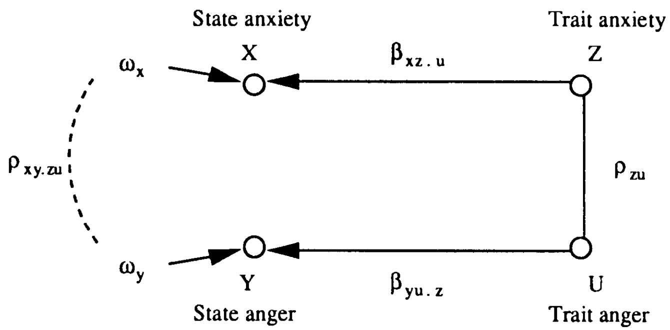

Figure 8: A research hypothesis defined with a structural equation model for personality characteristics implying $\beta_{xu.z} = \beta_{yz.u} = 0$

Table 4: Observed marginal correlations (lower half) and observed partial correlations given all remaining varaiables (upper half), and further data summaries, n=684 females; Data from Spielberger

Variable	X State anx	Y State ang	Z Trait anx	U Trait ang
X:= State anxiety	1	.45	.47	**−.04**
Y:= State anger	.61	1	**.03**	.32
Z:= Trait anxiety	.62	.47	1	.32
U:= Trait anger	.39	.50	.49	1
Mean	18.87	15.23	21.20	23.42
Standard Deviation	6.10	6.70	5.68	6.57

Table 5: Test results for conditional independencies of selected variable pairs, n=684

Pairs	Concurrent variables	Value of chi-square statistic	Degrees of freedom	Corresponding fractile or p-value
(X,Y)	XYZU	153.90	1	<. 001
(X,Z)	XYZU	171.51	1	<. 001
(X,U):=(State anx, Trait ang)	XYZU	1.22	1	**.268**
(Y,Z):=(State ang, Trait anx)	XYZU	0.33	1	**.572**
(Y,U)	XYZU	78.04	1	< .001
(Z,U)	XYZU	72.98	1	< .001
(Z,U)	ZU	189.73	1	< .001
(X,U)&(Y,Z)	XYZU	2.10	2	**.350**

Table 6: Observed marginal correlations (lower half) and observed partial correlations given all remaining variables (upper half), and further data summaries, n=588 males; Data from Spielberger

Variable	X State anx	Y State ang	Z Trait anx	U Trait ang
X:= State anxiety	1	.46	.43	**−.02**
Y:= State anger	.60	1	**.06**	.26
Z := Trait anxiety	.58	.43	1	.25
U:= Trait anger	.31	.41	.40	1
Mean	18.15	14.75	19.58	23.70
Standard Deviation	5.57	6.01	5.22	6.22

Table 7: Test results for conditional independencies of selected variable pairs, n=588

Pairs	Concurrent variables	Value of chi-square statistic	Degrees of freedom	Corresponding fractile or p-value
(X,Y)	XYZU	139.27	1	< .001
(X,Z)	XYZU	122.94	1	< .001
(X,U):=(State anx, Trait ang)	XYZU	0.30	1	**.590**
(Y,Z):=(State ang, Trait anx)	XYZU	2.10	1	**.143**
(Y,U)	XYZU	41.52	1	< .001
(Z,U)	XYZU	38.43	1	< .001
(Z,U)	ZU	100.58	1	< .001
(X,U)&(Y,Z)	XYZU	2.93	2	**.229**

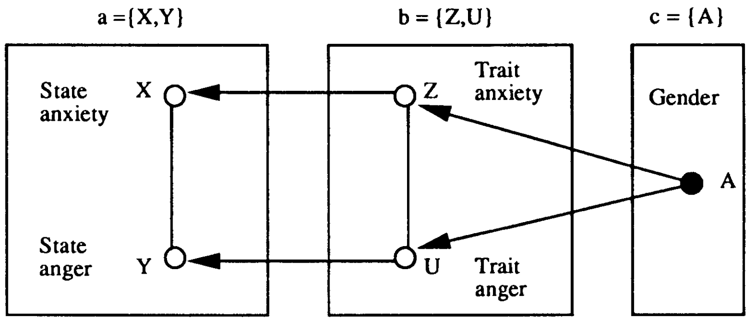

Figure 9: A research hypothesis expressed as chain graph having chain $\mathcal{C} = (a, b, c)$ and implying that the structure among X,Y,Z,U (as expressed in Figure 2) is not moderated by gender of respondent: $(X,Y) \perp\!\!\!\perp A \mid (Z,U)$, $X \perp\!\!\!\perp U \mid (Y,Z)$, and $Y \perp\!\!\!\perp Z \mid (X,U)$

Table 8: Test results for conditional independencies of selected variable pairs, n=1272

Pairs	Concurrent variables	Value of chi-square statistic	Degrees of freedom	Corresponding fractile or p-value
(X,Y)	XYZUA	293.17	2	$< .001$
(X,Z)	XYZUA	294.45	2	$< .001$
(X,U):= (State anx, Trait ang)	XYZUA	1.52	2	**.529**
(X,A):= (State anx, Gender)	XYZUA	3.50	5	**.626**
(Y,Z) := (State ang, Trait anx)	XYZUA	2.43	2	**.296**
(Y,U)	XYZUA	119.56	2	$< .001$
(Y,A):= (State ang, Gender)	XYZUA	6.45	5	**.264**
(Z,U)	XYZUA	111.41	2	$< .001$
(Z,A)	XYZUA	35.05	5	$< .001$
(U,A)	XYZUA	13.84	5	.017
(Z,U)	ZUA	290.31	2	$< .001$
(Z,A)	ZUA	44.15	3	$< .001$
(U,A)	ZUA	14.74	3	.002
(X,U)&(Y,Z)&(X,A)&(Y,A)	XYZUA	13.99	11	**.233**

Table 9: Studentised interactions estimated for level i=1 of A under the saturated model for data on personality characteristics, n=1272

Interaction		Involved variable pair			
type	name	(X,A):= (St anx,Gen)	(Y,A):= (St ang,Gen)	(X,U):= (St anx,Tr ang)	(Y,Z):= (St ang,Tr anx)
mixed	η^{AX}	.96			
linear	η^{AY}		.51		
mixed	ψ^{AX}	1.30			
quadratic	ψ^{AY}		**1.86**		
	ψ^{AXY}	−1.10	−1.10		
	ψ^{AXZ}	.04			
	ψ^{AXU}	−.25		−.25	
	ψ^{AYZ}		.68		.68
	ψ^{AYU}		.72		
pure	ψ^{XU}			1.07	
quadratic	ψ^{YZ}				−1.54

Table 10: Observed correlations (first row) and correlations estimated under the hypothesis of Figure 9 (second row) (compare also Appendix, Example 2)

	A := gender			
	females (n = 684)			
Variable	X State anx	Y State ang	Z Trait anx	U Trait ang
X:= State anxiety	1.00 1.00			
Y:= State anger	.61 **.61**	1.00 1.00		
Z:= Trait anxiety	.62 **.62**	.47 **.45**	1.00 1.00	
U:= Trait anger	.39 **.41**	.49 **.48**	.49 **.49**	1.00 1.00
	males (n = 588)			
Variable	X State anx	Y State ang	Z Trait anx	U Trait ang
X:= State anxiety	1.00 1.00			
Y:= State anger	.60 **.59**	1.00 1.00		
Z:= Trait anxiety	.59 **.58**	.43 **.39**	1.00 1.00	
U:= Trait anger	.31 **.35**	.41 **.43**	.40 **.40**	1.00 1.00

4.2 Blood pressure

In recent work on stress and coping with stressful events it is, typically, attempted to consider the interplay of physiological, emotional, contitutional and environmental variables. From a larger study (Hodapp et al., 1988) a subset of variables was selected for our purpose of investigating the association structure among several quantitative variables. Observations are available for 98 male respondents on

- variables for systolic and diastolic blood pressure, which are based on means of two measurements taken with a sphygmomanometer before and after completing questionnaires,
- variables for two emotions, for anxiety at work and for anger at work, which are defined as sum scores of questionnaires,
- two constitutional factors, on an index for the weight of a respondent defined as quotient of weight in kilogram to height in centimeters and on age, recorded in years.

We assume that the dependence of the physiological variables from both of emotional and constitutional factors is of interest, as well as the type of dependence of the emotions from the constitutional factors alone. This permits to define a dependence chain with three elements for the six variables such that the different chain elements refer to physiological, emotional and constitutional variables, respectively.

The hypothesis displayed as graph in Figure 10 states that the emotions are both direct influences for one of the blood pressure variables, and both of the latter depend on age and weight. Furthermore, both measures for emotions are expected to be dependent on age but not on weight. Observed summary statistics are given in Table 11. The many small values of correlation coefficients suggest that from this sample not all of the hypothesised dependencies will be established as being substantial.

Table 12 shows test results for the data in Table 11. A high value of a chi-square statistic, like 8.51 with one degree of freedom for the hypothesised only indirect relation of pair (Y,U) (having p-value .004) indicates a poor agreement of the hypothesis $\rho_{yu.xzvw} = 0$ with the observations. Given the poor fit for pair (Y,U) alone, the whole hypothesis of Figure 10 has to be revised.

Thus we try to decide for each of the sets of concurrent variables, separately, which relations can be regarded as being only indirect. The four pairs (X,Z), (X,W), (Y,Z) and (Y,V) appear to have zero partial correlations if looked at alone and there is still a good fit (1.94 with p-value .748) if zero partial correlations are requested for all four pairs, simultaneously.

A decision for pair (X,V) is less clear-cut, since looked at alone, there is neither evidence for a strong relation nor for a lack of linear relation (1.72 with p-value .186). The global test statistic for (X,V), (X,Z), (X,W), (Y,Z) and (Y,V) to have zero partial correlations, simultaneously, does not show a poor fit (4.19 having p-value .525), but the contribution of (X,V) to this statistic is not small (2.23 with p-value = .131). If one decides to regard all of these five pairs to have no direct relation then no further pair involving X or Y can be selected, in addition. This follows from the test result for pair (Y,W) and from pairs (X,Y), (X,U), (Y,U) all having larger partial correlations than pair (Y,W).

The value of the statistic for $\rho_{zw.uv} = 0$ in addition to having $\rho_{zv.uw} = 0$ is with 3.73 (having p-value .05) too large to speak for a good fit. These considerations lead to the revised hypothesis about the structure displayed in Figure 11. This graph shows that even when there are only a few variables, pictures of the structures may be drawn in a confusing way. It would be nice to have computer programs, to draw good pictures and to display, automatically, graphs of equivalent models.

Table 13 shows how well the standardised regression coefficients, the path coefficients computed from the observed covariance matrix are in agreement with those estimated under the hypothesis of Figure 11.

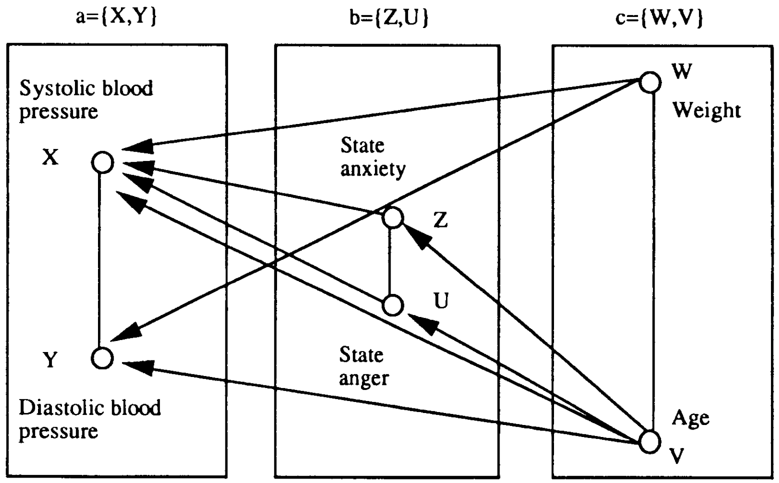

Figure 10: A research hypothesis concerning blood pressure and emotions expressed as chain graph having chain $\mathcal{C} = (a, b, c)$ and $Y \perp\!\!\!\perp (Z, U) \mid (X, V, W)$, $(Z, U) \perp\!\!\!\perp W \mid V$

Table 11: Observed marginal correlations and other summary statistics for n=98 males, Hodapp et al., 1988

Variable	X Syst	Y Diast	Z Anx	U Ang	W Wght	V Age
X := Systolic blood pressure	1					
Y := Diastolic blood pressure	.738	1				
Z := Anxiety at work	-.033	-.059	1			
U := Anger at work	.195	-.042	.353	1		
W:= Weight relative to height	.351	.317	-.102	.211	1	
V := Age	.270	.139	-.058	.283	.390	1
Mean	128.31	85.46	8.23	4.38	.42	32.74
Standard deviation	13.47	11.38	3.43	2.90	.04	11.67

Table 12: Test results for zero partial correlation given all of the remaining concurrent variables; to data of Table 11

Pairs	Concurrent variables	Value of chi-square statistic	Degrees of freedom	Corresponding fractile or p-value
(X,Y)	XYZUVW	76.35	1	<.001
(X,Z) := (Syst, Anx)	XYZUVW	0.67	1	**.581**
(X,U)	XYZUVW	8.14	1	.005
(X,V)	XYZUVW	1.72	1	.186
(X,W):= (Syst, Wght)	XYZUVW	0.25	1	**.625**
(Y,Z) := (Diast, Anx)	XYZUVW	0.58	1	**.549**
(Y,U)	XYZUVW	8.51	1	.004
(Y,V) := (Diast, Age)	XYZUVW	0.42	1	**.526**
(Y,W)	XYZUVW	2.78	1	.092
(Z,U)	ZUVW	17.06	1	<.001
(Z,V) := (Anx, Age)	ZUVW	1.36	1	**.241**
(Z,W)	ZUVW	2.00	1	.153
(U,V)	ZUVW	6.30	1	.012
(U,W)	ZUVW	2.55	1	.106
(V,W)	VW	16.20	1	<.001
(X,Z)&(X,W)&(Y,Z)&(Y,V)	XYZUVW	1.94	4	**.748**
(X,V) in addition	XYZUVW	2.23	1	.131
both of the above	XYZUVW	4.19	5	.525
(Y,W) in addition	XYZUVW	11.59	1	<.001
both of the above	XYZUVW	15.79	6	.015
(Z,W) in addition to (Z,V)	ZUVW	3.73	1	.005
both	ZUVW	5.10	2	.076

Table 13: Path coefficients as observed (first row) and as estimated (second row) under the hypothesis of Figure 11

Variable	X Syst	Y Diast	Z Anx	U Ang	W Wght	V Age
X := Syst	1.000	-.719	.057	-.212	-.036	-.093
	1.000	**-.733**	**.000**	**-.206**	**.000**	**-.071**
Y := Diast	-.752	1.000	-.054	.221	-.124	.047
	-.738	1.000	**.000**	**.216**	**-.139**	**.000**
Z := Anx			1.000	-.417	.143	.120
			1.000	**-.392**	**.185**	**.000**
U := Ang			-.383	1.000	-.155	-.245
			-.362	1.000	**-.169**	**-.203**
W:= Weight					1.000	-.390
					1.000	**-.390**
V := Age					-.390	1.000
					-.390	1.000

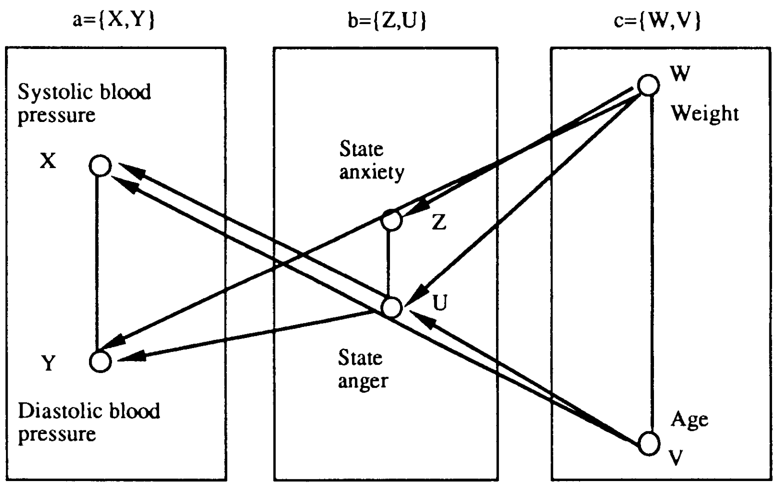

Figure 11: A revised research hypothesis concerning blood pressure and emotions expressed as chain graph having chain $\mathcal{C} = (a, b, c)$

5 A case study for qualitative variables: perinatal mortality

Since the analysis of only categorical data is well developed (see, e.g., Bishop et al.), we give only a simple example, one which indicates that a conditional independence may show up even when the total sample size becomes extremely large.

A hypothesis concerning the relative importance of two alternative potential influences for perinatal mortality is displayed in Figure 12. Definitions for the categories of these variables and of a related risk factor are presented in Table 15. The increase in risk (Table 16) for perinatal mortality by a factor of more than four for those women, who are unable to report the survival status of their previous child, is to be compared with the risks for women never pregnant before (Table 17) and with those due to known risk factors like vaginal bleeding (Table 18). The test results (Table 14) as well as inspection of standardised interactions (Table 19) indicate clearly that skin colour is by far the inferior of the two considered influences for perinatal mortality.

A further known aspect of analyses with very large sample sizes is illustrated with the marginal table (Table 20) concerning survival status of previous child and skin color: though no appreciable differences in rates are observed, the studentised interactions become rather large. The reason is that this standardisation is sample size-dependent. This is just one instance illustrating that statistical significance need not coincide with subject matter relevance.

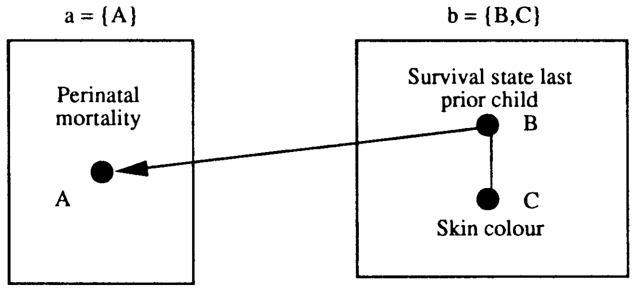

Figure 12: A research hypothesis expressed with a graph having chain $\mathcal{C} = (a, b)$ implying $\Pi_{i|jk} = \Pi_{i|j}$

Table 14: Tests for conditional independencies, 13801 women, NIH

Pair	Concurrent variables	Value of chi-square statistic	Degrees of freedom	Corresponding fractile or p-value
(A,B)	ABC	279.12	8	<.001
(A,C)	ABC	6.03	5	**.302**
(B,C)	ABC	71.99	8	<.001
(B,C)	BC	68.81	4	<.001

Table 15: Definition of categories of variables in the mortality data

A:	Perinaltal death
i=1:	yes
i=2:	no
B:	Survival state of last prior child
j=1:	living
j=2:	child death
j=3:	fetal death
j=4:	neonatal death
j=5:	unknown
C:	Skin colour of woman
k=1:	light
k=2:	dark
D:	Occurrence of vaginal bleeding
l=1:	never during pregnancy
l=2:	first trimester
l=3:	second trimester
l=4:	third trimester

Table 16: Counts and other data summaries for perinatal deaths (A), survival state of last prior child (B), and skin colour of woman(C); Data for 13801 women, National Institutes of Health (1972) p.187

Levels			Observed	Observed	Estimates under $\Pi_{i\|jk} = \Pi_{i\|j}$		
A	B	C	count	‰-rate	count	‰-rate	relative risk
1	1	1	270	28.7	297.5	31.6	
2	1	1	9148		9120.5		
1	2	1	3	27.0	3.4	30.9	1
2	2	1	108		107.6		
1	3	1	134	74.0	132.8	73.3	2.3
2	3	1	1678		1679.2		
1	4	1	17	89.5	19.3	101.5	3.2
2	4	1	173		170.7		
1	5	1	56	125.8	59.3	133.4	4.2
2	5	1	389		385.7		
1	1	2	371	34.1	343.5	31.6	
2	1	2	10502		10529.5		
1	2	2	5	33.6	4.6	30.9	1
2	2	2	144		144.4		
1	3	2	154	72.2	155.2	73.3	2.3
2	3	2	1963		1961.8		
1	4	2	37	108.2	34.7	101.5	3.2
2	4	2	305		307.3		
1	5	2	46	143.8	42.7	133.4	4.2
2	5	2	274		277.3		

Table 17: Counts and other data summaries for perinatal deaths (A) and skin colour (C), women never pregnant before; Data for 13438 women, NIH, p.187

Levels A	Levels C	Observed count	Observed ‰-rate
1	1	188	26.6
2	1	6884	
1	2	232	36.4
2	2	6134	

Table 18: Counts and other data summaries for perinatal deaths (A) and time of first occurrence of vaginal bleeding during pregnancy (D) for woman with light skin colour; 19048 women, NIH, p.399

Levels A	Levels D	Observed count	Observed ‰-rate
1	1	304	21.4
2	1	13876	
1	2	172	77.5
2	2	2047	
1	3	113	128.9
2	3	764	
1	4	79	44.6
2	4	1693	

Table 19: Studentised interactions under the saturated model for mortality data; 13801 women, NIH

Levels A	Levels B	Levels C	Two-factor λ's	Levels A	Levels B	Levels C	Three-factor λ's
1	1		**−8.12**	1	1	1	−.17
1	2		**−2.57**	1	2	1	−.12
1	3		1.72	1	3	1	.87
1	4		**3.34**	1	4	1	−.21
1	5		7.13	1	5	1	−.02
1		1	−.89				
	1	1	.08				
	2	1	−.55				
	3	1	.90				
	4	1	**−3.00**				
	5	1	**4.30**				

Table 20: Counts and other data summaries for survival state of last prior child (B) and skin colour of woman (C); 13801 women, NIH

Levels B	Levels C	Observed count	Observed % -rate	estimates under $\Pi_{j\|k} = \Pi_j$ % -rate	estimates under $\Pi_{j\|k} = \Pi_j$ studentised λ^{BC}
1	1	9418	78.6	78.8	.75
2	1	111	0.9	1.0	−1.21
3	1	1812	15.1	15.3	.35
4	1	190	1.6	2.1	**−5.36**
5	1	445	3.7	3.0	**7.56**
1	2	10873	78.8		
2	2	149	1.1		
3	2	2117	15.3		
4	2	342	2.5		
5	2	320	2.3		

6 A case study with both qualitative and quantitative variables

This last section presents first analyses of a type of structure which is the most interesting in the sense that adequate statistical models to analyse it have been lacking in the past.

In Kohlmann et al. (1988) it is well described how psychologists develop theories concerning possible consequences for the children if parents employ particular educational styles. We use here just four of their variables, all of which are observed as sum scores of questionnaires. The relevant part of the theory concerning the four variables, for which summary statistics are displayed in Table 21, may be summarised as follows.

Inconsistent behaviour of a parent is expected to increase trait anxiety in the child. If a particular coping strategy with stress, called high sensitisation, is preferred, than more anxiety is expected to be reported by the child than if sensitisation is low. It is further hypothesised that supportive behavior of one parent may decrease the consequences of inconsistent behavior of the other parent. A direct relation between the coping strategy and the personality characteristic anxiety is assumed, while for a prediction of sensitisation the supportive behavior of a parent is considered to be not directly relevant.

One translation of parts of these expectations into a graphical chain model is given with Figure 6. Unfortunately, no computer programs are available yet to estimate and test all parameters in this model. Also, more research is needed on easily interpretable measures of association.

We have proceeded to analyse only undirected relations and found the structure displayed in Figure 14 to be well-fitting. Whether a sensible subject-matter interpretation coresponding to this structure may be found still needs to be discussed with psychologists.

The decision on the good fit was based on the usual tests for variable pairs (compare Table 22) and on the more detailed analyses in terms of studentised interactions (compare Table 23). The latter reveal that a good fit to indirect relations can at most be assumed for two variable pairs, for (A,B) and for (B,Y). All other pairs have at least one relatively large studentised interaction, one with value larger than two.

Qualitatively similar results on the association of pair (A,B) are obtained in an analysis of anxiety (X) as response and A,Y, and B as influences within the context of a linear regression with indicator variables. Such an analysis

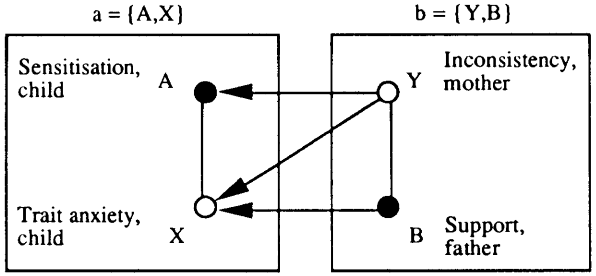

Figure 13: A research hypothesis for the educational styles defined with a graph having chain $\mathcal{C} = (a, b)$, implying $A \perp\!\!\!\perp B \mid (X, Y)$ or $\Pi_{i|xyj} = \Pi_{i|xy}$

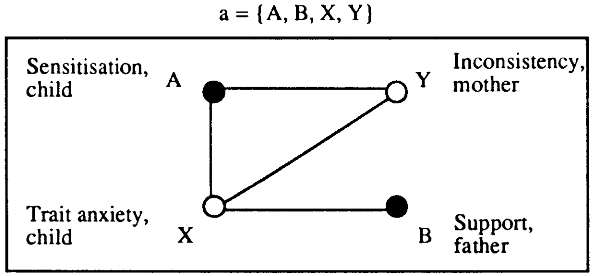

Figure 14: A structure present in the data on educational styles, reflected in the undirected graph, implying $B \perp\!\!\!\perp (A, Y) \mid X$ or $\Pi_{j|xyi} = \Pi_{j|x}$

is possible since the test for homogeneity of the residual variance of X given A,Y,B (Table 24) indicates a reasonable agreement with this homogeneity, i.e. with an essential assumption in regressions with indicator variables. In this regression the interactions ABY and AB can be set to zero while the interactions AY and BY cannot be set to zero . This implies for a corresponding joint CG-distribution that the interactions involving pair (A,B) are zero while the three-factor interactions AXY and BXY are nonzero.

The observed summary statistics displayed in Table 21 as well as the ones obtained after assuming conditional independence just for the pair (A,B) (no direct dependence of the coping strategy sensitisation on support), displayed in Table 25, show an amazing agreement to the hypothesised strengths and directions of associations.

Table 21: Simple correlation coefficients and further summary statistics for the data on educational styles, n=117; Data from Kohlmann et. al. (1988)

	B:= Supportive behavior father							
	low				high			
	A:= Sensitisation, child				A:= Sensitisation, child			
	low		high		low		high	
Continuous variables	X	Y	X	Y	X	Y	X	Y
X:= Anxiety, child	1		1		1		1	
Y:= Inconsistency, mother	.80	1	.54	1	.48	1	.25	1
Mean	28.55	23.36	33.62	25.41	26.10	21.20	30.81	23.48
Standard deviation	8.15	6.98	6.42	6.59	4.30	4.43	5.24	5.81
Count	22		39		29		27	

Table 22: Test results for conditional independencies of selected variable pairs, n=117

Pairs	Concurrent variables	Value of chi-square statistic	Degrees of freedom	Corresponding fractile or p-value
(A,X)	ABXY	23.22	6	$< .001$
(A,Y)	ABXY	7.58	6	**.270**
(A,B):=(Sens,Sup)	ABXY	4.46	6	**.616**
(X,Y)	ABXY	45.09	4	$< .001$
(B,X)	ABXY	10.61	6	.101
(B,Y):=(Sup,Inc)	ABXY	2.78	6	**.837**
(B,Y)	BY	8.03	2	.018
(A,B)&(B,Y)	ABXY	7.12	9	**.625**
(A,B)&(B,Y)&(A,Y)	ABXY	14.30	12	
(A,B)&(B,Y)&(B,X)	ABXY	22.42	11	

Table 23: Studentised interactions estimated for levels i=1 of A and j=1 of B under the saturated model for data on educational styles, n=117

Interaction		Involved variable pair					
type	name	(A,B):= (Sens,Sup)	(A,X):= (Sens,Anx)	(A,Y):= (Sens,Inc)	(B,X):= (Sup,Anx)	(B,Y):= (Sup,Inc)	(X,Y):= (Anx,Inc)
pure discrete	λ^{AB}	.72					
mixed linear	η^{AX}		−.49				
	η^{BX}				**−2.07**		
	η^{ABX}	−1.01	−1.01		−1.01		
	η^{AY}			.19			
	η^{BY}					−1.22	
	η^{ABY}	−.14		−.14		−.14	
mixed quadratic	ψ^{AX}		1.47				
	ψ^{BX}				−1.27		
	ψ^{ABX}	−.94	−.94		−.94		
	ψ^{AY}			**2.19**			
	ψ^{BY}					−.29	
	ψ^{ABY}	−.37		−.37		−.37	
	ψ^{AXY}		**−2.07**	**−2.07**			**−2.07**
	ψ^{BXY}				−.75	−.75	−.75
	ψ^{ABXY}	.15	.15	.15	.15	.15	.15
pure quadratic	ψ^{XY}						**−4.59**

Table 24: Tests against the saturated model in regressions with indicator variables for educational styles data with X:= Anxiety as response and with influences: Y:= Inconsistency, mother, A:= Sensitisation, child and B:= Support,father

GLIM-notation for model	Coefficient of determination R^2	Value of chi-square statistic	Degrees of freedom
$A*B*Y$	.482	5.35[1]	3
$A*B+A*Y+B*Y$	.481	5.61	4
$A*Y+B*Y$	.481	5.65	5
$A*Y+B$	.455	11.53	6
$B*Y+A$	.458	10.68	6

1) The value of the test statistic for variance homogeneity is 5.35, the value for any other model (M) is obtained as: $5.35 + n\log\{(1-R^2_M)/(1-R^2_{A*B*Y})\}$ with n = 117

Table 25: Estimated correlations and other summaries under the hypothesis $\Pi_{i|jxy} = \Pi_{i|xy}$, n=117

	B:= Supportive behavior father							
	low				high			
	A:= Sensitisation, child				A:= Sensitisation, child			
	low		high		low		high	
Continuous variables	X	Y	X	Y	X	Y	X	Y
X:= Anxiety, child	1		1		1		1	
Y:= Inconsistency, mother	.78	1	.52	1	.56	1	.23	1
Mean	28.92	22.69	34.27	25.98	26.37	21.64	30.21	22.91
Standard deviation	7.47	6.58	6.39	6.613	5.05	4.82	4.89	5.57
Count	24.25		36.75		26.75		29.25	

A Illustration for the derivation of block-recursive linear regression equations

Let the positive definite covariance matrix Σ of the random vector X and its inverse, the concentration matrix Σ^{-1}, be partitioned according to the definition of the chain elements in $\mathcal{C} = (a, b, c)$

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} & \Sigma_{ac} \\ \Sigma_{ba} & \Sigma_{bb} & \Sigma_{bc} \\ \Sigma_{ca} & \Sigma_{cb} & \Sigma_{cc} \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} \Sigma^{aa} & \Sigma^{ab} & \Sigma^{ac} \\ \Sigma^{ba} & \Sigma^{bb} & \Sigma^{bc} \\ \Sigma^{ca} & \Sigma^{cb} & \Sigma^{cc} \end{pmatrix}.$$

Note known relations (see, e.g., Wermuth, 1988b) such as

$$\Sigma^{bb.a} = (\Sigma_{bb.c})^{-1}$$

with

$$\begin{aligned} \Sigma^{bb.a} &= \Sigma^{bb} - \Sigma^{ba}(\Sigma^{aa})^{-1}\Sigma^{ab}, \\ \Sigma_{bb.c} &= \Sigma_{bb} - \Sigma_{bc}\Sigma_{cc}^{-1}\Sigma_{ca}. \end{aligned}$$

For the given order (a,b,c) there exists a unique block-triangular decomposition of Σ^{-1}:

$$\Sigma^{-1} = A^T T^{-1} A$$

with

$$T^{-1} = \begin{pmatrix} \Sigma^{aa} & 0 & 0 \\ 0 & \Sigma^{bb.a} & 0 \\ 0 & 0 & \Sigma^{cc.ab} \end{pmatrix}, \quad A = \begin{pmatrix} I_a & (\Sigma^{aa})^{-1}\Sigma^{ab} & (\Sigma^{aa})^{-1}\Sigma^{ac} \\ 0 & I_b & (\Sigma^{bb.a})^{-1}\Sigma^{bc.a} \\ 0 & 0 & I_c \end{pmatrix}.$$

Block-recursive concentration equations are defined (Wermuth, 1988) as

$$B^*(X - \mu) = \epsilon^* \qquad (1)$$

with

$$B^* = T^{-1}A = \begin{pmatrix} \Sigma^{aa} & \Sigma^{ab} & \Sigma^{ac} \\ 0 & \Sigma^{bb.a} & \Sigma^{bc.a} \\ 0 & 0 & \Sigma^{cc.ab} \end{pmatrix}, \quad B^* \begin{pmatrix} \mu_a \\ \mu_b \\ \mu_c \end{pmatrix} = \begin{pmatrix} h^a \\ h^{b.a} \\ h^{c.ab} \end{pmatrix}$$

where for instance $h^{b.a}$ are the linear canonical characteristics obtained after having marginalised over the variables in set a. These equations imply

$$\begin{aligned} E(\epsilon^*) &= B^*(E(X) - \mu) \\ &= 0 \qquad (2) \\ var(\epsilon^*) &= T^{-1}A\ var(X) A^T T^{-1} \\ &= T^{-1}. \end{aligned}$$

Then, block-recursive linear regression equations are obtained by dividing each of the concentration equations by a precision, more precisely, by the corresponding diagonal element of B^*. This gives

$$B(X - \mu) = \epsilon, \qquad (3)$$

where B contains partial regression coefficients and $B\mu$ intercepts of regression lines

Example 1: For $a = \{X, Y\}$, $b = \{Z, U\}$, $c = \emptyset$

$$B^* = \begin{pmatrix} \sigma^{xx} & \sigma^{xy} & \sigma^{xz} & \sigma^{xu} \\ \sigma^{yx} & \sigma^{yy} & \sigma^{yz} & \sigma^{yu} \\ 0 & 0 & \sigma^{zz.xy} & \sigma^{zu.xy} \\ 0 & 0 & \sigma^{uz.xy} & \sigma^{uu.xy} \end{pmatrix}, \quad B^* \begin{pmatrix} \mu_x \\ \mu_y \\ \mu_z \\ \mu_u \end{pmatrix} = \begin{pmatrix} h^x \\ h^y \\ h^{z.xy} \\ h^{u.xy} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -\beta_{12.34} & -\beta_{13.24} & -\beta_{14.23} \\ -\beta_{21.34} & 1 & -\beta_{23.14} & -\beta_{24.13} \\ 0 & 0 & 1 & -\beta_{34} \\ 0 & 0 & -\beta_{43} & 1 \end{pmatrix}, \quad B\mu = \begin{pmatrix} \alpha_{1.234} \\ \alpha_{2.134} \\ \alpha_{3.4} \\ \alpha_{4.3} \end{pmatrix}.$$

Example 2: For $a = \{X, Y\}$, $b = \{Z, U\}$ and $c = \{A\}$, the saturated block-recursive concentration equations are given by

$$B^*(i) = \begin{pmatrix} \sigma^{xx}(i) & \sigma^{xy}(i) & \sigma^{xz}(i) & \sigma^{xu}(i) \\ \sigma^{yx}(i) & \sigma^{yy}(i) & \sigma^{yz}(i) & \sigma^{yu}(i) \\ 0 & 0 & \sigma^{zz.xy}(i) & \sigma^{zu.xy}(i) \\ 0 & 0 & \sigma^{uz.xy}(i) & \sigma^{uu.xy}(i) \end{pmatrix}, B^* \begin{pmatrix} \mu_x \\ \mu_y \\ \mu_z \\ \mu_u \end{pmatrix} = \begin{pmatrix} h^x(i) \\ h^y(i) \\ h^{z.xy}(i) \\ h^{u.xy}(i) \end{pmatrix}.$$

This is also a saturated model belonging to the hypothesis of Figure 8. The block-recursive linear regression equations corresponding to the hypothesis of Figure 8 are then given by

$$B(i) = \begin{pmatrix} 1 & -\beta_{xy.zu} & -\beta_{xz.yu} & 0 \\ -\beta_{yx.zu} & 1 & 0 & -\beta_{yu.xz} \\ 0 & 0 & 1 & -\beta_{zu}(i) \\ 0 & 0 & -\beta_{uz}(i) & 1 \end{pmatrix}, \quad B\mu = \begin{pmatrix} \alpha_{x.yzu} \\ \alpha_{y.xzu} \\ \alpha_{z.u}(i) \\ \alpha_{u.z}(i) \end{pmatrix}$$

In contrast to traditional structural equations (Goldberger, 1964) no overparametrisation occurs in block-recursive linear regression equations. The reason is that the particular definition of B^* (compare Equation 1) and hence of B completely determines the covariance matrix of the residuals ϵ (compare Equation 2). Also, as a consequence no problems of identification can occur if Σ and the observed covariance matrix are positive definite. Maximum-likelihood estimates of structural regression equations can, in general, not be found by minimising residuals in each equation, separately. Instead an iterative algorithm (Frydenberg and Edwards, 1988) is needed.

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