The first Rogers-Ramanujan identity:

$$\sum_{m\geq 0} \frac{q^{m^2}}{(1-q)\dots(1-q^m)} = \prod_{m\geq 0} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}.$$

A combinatorial interpretation:

The number of partitions of n with parts differing by at least 2 is equal to the number of partitions of n with parts congruent to 1 or 4 (mod 5).

The Involution Principle

Let A and B be disjoint sets, such that

$A = A^{+} \cup A^{-}$ and $B = B^{+} \cup B^{-}$.

Let $f : A \to B$ be a bijection such that

$f(A^{+}) = B^{+}$, $f(A^{-}) = B^{-}$.

$\alpha : A \to A$ is a bijection and for all $a \in A$ either

(i) $\alpha(a) = a$ and $a \in A^{+}$ or

(ii) $\alpha(a) \neq a$, and $\quad a \in A^{+} \Leftrightarrow \alpha(a) \ \in \ A^{-}$.

Similarly for $\beta : B \to B$.

Let F_α and F_β be the fixed point sets of α and β, respectively.

$|A^+| - |F_\alpha| = |A^-|$, $|B^+| - |F_\beta| = |B^-|$,

$|A^-| = |B^-|$ and $|A^+| = |B^+|$, so $|F_\alpha| = |F_\beta|$.

$\alpha^* = f \circ \alpha, \qquad \beta^* = f^{-1} \circ \beta.$

α^* maps A one-to-one onto B.

β^* maps B one-to-one onto A.

Let $a \in F_\alpha$. We form a sequence $a = a_1, b_1, a_2, b_2, \dots$, by first applying α^*, then β^*, then α^*, etc., i.e., $a_{i+1} = \beta^*(b_i)$ and $b_i = \alpha^*(a_i)$ for $i = 0, 1, \dots$.

$a_1, b_1, \dots$ is called the *iterated (α, β)-sequence* associated with a.

There are two basic facts about such sequences:

(I) If $a \in F_\alpha$, then there is a least n, denoted by n_a, such that $b_n \in F_\beta$.

(II) If $a, a' \in F_\alpha$ with iterated (α, β)-sequences

$$a = a_1, b_1, \dots, a_{n_a}, b_{n_a},$$

$$a' = a'_1, b'_1, \dots, a'_{n'_a}, b'_{n'_a},$$

then $a \neq a'$ implies $b_{n_a} \neq b'_{n'_a}$.

We define a map

$$I^f_{\alpha,\beta} : F_\alpha \to F_\beta,$$

which we called the *iterated (α, β)-map*, by

$$I^f_{\alpha,\beta}(a) = b_{n_a}$$

for all $a \in F_\alpha$.

Theorem 1.(*Garsia-Milne*)

Let $A = A^{+} \cup A^{-}$, $B = B^{+} \cup B^{-}$, $f : A \to B$, $\alpha : A \to A$, and $\beta : B \to B$ be as above.

Then the iterated (α, β)-map $I^{f}_{\alpha,\beta}$ is a bijection between F_{α} and F_{β}.

The first Rogers-Ramanujan identity:

$$\sum_{m\geq 0} \frac{q^{m^2}}{(1-q)\dots(1-q^m)} = \prod_{m\geq 0} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}.$$

Some more identities:

$$\sum_{p=-\infty}^{\infty} z^p q^{(p^2-p)/2} \prod_{m\geq 1} \frac{1}{1-q^m} = \prod_{i\geq 0}(1+zq^i) \prod_{j\geq 1}(1+\frac{q^j}{z}),$$

$$\prod_{m\geq 1}(1-q^m) = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3/2)k^2-(1/2)k}.$$

A *partition* Π *of* n is a multiset $\Pi = \{n_1 * a_1, \dots, n_k * a_k\}$.

This is also denoted by $\Pi \vdash n$.

$W(\Pi) = \sum_{i=1}^{k} n_i a_i$ is called a weight of Π.

$\mathbf{A} = < A_i >_{i \in \omega}$ is a sequence of multisets.

$S_{\mathbf{A}}(\Pi)$ is the set of indices s.t. $A_i \subseteq \Pi$, i.e., $S_{\mathbf{A}}(\Pi) = \{i \mid A_i \subseteq \Pi\}$.

$P_n(\mathbf{A}) = \{\Pi \mid \Pi \vdash n \text{ and } S_{\mathbf{A}}(\Pi) = \emptyset\}$.

Theorem 2.

Suppose $\mathbf{A} = \{A_i\}_{i\in\omega}$ and $\mathbf{B} = \{B_i\}_{i\in\omega}$ are sequences of distinct nonempty multisets such that for all finite sets $S \subseteq \omega$, we have $W(\bigcup_{i\in S} A_i) = W(\bigcup_{i\in S} B_i)$.

Then $|P_n(\mathbf{A})| = |P_n(\mathbf{B})|$.

Proof (Theorem 2).

We shall construct a bijection between $P_n(\mathbf{A})$ and $P_n(\mathbf{B})$ for any fixed n using the involution principle (Theorem 1).

$A = \{(\Pi, S) \mid \Pi \vdash n \quad \text{and} \quad S \subseteq S_{\mathbf{A}}(\Pi)\}$,
$B = \{(\Pi, S) \mid \Pi \vdash n \quad \text{and} \quad S \subseteq S_{\mathbf{B}}(\Pi)\}$,

where the sign of (Π, S) is $(-1)^{|S|}$.

$a_{\Pi} = \max(S_{\mathbf{A}}(\Pi))$ if $S_{\mathbf{A}}(\Pi) \neq \emptyset$,

$b_{\Pi} = \max(S_{\mathbf{B}}(\Pi))$ if $S_{\mathbf{B}}(\Pi) \neq \emptyset$.

If $S_{\mathbf{A}}(\Pi) \neq \emptyset$ then
$\alpha(\Pi, S) = (\Pi, S - \{a_{\Pi}\})$, if $a_{\Pi} \in S$,
$\alpha(\Pi, S) = (\Pi, S \cup \{a_{\Pi}\})$, if $a_{\Pi} \notin S$.

If $S_{\mathbf{A}}(\Pi) = \emptyset$ then $\alpha(\Pi, S) = (\Pi, \emptyset)$.

α is a sign-reversing involution on A with fixed point set $F_{\alpha} = \{(\Pi, \emptyset) \mid \Pi \in P_n(\mathbf{A})\}$.

The involution β is defined similarly and has analogous properties.

The sign preserving map $f : A \to B$ satisfies:

If $(\Pi, S) \in A$, then $f(\Pi, S) = (\lambda, S)$, where $\lambda = [\Pi - (\bigcup_{i \in S} A_i)] \cup [\bigcup_{i \in S} B_i]$.

Thus by Theorem 1, the iterated (α, β)-map gives a bijection between $P_n(\mathbf{A}) \times \{\emptyset\}$ and $P_n(\mathbf{B}) \times \{\emptyset\}$ which may be regarded as a bijection between $P_n(\mathbf{A})$ and $P_n(\mathbf{B})$.

Theorem 3.

Suppose $\mathbf{A} = \{A_i\}_{i\in\omega}$ and $\mathbf{B} = \{B_i\}_{i\in\omega}$ are sequences of distinct nonempty multisets.

Then

$|P_n(\mathbf{A})| = |P_n(\mathbf{B})|$ for all n iff the sequences $\{W(A_i)\}_{i\in\omega}$ and $\{W(B_i)\}_{i\in\omega}$ are rearrangements of each other.

Corollary 2.1 (Euler). The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

Corollary 2.2 (Glaisher). The number of partitions of n with no part divisible by d equals the number of partitions of n in which no part is repeated d or more times.

Proof (Corollary 2.2). The number of partitions of n with no parts divisible by d equals $P_n(\mathbf{A}_d)$, where $\mathbf{A}_d =< d, 2d, 3d, \ldots >$ and the number of partitions of n with no part repeated d or more times equals $P_n(\mathbf{B}_d)$, where

$$\mathbf{B}_d =< \{\underbrace{1, \ldots, 1}_{d\ times}\}, \{\underbrace{2, \ldots, 2}_{d\ times}\}, \{\underbrace{3, \ldots, 3}_{d\ times}\}, \ldots >=$$

$$< \{d * 1\}, \{d * 2\}, \{d * 3\}, \ldots > .$$

Glaisher constructed the following bijection:

$$\theta : P_n(\mathbf{A}_d) \to P_n(\mathbf{B}_d).$$

If $\Pi \in P_n(\mathbf{A}_d)$, then
$\Pi = \{n_1 * a_1, n_2 * a_2, \ldots n_k * a_k\}$, where $d \nmid a_i$ for any i.

$n_i = e_0^i d^0 + e_1^i d^1 + \ldots + e_{\ell_i}^i d^{\ell_i}$, where $0 \le e_j^i \le d - 1$ for $j = 0, 1, \ldots, \ell_i$.

$\theta(\Pi)$ is the partition that results by replacing each sequence of parts $n_i * a_i$ in Π by the sequence of parts $\{e_0^i * d^0 a_i\}, \{e_1^i * d^1 a_i\}, \ldots,$ $\{e_{\ell_i}^i * d^{\ell_i} a_i\}$.

Theorem 4.

The iterated (α, β)-map given in Theorem 2 between $P_n(\mathbf{A}_d)$ and $P_n(\mathbf{B}_d)$ is the Glaisher bijection θ for any fixed d.

Corollary 2.3 (Schur).

(i) The number of partitions of n into parts $\equiv \pm 1 \pmod 6$ equals

(ii) the number of partitions of n into distinct parts $\equiv \pm 1 \pmod 3$ and

(iii) the number of partitions of n in which the difference of any two parts is at least 3 and in which no consecutive multiples of 3 occur.

Proof (Corollary 2.3 i), ii)).

The partitions of type (i) are just $P_n(\mathbf{A})$ where

$$\mathbf{A} =< \{2\},\{3\},\{4\},\{6\},\{8\},\{9\},\{10\},\{12\},\ldots >.$$

For the partitions of type (ii), we must eliminate the subsets $\{1,1\},\{2,2\},\ldots$, to leave only distinct parts and multisets $\{3\},\{6\},\{9\},\ldots$, to leave only parts $\equiv \pm 1 \pmod 3$.

Thus the partitions of type (ii) are just $P_n(\mathbf{B})$, where

$$\mathbf{B} =< \{1,1\},\{3\},\{2,2\},\{6\},\{4,4\},\{9\},\{5,5\},\ldots >.$$

Corollary 2.4 (Andrews).

Let M_1 and M_2 be two sets of positive integers.

Let $2M_1 = \{j \mid (j/2) \in M_1\}$.

Then the number of partitions of n into distinct parts taken from M_1 equals the number of partitions of n into parts taken from M_2 if

$2M_1 \subseteq M_1$ and $M_2 = M_1 - 2M_1$.

Corollary 2.5 (Subbarao).

Let S_1 and S_2 be two sets of positive integers. Let $dS_1 = \{j \mid (j/d) \in S_1\}$. Then the number of partitions of n into parts taken from S_2, equals the number of partitions of n with parts taken from S_1, where no part is repeated d or more times if $dS_1 \subseteq S_1$ and $S_2 = S_1 - dS_1$.

Pairs (S_1, S_2) such that the number of partitions of n with parts taken from S_1 with no part repeated r or more times equals the number of partitions of n with parts taken from S_2 are called *Eulerian pairs of order* r by Subbarao.

The full statement of Andrews' results:

(M_1, M_2) is the Eulerian pair of order 2 iff $2M_1 \subseteq M_1$ and $M_2 = M_1 - 2M_1$.

The full statement of Subbarao's results:

(S_1, S_2) is the Eulerian pair of order r ($r \geq 2$) iff $dS_1 \subseteq S_1$ and $S_2 = S_1 - dS_1$.

Corollary 2.6.

(i) The number of partitions of n whose parts $\equiv \pm 1$, 4 (mod 5) do not differ by exactly 8, equals

(ii) the number of partitions of n whose parts $\equiv \pm 2$, 3 (mod 5) do not differ by exactly 6, equals

(iii) the number of partitions of n whose parts $\equiv \pm 2$, 3 (mod 5) do not differ by exactly 4, equals

(iv) the number of partitions of n whose parts $\equiv \pm 1$, 4 (mod 5) do not differ by exactly 2, equals

(v) the number of partitions of n whose parts $\equiv 0$ (mod 5), equals

(vi) the number of partitions of n with no multiples of 10.

Corollary 2.7. For $k \geq 3$, the number of partitions of n with no odd multiples of $\binom{k}{2}$ equals the number of partitions of n with no consecutive parts $\equiv 1, 2, \ldots, k-1 \pmod{k}$, respectively.

Corollary 2.8. For $k \geq 3$, the number of partitions of n with no odd multiples of $k^2 - k$ equals the number of partitions of n with no consecutive repeated parts $\equiv 1, 2, \ldots, k-1 \pmod{k}$, respectively.

Corollary 2.9. The number of partitions of n with no consecutive even parts equals the number of partitions of n with no consecutive repeated parts.

Corollary 2.10. The number of partitions of n with no consecutive parts repeated and no part repeated more than 3 times equals the number of partitions of n whose even parts differ by least 4.

Corollary 2.11. The number of partitions of n with no consecutive odd parts repeated and no odd part repeated more than 3 times equals the number of partitions of n whose even parts $\equiv 2 \pmod 4$ differ by at least 8.

Corollary 2.12 (Kitaev).

The number of partitions of n with no part i repeated i or more times, equals

the number of partitions of n where no part is a perfect square.

Proof (Corollary 2.12).

$\mathbf{A} =< \{1\}, \{2, 2\}, \{3, 3, 3\}, \{4, 4, 4, 4\}, \ldots >$,

$\mathbf{B} =< \{1\}, \{4\}, \{9\}, \{16\}, \ldots >$.

Corollary 2.2000 (Kitaev).

The number of partitions of n whose parts $\equiv 200, 800 \pmod{1000}$ do not differ by exactly 1600, equals

the number of partitions of n with no multiples of 2000.

Proof (Corollary 2.2000).

$\mathbf{A} = < \{200, 1800\}, \{1200, 2800\}, \ldots >$,

$\mathbf{B} = < \{2000\}, \{4000\}, \{6000\}, \ldots >$.

Bijection Between Classes of Words Omitting Certain Words.

Let $X = \{x_1, \dots, x_\ell\}$ be a finite alphabet.

X^* is the set of all finite words with letters from X.

If $v = w_1 u w_2$ then u is a *factor* of v.

u is a factor of v *starting at i* if the length of w_1, $\ell(w_1)$, equals $i - 1$.

$\mathbf{A} =< \alpha_1, \alpha_2, \dots >$ is a sequence of words ($\mathbf{A}$ may be either finite or infinite).

$W_n(\mathbf{A})$ is the set of all $w \in X^*$ of length n that do not have any α_i as a factor.

w_2 *overlaps* w_1 *starting at* k if $w_1 = uv$ and $w_2 = vw$, where v is nonempty word and $\ell(u) = k - 1$.

Let $\mathbf{A} =< \alpha_1, \alpha_2, \ldots >$ and $\mathbf{B} =< \beta_1, \beta_2, \ldots >$. $\mathbf{A}$ and $\mathbf{B}$ have the *consistent overlapping property* if

(1) For all i, $\ell(\alpha_i) = \ell(\beta_i)$.

(2) For all i, j, and k, α_i is a factor of α_j starting at k iff β_i is a factor of β_j starting at k.

(3) For all i, j, and k, α_i overlaps α_j starting at k iff β_i overlaps β_j starting at k.

Theorem 6.

Suppose

$\mathbf{A} =< \alpha_1, \alpha_2, \ldots >$ and $\mathbf{B} =< \beta_1, \beta_2, \ldots >$

are sequences of distinct words from X^* with the consistent overlapping property, then

for all n, $|W_n(\mathbf{A})| = |W_n(\mathbf{B})|$.

REFERENCES

1. JEFFREY B. REMMEL, Bijective Proofs of Some Classical Partition Identities, *J. Combin. Theory Ser. A* **33** (1982), 273-286.

2. A.M. GARSIA, AND S.C. MILNE, A Rogers-Ramanujan bijection, *J. Combin. Theory Ser. A* **31** (1981), 289-339.

See these papers for other references!