A finite sequence α_1, α_2, ... , α_m of vertices of the n-dimensional unit cube C^n is called a *2-separated path*, if

1) α_i and α_{i+1} differ in one coordinate,

2) α_i and α_j differ in at least two coordinates, if $|i-j| > 1$.

Let $L(n)$ be the maximal length of a 2-separated path in C^n.

What is $L(n)$ for each n?

In 1963 Vasilev obtained that $L(n) \gtrsim 2^n/n$.

It means that $\lim_{n\to\infty} (L(n) - 2^n/n) > 0$.

In 1967 it was proved by Danzer and Klee that

$L(n) \gtrsim 2^{n+1}/n$.

But in 1963, Vasilev obtained the stronger result that

$L(n) \gtrsim \frac{const \cdot 2^n}{B(n)}$,

where $B(n)$ is defined by

$$1 \leq \underbrace{\log_2(\log_2(\ldots(\log_2 n)\ldots))}_{B(n)\ times} < 2$$

It was proved by Evdokimov in 1969 that

$L(n) \gtrsim const \cdot 2^n$.

Since we have the obvious inequality $L(n) < 2^n$, the result gives us the order of the function $L(n)$.

The latest upper bound was found by A. Lukito:

$$L(n) < \frac{2n-5}{4n-9} \cdot 2^n, \ for \ n > 10.$$

The latest lower asymptotic bound (known to me) was found by R. Ropling-Lenhart:

$$L(n) \gtrsim \frac{5}{24} \cdot 2^n.$$

That is,

$$\frac{1}{2} \cdot 2^n \gtrsim L(n) \gtrsim \frac{5}{24} \cdot 2^n.$$

Let α_1, α_2, ... be a finite or infinite sequence of vertices of C^n, such that successive vertices differ in exactly one coordinate. We associate this sequence to a word $X = x_{i_1}x_{i_2}x_{i_3}\ldots$ in an alphabet $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$, where i_j is the number of the coordinate in which α_j and α_{j+1} differ.

Conversely, we associate to an arbitrary finite or infinite word $X = x_{i_1}x_{i_2}x_{i_3}\ldots$, a sequence of vertices of C^n which is obtained by the rule:

$\alpha_1 = (0, \ldots, 0)$. (but it does not matter where we start),

$\alpha_{j+1} = \alpha_j \oplus e_k$, if x_k is the j-th letter of X.

Here $e_k = (0, ..., 0, \underbrace{1}_{k}, ..., 0)$ and $\oplus$ is coordinate-wise addition modulo two.

Def. *A segment of length* i in a word is any i consecutive letters of the word.

Def. A segment is a

- *0-segment*, if it contains no letter an odd number of times.
- *1-segment*, if it contains exactly one letter an odd number of times.
- *2-segment*, otherwise.

Def. *A chain* is a word W, such that each segment of W is a 2-segment.

For any n, find the maximal length of a chain in an n-letter alphabet.

The *length* of a word X is denoted by $L(X)$.

Def. If X' is an initial segment of X, we say that X' is a *beginning* of X.

Def. If X' is a final segment of X, we say that X' is an *end* of X.

Def. Let $X = x_1 x_2 \ldots x_m$ and $Y = y_1 y_2 \ldots y_m$.

Then the *Shuffle product* of X and Y is

$$X \star Y = x_1 y_1 x_2 y_2 \ldots x_m y_m.$$

Def. Let Y' be the segment $y_i y_{i+1} \dots y_{j-1}$ of Y. A disposition of Y' in $X \star Y$ can be drawn schematically as following:

$$\dots x_{i-1} y_{i-1} x_i [y_i \dots x_{j-1} y_{j-1}] x_j y_j \dots.$$

A word X is called *asymmetric* with respect to Y', if in the word $X \star Y$ we have the following conditions:

1) $x_i \neq x_j$;

2) $x_{i-1} \neq x_j$;

3) $x_i \neq x_{j-1}$.

Conditions 1), 2), 3) are *conditions of asymmetry*.

Def. Let S be a set of segments of a word Y. A word X is called *asymmetric with respect to* S, if X is asymmetric with respect to each segment of S.

Suppose the i-th condition of asymmetry holds for a word X with respect to segments of a set S. Then X is i-*asymmetric* with respect to S.

Lemma 1. Let X be a word in an alphabet **X**, Y a word in an alphabet **Y** and Z a word in the alphabet $\{z\}$, such that

1) $L(X) = L(Y) = L(Z)$,

2) The alphabets **X** and **Y** have no common letters and do not contain z,

3) The word Y does not contain 0-segments,

4) The word X is asymmetric with respect to the set of all 1-segments of the word Y.

Then the word $W = (X \star Y) \star (Z \star X)$ is a chain in the alphabet $\mathbf{X} \cup \mathbf{Y} \cup \{z\}$.

Our aim now is to construct, for any n, sequences of words $X_1, X_2, X_3 \ldots$ and $Y_1, Y_2, Y_3 \ldots$ such that:

- for any k, words X_k and Y_k satisfy the conditions of Lemma 1,
- $L(W_k) = const \cdot 2^n$,

where n is the number of letters of the alphabet $\mathbf{X_k} \cup \mathbf{Y_k} \cup \{z\}$.

Here $\mathbf{X_k}$ ($\mathbf{Y_k}$) is the alphabet of the letters in X_k (Y_k).

A construction of the words $Y_1, Y_2, Y_3 \ldots$.

We define the sequence $Y_1, Y_2, Y_3 \ldots$ by an induction scheme:

- $Y_1 = y_1$,
- $Y_{k+1} = Y_k \; y_{k+1} \; Y_k \quad (k = 1, 2 \ldots)$.

Y_k consists of the letters $\{y_1, y_2, \ldots y_k\}$ and $L(Y_k) = 2^k - 1$.

We define a sequence of sets $M_1, M_2 \ldots$ by the following.

M_1 has a segment y_1 of the word Y_1, that is M_1 has Y_1.

Suppose a set M_k of segments of the word Y_k has been already defined, then the set M_{k+1} of segments of the word Y_{k+1} consists of

a) all segments from the left Y_k which belong to M_k,

b) all segments from the right Y_k which belong to M_k,

c) all segments of odd length the middle letter of which is y_{k+1} (including the word Y_{k+1} itself).

Lemma 2. For any k, the word Y_k does not contain 0-segments. M_k is a set of all 1-segments of the word Y_k.

Lemma 3. If some word is 2-asymmetric with respect to the set M_k then it is 3-asymmetric with respect to M_k.

We partition M_k into two sets M_k' and M_k''.

We set $M_1' = \{y_1\}$.

Suppose a subset M_k' of the set M_k has been already defined, then M_{k+1}' consists of

a) all segments of the word Y_{k+1} which are contained in the left Y_k and in M_k'.

b) all segments of the word Y_{k+1} which are contained in the right Y_k and in M_k'.

c) the word Y_{k+1} itself.

Let $M_k'' = M_k \setminus M_k'$ for any k.

Remark. It follows from the construction that M'_k is all 1-segments which coincide with one of the words $Y_1, Y_2, \ldots Y_k$ and therefore the segments of the set M'_k have length $2^m - 1$.

A construction of the words $X_1, X_2, X_3 \ldots$.

Let A_k be a beginning segment of length $2^k - 1$ of the sequence $abababab\ldots$.

Lemma 4. For any k the word A_k is 1-asymmetric with respect to the set M_k.

Let B_k be a beginning segment of length $2^k - 1$ of the sequence $cdecde\ldots$.

Lemma 5. For any k the word B_k is 2-asymmetric with respect to the set M'_k.

We define sequences of words $C_1, C_2, C_3 \ldots$ and $D_1, D_2, D_3 \ldots$ by induction:

$$C_1 = f, \qquad\qquad D_1 = g,$$

$$C_{k+1} = C_k f D_k, \quad D_{k+1} = C_k g D_k$$

$$k = 1, 2 \ldots$$

Lemma 6. For any k the words C_k and D_k are 2-asymmetric with respect to the set M_k''.

For any k and $i = 1, 2, 3, \ldots, 2^k - 1$ we define the i-th letter of the word X_k as the triple $(a_i\ b_i\ c_i)$, where a_i, b_i, c_i are the i-th letters of the words A_k, B_k, C_k respectively.

Since $a_i \in \{a, b\}$, $b_i \in \{c, d, e\}$, $c_i \in \{f, g\}$ we have that for any k the alphabet $\mathbf{X_k}$ of the word X_k consists of at most 12 letters.

Lemma 7. For any k the word X_k is asymmetric with respect to the set of all 1-segments of the word Y_k.

Proof of Lemma 7 follows from Lemmas 3-6.

Theorem. $L(n) > const \cdot 2^n$.

Proof. From Lemmas 2 and 7 it follows that for any k the words $X_k,\ Y_k,\ Z_k = z^{(2^k-1)}$ satisfy the conditions of Lemma 1.

Since $L(X_k) = L(Y_k) = L(Z_k) = 2^k - 1$ then $L(W) = 2^{k+2} - 4$. The alphabet $X_k \cup Y_k \cup \ \{z\}$ consists of $12 + k + 1 = k + 13$ letters. If $k + 13 = n$ then $L(W) = 2^{n-11} - 4$.

The theorem is proved.

$n = 6, \quad k = 2, \quad 2^k - 1 = 3.$

$A_2 = aba,\ B_2 = cde,\ C_2 = ffg.$

$X_2 = (acf)(bdf)(aeg) = x_1 x_2 x_3$

$Y_2 = y_1 y_2 y_1$

$Z_2 = zzz.$

$X_2 \star Y_2 = x_1 y_1 x_2 y_2 x_3 y_1,$

$Z_2 \star X_2 = z x_1 z x_2 z x_3,$

$W = (X_2 \star Y_2) \star (Z_2 \star X_2) = x_1 z y_1 x_1 x_2 z y_2 x_2 x_3 z y_1 x_3.$

If $x_1 = 1, x_2 = 2, x_3 = 3, y_1 = 4, y_2 = 5, z = 6$ then

$W = 164126523643,\ |W| = 12.$

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