Strong Markov property of Poisson processes and Slivnyak formula

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Summary. We discuss strong Markov property of Poisson point processes and the related stopping sets. Viewing Poisson process as a set indexed random field, we demonstrate how the martingale technique applies to establish the analogues of the classical results: Doob's theorem, Wald identity in this multi-dimensional setting. In particular, we show that the famous Slivnyak-Mecke theorem characterising the Poisson process is a consequence of the strong Markov property.

Keywords: Poisson point process, Slivnyak formula, Strong Markov property, Gamma-type result

1 Filtrations and stopping sets

To outline the idea of this paper, let us start with an example of a temporal stochastic process, i. e. a random function $\xi_\bullet(\omega) = \{\xi_t(\omega)\}_{t\geq 0}$, $\omega \in \Omega$ indexed by one-dimensional parameter $t \geq 0$ which we refer as *time*. Surely, this map from sample space Ω into the appropriate function space over $[0,\infty)$ should be measurable with respect to a suitably chosen σ-algebra. However, such a definition is usually too general as it does not describe the temporal evolution of $\xi_\bullet$. Therefore it is useful to define a growing sequence of σ-algebras $\mathcal{F}_{[0,s]}$ of subsets of Ω representing the process' history up to time s, and impose the condition that the restriction of $\xi_\bullet$ onto time interval $[0,s]$, i. e. the function $\{\xi_t(\omega)\}_{t\in[0,s]}$, should be $\mathcal{F}_{[0,s]}$-measurable for all $s \geq 0$. Of course, this is a stronger notion of measurability for the random function which is called *progressive measurability*. The system of growing σ-algebras $\mathcal{F}_{[0,s]}$ is called *filtration*.

One of the central notions for temporal processes is the *stopping time*. It is a random variable τ such that event $\{\omega \in \Omega : \tau(\omega) \leq s\}$ is $\mathcal{F}_{[0,s]}$-measurable for all $s \geq 0$. In words, the fact that τ is observed before time s is defined only by the history $\mathcal{F}_{[0,s]}$ up to time s only. With every stopping time one may associate the corresponding stopping σ-algebra which is the collection of events

$$\mathcal{F}_\tau = \{\Sigma \in \mathcal{F}_{[0,\infty]} :\ \Sigma \cap \{\omega :\ \tau(\omega) \le s\} \in \mathcal{F}_{[0,s]} \text{ for all } s \ge 0\}\,. \tag{1}$$

The main object of our study here are point processes in a general space. We shall see how far we can mimic the above objects in this intrinsically multi-dimensional setting. We treat point processes as random countable measures and as we will see, their usual definition actually assumes the progressive measurability. Specifically, let X be a locally compact separable topological space (LCS-space) which we call a *phase space* of the process and $\mathcal{B}$ be its Borel σ-algebra. X plays the role of the index set $[0,\infty)$ above – we typically consider $X = \mathbb{R}^d$ for simplicity. Let $\mathcal{N}$ be a set of counting measures on $\mathcal{B}$, so that a measure $\phi \in \mathcal{N}$, if $\phi(B) \in \{0,1,2,\dots\} = \mathbb{Z}_+$ for any Borel B. Any such measure can be represented as the sum of unit masses: $\phi = \sum_i \delta_{x_i}$, where x_i are not necessarily different and $\delta_x(B) = 1\!\!1_{x\in B}$. We call the support points *particles*.

A point process $N = N(\omega)$ is a $[\mathcal{F}, \Xi]$-measurable mapping from some abstract probability space $(\Omega, \mathcal{F}, \mathbf{P})$ into the measurable space $[\mathcal{N}, \Xi]$ of counting measures. σ-algebra Ξ is generated by the sets of the type $\{\phi \in \mathcal{N} : \phi(B) = k\},\ B \in \mathcal{B},\ k \in \mathbb{Z}_+$. This is a natural definition of measurability for point processes as this makes the events of type $\{\omega \in \Omega :\ N(\omega, B) = k\}$ measurable. Often $[\Omega, \mathcal{F}]$ is taken to be $[\mathcal{N}, \Xi]$ itself and N is identity mapping. Such processes are called *canonically defined*. From now on we consider canonically defined processes and write ϕ (a point configuration) instead of ω to stress that and give up notation Ξ in favour of $\mathcal{F}$.

The *intensity measure* of a point process $N = N(\phi)$ defined on Borel $B \in \mathcal{B}$ as $\lambda_N(B) = \mathbf{E}N(B)$. The *Campbell measure* is a measure $\mathcal{C}(d\phi\, dx)$ on $\mathcal{F} \otimes \mathcal{B}$ defined on $\Sigma \times B$ as $\mathcal{C}(\Sigma \times B) = \mathbf{E}N(\phi, B)\, 1\!\!1_{\phi\in\Sigma}$. We observe that $\mathcal{C}(\Sigma \times \bullet)$ as a measure on $\mathcal{B}$ is absolutely continuous with respect $\lambda_N(dx)$, thus there exists a Radon-Nikodym derivative $\mathbf{P}_N^x(\Sigma)$ which is a measurable function of $x \in X$, but which can also be chosen to be a probability measure on $[\mathcal{N}, \mathcal{F}]$ called the *Palm distribution* corresponding to N at x. By definition the following identity called *refined Campbell theorem* holds:

$$\mathbf{E}\int F(\phi, x)\, N(dx) = \int \mathbf{E}_N^x F(\phi, x)\, \lambda_N(dx) \tag{2}$$

for any measurable function F. The Palm measure $\mathbf{P}_N^x$ is concentrated on configurations ϕ such that $\phi(\{x\}) > 0$ and can be regarded as a distribution of a random configuration conditioned on having a particle at x.

Let $\mathbb{F}, \mathbb{K}$ be the system of closed and compact subsets of X respectively. Then for every $K \in \mathbb{K}$ one may define the σ-algebra $\mathcal{F}_K$ which is generated by the sets $\{\phi \in \mathcal{N} :\ \phi(B \cap K) = k\},\ B \in \mathcal{B},\ k \in \mathbb{Z}_+$. Similarly to one-dimensional case, the following properties allow us to call the system $\{\mathcal{F}_K\},\ K \in \mathbb{K}$ a *filtration*:

- monotonicity: $\mathcal{F}_{K_1} \subseteq \mathcal{F}_{K_2}$ for any two compact $K_1 \subseteq K_2$;
- continuity from above: $\mathcal{F}_K = \cap_{n=1}^{\infty} \mathcal{F}_{K_n}$ if $K_n \downarrow K$.

By construction, the restriction of the point process N onto K is $\mathcal{F}_K$-measurable, so N is automatically progressively measurable and $\{\mathcal{F}_K\},\ K \in \mathbb{K}$ is thus the *natural filtration* associated with the process. We see a complete analogy when one-dimensional parameter t – time is replaced now by a compact set K. To pursue this analogy we need a notion of a random compact set which supersedes a random time.

A *random closed set* $\mathcal{N}$ is a measurable mapping $\mathcal{N}: \ [\mathcal{N}, \mathcal{F}] \mapsto [\mathbb{F}, \sigma_f]$, where σ_f is the σ-algebra generated by the system $\{F \in \mathbb{F}: \ F \cap K \neq \emptyset\},\ K \in \mathbb{K}$.

A random compact set $S = S(\phi)$ is called a *stopping set* (more precisely, $\{\mathcal{F}_K\}$-stopping set) if the event $\{\phi : S(\phi) \subseteq K\}$ is $\mathcal{F}_K$ measurable for all $K \in \mathbb{K}$. It is a natural generalisation of the notion of a stopping time: knowing the configuration of $N(\phi)$ inside a compact K is sufficient to conclude whether $S(\phi) \subseteq K$ or not.

Similarly to (1), with each stopping set S there associated a *stopping σ-algebra*:

$$\mathcal{F}_S = \{\Sigma \in \mathcal{F}: \ \Sigma \cap \{\phi : S(\phi) \subseteq K\} \in \mathcal{F}_K \text{ for all } K \in \mathbb{K}\}.$$

It can be shown that

$$S(\phi) = S(\phi|_{S(\phi)}) \text{ and } F(\phi) = F(\phi|_{S(\phi)}) \tag{3}$$

if F is $\mathcal{F}_S$-measurable. Here and afterwards, $\phi|_B(\bullet) = \phi(B \cap \bullet)$ denotes restriction of a counting measure ϕ onto B This stems from Proposition 3 of [7, Prop. 3] on the structure of the stopping σ-algebra and reflects the fact that to decide whether or not S is a stopping set, one only needs to know configuration in S itself. Since non-random compacts are also stopping sets, then (6) also covers (4).

Perhaps, the simplest of stopping set is based on the stopping time: if τ is a finite stopping time in 1D case, then the set $[0, \tau]$ is a compact $\{\mathcal{F}_{[0,s]}\}$-stopping set. More complex examples. Assume that X is a metric space and $N(X) \geq k$ almost surely for some $k \geq 1$. Then the smallest closed ball $B(x_0)$ centred in a given point x_0 containing k points of the process inside is a stopping set. Indeed, given realisation $N(\phi)$, start 'growing' a ball from x_0 increasing its radius from 0 to infinity and stop when it first accumulates k points (or maybe more at once, when the process points are not always in a general position or may overlap). Then whatever compact K is considered, either we stop before this growing ball touches the complement $K^{\mathbf{c}}$, so that $B(x_0) \subseteq K$, or we reach $K^{\mathbf{c}}$ and thus $B(x_0) \nsubseteq K$. Either way, we only used point configuration inside K to decide whether or not $B(x_0) \subseteq K$, i. e. this event is $\mathcal{F}_K$-measurable.

This observation actually shows a very useful way to establish the stopping property: if there is a one-parameter sequence of growing compact sets which eventually leads to construction of the random compact, then this compact is a stopping set. Consider $X = \mathbb{R}^2$ and $N(\phi)$ containing almost surely at least one particle in each of the four quadrants. Assume also that N does not contain multiple points and that all the particles are in a general position (no three points are aligned and no four points lie on a circle). Construct the Voronoi cell centred in the origin O with respect to $N(\phi) \cup \{O\}$. It consists of the points which are closer to O than to any particle from $N(\phi)$. Its vertices are the centres of the balls which have the origin and exactly two particles of $N(\phi)$ on their boundaries and no point of $N(\phi)$ inside. The union $F(\phi)$ of these balls is known as the *Voronoi flower* or fundamental region as the geometry of the cell is completely determined by F. Let us show that F is a stopping set.

Let S_0 be the largest disk centred on the positive x-axis passing through the origin and one of the particles (call it x_1), *and* not having any particles in its interior (see Figure 1). The right bisector of O and x_1 can be seen on the figure; it is the side of the Voronoi polygon cut by the positive x-axis. Now consider the continuum of disks passing through O and x_1, with centre moving upward along this right bisector. Stop when this 'growing' disk first hits another particle (which is labelled x_2). This disk is B_1. In a similar fashion, we move a circle-centre along the next right-bisector, stopping the growing disk (which passes through O and x_2) when it hits another particle, x_3. The last of these constructions stops when x_1 is encountered by a growing disk. This algorithm successfully constructs the Voronoi flower $F = S_0 \cup B_1 \cup \dots B_n$, if the cell has n sides.

A *Poisson process* with intensity measure $\lambda(dx)$ is a point process Π with the following two properties: the variables $\Pi(B_1), \dots, \Pi(B_k)$ are mutually independent for disjoint $B_1, \dots, B_k$ for any k; and $\Pi(B)$ follows Poisson distribution with parameter $\lambda(B)$. As a result, for any Borel set B and any functional $F(\phi)$, $\phi \in \mathcal{N}$ one has:

$$\begin{aligned}\int F(\phi)\,\mathbf{P}(d\phi) &= \int F(\phi|_B + \phi|_{B^c})\,\mathbf{P}(d\phi)\\ &= \iint F(\phi|_B + \phi'|_{B^c})\,\mathbf{P}(d\phi)\,\mathbf{P}(d\phi')\\ &= \iint F(\phi + \phi')\,\mathbf{P}_B(d\phi)\,\mathbf{P}_{B^c}(d\phi')\,, \end{aligned} \quad (4)$$

where $\mathbf{P}_B$ is the restriction of $\mathbf{P}$ onto the σ-algebra $\mathcal{F}_B$. The property (4) reflects complete independence of the Poisson process distribution due to which $\mathbf{P} = \mathbf{P}_B \otimes \mathbf{P}_{B^c}$. In particular, a Poisson process is a Markov process. Therefore it also possesses the *strong Markov property*:

$$\int F(\phi)\,\mathbf{P}(d\phi) = \iint F(\phi|_{S(\phi)} + \phi'|_{S^c(\phi)})\,\mathbf{P}(d\phi)\,\mathbf{P}(d\phi') \quad (5)$$

for every compact stopping set S, see [5], Theorem 4.

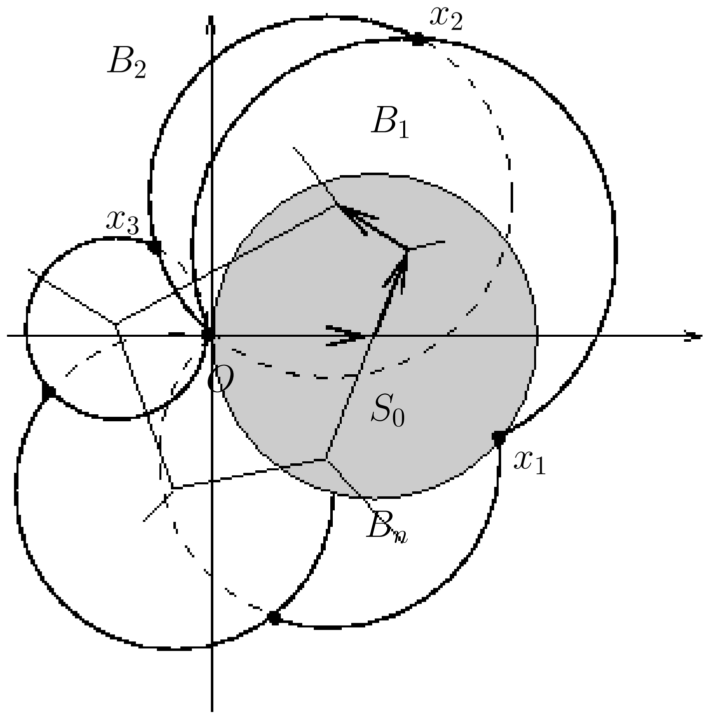

Fig. 1. Incremental construction of the Voronoi flower. Stopping set S_0 is shaded. Direction of the circle-centre move is shown by arrows.

Relation (5) can also be expressed as

$$\mathbf{E}[F(\Pi) \bigm| \mathcal{F}_S](\phi|_{S(\phi)}) = \mathbf{E}_{S^{\mathsf{c}}(\phi)} F(\phi|_{S(\phi)} + \Pi) \tag{6}$$

(more exactly, this is one of versions of the conditional expectation).

2 Slivnyak theorem for locally defined processes

It is common in stochastic geometry and other applications to have another point process Φ which is defined as a function of the reference process N. For instance, Φ may be the process of vertices of the Voronoi tessellation constructed with respect to planar process N. The way this process is constructed uses only local information to decide where the positions of Φ-particles are. Assume for simplicity that N is simple, i. e. with probability 1 it does not contain multiple particles. Then, given a configuration ϕ of the reference process N, the points of $\Phi(\phi)$ have the following identifying property: $x \in \Phi(\phi)$ if and only if there is a ball centred at x which contains exactly 3 ϕ-particle on its boundary and no ϕ-particle inside. A way to establish, if $x \in \Phi(\phi)$ is simple: start 'blowing' a ball centred at x until it hits a ϕ-particles. Call that inflated ball with at least one particle on the boundary $S(x,\phi)$. As we already discussed above, $S(x,\phi)$ is a stopping set. Then we just count how many ϕ-particles are on the boundary, if there are 3 of them, then $x \in \Phi(\phi)$, otherwise $x \notin \Phi(\phi)$. With this example in mind, call a point process $\Phi(\phi)$ *locally defined* if for every $x \in X$ there is a compact stopping set $S(x,\phi)$ such that the event $\{x \in \Phi(\phi)\}$ is $\mathcal{F}_{S(x)}$-measurable.

Now we are ready to formulate our main result.

Theorem 1. *Let Φ be a locally defined point process on the canonical probability space of a Poisson process with distribution $\mathbf{P}$ and $S(x,\phi)$ be the corresponding defining family of stopping sets. Then for λ_Φ-almost all $x \in X$ and a measurable function $F(\phi)$ one has*

$$\mathbf{E}^x_\Phi F = \int F(\phi)\, \mathbf{P}^x_\Phi(d\phi) = \iint F\big(\phi|_{S(x,\phi)} + \phi'|_{S^c(x,\phi)}\big)\, \mathbf{P}^x_\Phi(d\phi)\, \mathbf{P}(d\phi')\,. \quad (7)$$

Proof. The statement of the theorem is equivalent to the fact that for all $B \in \mathcal{B}$ one should have

$$\iint F(\phi)\, \mathbb{1}_B(x)\, \mathbf{P}^x_\Phi(d\phi)\, \lambda_\Phi(dx) = \iiint F\big(\phi|_{S(x,\phi)} + \phi'|_{S^c(x,\phi)}\big)\, \mathbb{1}_B(x)\, \mathbf{P}^x_\Phi(d\phi)\, \mathbf{P}(d\phi')\,.$$

By the Campbell theorem (2), this is equivalent to

$$\iint F(\phi)\, \mathbb{1}_B(x)\, \Phi(\phi, dx)\, \mathbf{P}(d\phi) = \iiint F\big(\phi|_{S(x,\phi)} + \phi'|_{S^c(x,\phi)}\big)\, \mathbb{1}_B(x)\, \Phi(\phi, dx)\, \mathbf{P}(d\phi)\, \mathbf{P}(d\phi')\,. \quad (8)$$

Apply identity (5) to the left hand side of (8). By the local definition of Φ and by (3) one has $\Phi(\phi|_{S(x,\phi)} + \phi'|_{S^c(x,\phi)}) = \Phi(\phi|_{S(x,\phi)}) = \Phi(\phi)$. The result is indeed the right hand side. The proof is complete.

A few remarks are now in order.

A result similar to (7) was first established in [4] for the above example of the nodes of the Voronoi tessellation constructed with respect to a stationary Poisson process. The proof there uses particular geometric properties of the empty Delaunay disks ($S(x)$ in our notation) and cannot be ported to our general setting. In this above form, the result was shown in [1] for the case of stationary processes. In the stationary case the Palm distribution is just no longer a function of x, so it is covered by the same identity (7).

Consider the case when the Poisson process Π is simple and Φ coincides with Π itself. It is trivially locally defined: the stopping sets $S(x)$ are just the singletons $\{x\}$. Now the formula (7) transforms into

$$\int F(\phi)\,\mathbf{P}^x(d\phi) = \int F(\delta_x + \phi')\,\mathbf{P}(d\phi') = \int F(\phi + \delta_x)\,\mathbf{P}(d\phi) \tag{9}$$

which is exactly the Slivnyak's theorem, see [6] and [3]. So this Slivnyak-Mecke characterising formula is no more than another face of the Strong Markov property of the Poisson process.

The proof of the theorem used only the strong Markov property (5) of the Poisson process distribution $\mathbf{P}$ which, in turn, was a consequence of the complete independence property (4). Thus Theorem 1 also holds for completely independent point processes. Such processes are, in fact, a superposition of two independent point processes: a counting measure concentrated on a non-random at most countable set of atoms and a Poisson process with a diffuse intensity measure, see [2, Theorem 2.4.VIII]. This Poisson process is thus simple. We saw, however, that when the first component is absent, the theorem implies identity (9) which *characterises* Poisson point process distribution, as was proved in [3]. Thus, as a by-product we have shown that there is no simple complete independent point process other than Poisson.

Let us also mention another generalisation of idea of locally defined point processes to higher dimensional random sets. For simplicity of formulations, we deal only with the phase space $X = \mathbb{R}^d$.

Consider an n-dimensional ($n < d$) random fiber process, i.e. a random closed set Φ on the Poisson process' probability space $[\mathcal{N}, \Xi]$ such that its n-dimensional *intensity measure* $\lambda_\Phi(\,\bullet\,) = \mathbf{E}H^n(\,\bullet \cap \Phi)$ is non-trivial and σ-finite (H^n is the n-dimensional Hausdorff measure in $\mathbb{R}^d$).

As above, call Φ locally defined if for every $x \in X$ there is a compact stopping set $S(x, \phi)$ such that the event $\{x \in \Phi(\phi)\}$ is $\mathcal{F}_{S(x)}$-measurable. A visual example may provide the collection of n-dimensional edges of the Voronoi cells constructed with respect to the particles of the process. A point x belongs to n-dimensional edge if and only if the glowing ball centred at x will hit at least $d - n + 1$ particles at once, see Figure 2.

Similarly to point process case, one may introduce the Campbell measure $\mathcal{C}(\Sigma \times B) = \mathbf{E}\,\mathbb{1}_\Sigma\, H^n(B \cap \Phi)$ and its Radon-Nikodym derivative

$$\mathbf{P}^x_\Phi(\Sigma) = \frac{d\mathcal{C}(\Sigma \times \bullet)}{d\lambda_\Phi}(x), \quad \Sigma \in \Xi\,,$$

which is called the Palm probability (more exactly, its version which is a probability measure on Ξ).

Now, the proof of Theorem 1 can be carried through to give us a similar result:

Theorem 2. *Assume that the fiber process Φ is locally defined. Then Formula (7) holds for Φ and its Palm distribution.*

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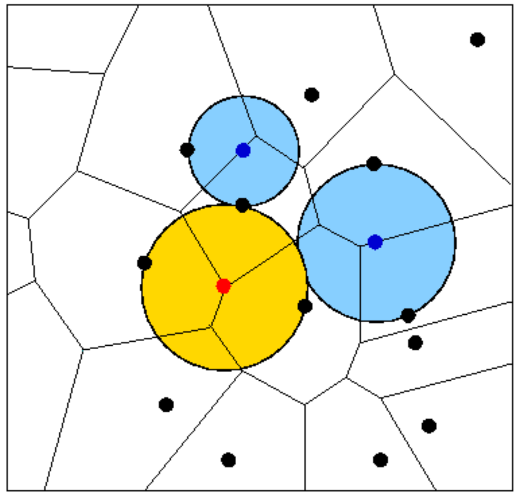

Fig. 2. Edges of the Voronoi cells and the corresponding defining stopping sets.

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