Student (1999). Vol.3. No.1, 41-43

Proof Without Words: Sum of Squared Integers

Nanny Wermuth and Hans-Jürgen Schuh

Johannes-Gutenberg Universität, Mainz, Germany

When a treatment effect is to be juged from a random sample on paired observations, like measurements taken for each patient before and after treatment, the judgement may be based on rankings of the size of differences (Wilcoxon, F. (1945). Individual comparisons by ranking methods. *Biometrics*, **1**, 80-83.).

In samples of n nonzero differences the signed rank statistic, W, is a sum of variables which have values $+i$, $-i$, for $i = 1, \ldots, n$, where i denote ranks assigned to absolute values of differences. Here is a small example how to compute an observed w of W and related quantities.

l	x_l	y_l	$x_l - y_l$ $= d_l$	$\|d_l\|$	Rank $\|d_l\|$ i	Signed rank: r_i
1	10	5	5	5	4	4
2	12	10	2	2	1	1
3	5	9	-4	4	3	-3
4	9	6	3	3	2	2

$$w = \sum r_i = 4,\ w_{pos} = 7,\ w_{neg} = 3,\ w_{smaller} = 3.$$

In the case of no treatment effect positive and negative signs are equally probable for each rank, therefore the mean and variance of W is obtained as $E(W) = 0$ and $var(W) = \sum i^2$ (Mosteller, F. & Rourke, R.E.K. (1973). *Sturdy Statistics*. Reading: Addison-Wesley.).

For large n the test statistics is based, equivalently, either on the negative ranks W_{neg} or the positive ranks W_{pos}, i.e. for ease of computation the one based on the smaller number of signs can be chosen. An approximately standard normal test statistic results as:

$$\frac{W_{pos} - E(W_{pos})}{var(W_{pos})^{1/2}} = \frac{W_{pos} - n(n+1)/4}{\sqrt{(2n+1)(n+1)n/24}},$$

where

$$\begin{aligned} W_{pos} &= W + W_{neg} \\ 2W_{pos} &= W + W_{neg} + W_{pos} = W + \sum i \\ E(W_{pos}) &= \frac{1}{2}\left[E(W) + \sum i\right] = \frac{1}{2}\sum i \\ var(W_{pos}) &= \frac{1}{4}var(W) = \frac{1}{4}\sum i^2. \end{aligned}$$

To see that

$$\sum i = \frac{n(n+1)}{2}$$

an anecdote about C.F. Gauss can be used. To keep Carl Friedrich, at age of 7, quiet for some time his teacher asked him to add the numbers from 1 to 500. However this he did fast as follows:

$$\begin{array}{cccccc} 1 & 2 & 3 & \cdots & 498 & 499 & 500 \\ 500 & 499 & 498 & \cdots & 3 & 2 & 1 \\ \hline 501 & 501 & 501 & \cdots & 501 & 501 & 501 \end{array}$$

giving

$$2\sum_{i=1}^{i=500} i = 500 \cdot 501,$$

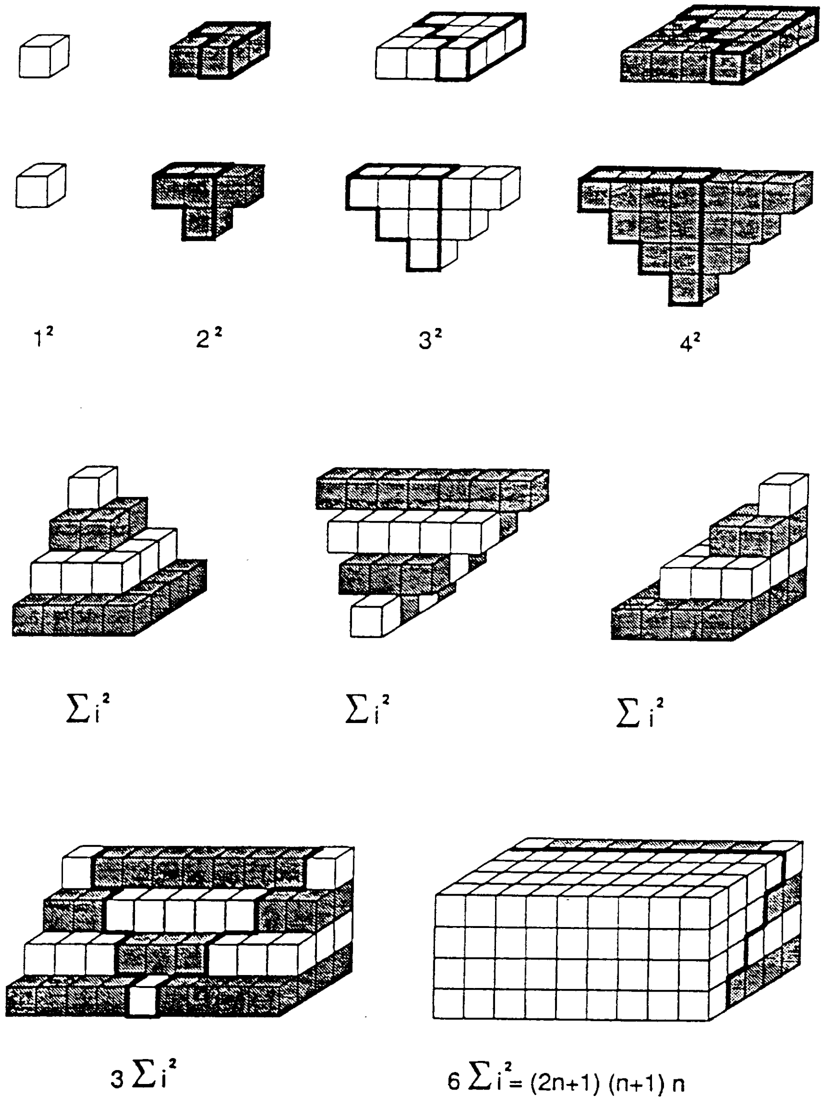
so that, in general

$$2\sum i = n(n+1).$$

The value of $\sum i^2$ is derived in the following 'Proof without Words'.

Proof Without Words: Sum of Squared Integers

$$\sum i^2 = (2n+1)(n+1)n/6$$



— NANNY WERMUTH, HANS-JÜRGEN SCHUH
JOHANNES–GUTENBERG UNIVERSITÄT
D-55099 MAINZ, GERMANY