On Pattern Avoidance in Permutations

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The Classical Definition of Pattern Avoidance

S_n is the symmetric group on $\{1, 2, \ldots, n\}$.
A *pattern* is a permutation on $\{1, 2, \ldots, r\}$ where $r \leq n$.

A permutation $\sigma \in S_n$ *avoids* the pattern $p \in S_r$ if there is no $1 \leq i_1 < i_2 < \ldots < i_r \leq n$ such that $p = [\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)]$ in reduced form.

The *reduced form* of a permutation σ on a set $\{j_1, j_2, \ldots, j_n\}$, where $j_1 < j_2 < \ldots < j_n$ is the permutation $\sigma_1 \in S_n$ obtained by renaming j_1 by 1, $\ldots$, j_n by n.

Let R be a set of patterns.

Let $S_n(p)$ be the set of all permutations in S_n which avoid the pattern p.

Then $S_n(R) = \bigcap_{p \in R} S_n(p)$.

An extreme case is $S_n(\emptyset) = S_n$ for all $n \geq 1$.

$N_n(R)$ is the number of elements of $S_n(R)$.

A function $f : \mathbf{N} \to \mathbf{C}$ is called P-*recursive* if there exist polynomials P_0, $P_1, \ldots, P_k \in \mathbf{C}[n]$, so that

$$P_k(n)f(n+k) + P_{k-1}(n)f(n+k-1) + \ldots + P_0(n)f(n) = 0$$

for all $n \in \mathbf{N}$.

Questions about $S_n(R)$:

1. Formula for $N_n(R)$;

2. Generating function for $N_n(R)$, that is, $f_R(x) = \sum_i N_i(R)x^i$;

3. P-recursiveness of $N_n(R)$;

4. Existence of c such that $N_n(R) < c^n$;

5. Relations to other combinatorial structures;

6. Equidistribution: $S_n(R) = S_n(R')$, for all n?

Content

1. **Permutations with Single Restrictions**

   $R = \{p\}$, where either $p \in S_3$ or $p \in S_4$;
   More generally $N_{n,r}(p) =$ the number of n-permutations with exactly r occurrences of a pattern p.

2. **Permutations with Double Restrictions**

   $R = \{p, q\}$, where either p, $q \in S_3$ or $p \in S_3$, $q \in S_4$;

3. **Permutations with More than two Restrictions from S_3**

1. Permutations with Single Restrictions

Theorem 1. *[Knuth] For all $n \geq 1$, and for all $p \in S_3$:*

$$N_n(p) = \frac{1}{n+1}\binom{2n}{n},$$

the nth Catalan number.

A *stack* is a linear list for which all insertions and deletions are made at one end of the list.

A stack represented as a railway switching network.

With a stack we always remove the "youngest" item currently in the list.

The *stack-sorting operation* Π on the set of all n-permutations is the following:

Let $\pi = \pi_L n \pi_R$ be an n-permutation, π_L and π_R respectively its subword before and after n. Let $\Pi(\pi) = \Pi(\pi_L)\Pi(\pi_R)n$.

A permutation π is called t*-stack sortable* if $\Pi^t(\pi)$ is the identity permutation.

Proposition.

A permutation is 1-stack sortable iff it avoids the pattern 231.

For any $\sigma \in S_n$ its

reversal $\sigma^r \in S_n$ is given by $\sigma^r(i) = \sigma(n+1-i)$;

complement $\sigma^c \in S_n$ is the permutation $\sigma^c(i) = n+1-\sigma(i)$;

inverse σ^{-1} is the usual group theoretic inverse permutation.

Let $\quad RSFS \equiv Rodica\ Simion\ and\ Frank\ W.\ Schmidt.$

Lemma 1. *[RSFS] Let* $R = \{p\} \subset S_3$. *If* $\sigma \in S_n(R)$, *then*

1. $\sigma^r \in S_n(R^r)$, *where* $R^r = \{p^r\}$;
2. $\sigma^c \in S_n(R^c)$, *where* $R^c = \{p^c\}$;
3. $\sigma^{-1} \in S_n(R^{-1})$, *where* $R^{-1} = \{p^{-1}\}$.

$E_n(R)$ is the number of even permutations in $S_n(R)$;

$O_n(R)$ is the number of odd permutations in $S_n(R)$;

$I_n(R)$ is the number of involutions in $S_n(R)$;

$EI_n(R)$ is the number of even involutions in $S_n(R)$;

$OI_n(R)$ is the number of odd involutions in $S_n(R)$;

$$C_k = \begin{cases} \frac{1}{k+1}\binom{2k}{k}, & \text{if } k \geq 0 \text{ and } k \text{ is an integer;} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2. *[RSFS] For every $n \geq 1$,*

$$E_n(132) - O_n(132) = C_{(n-1)/2}.$$

Since $sign(\sigma^r) = (-1)^{\binom{n}{2}} sign(\sigma) = sign(\sigma^c)$, the following is immediate:

Corollary. For every $n \geq 1$,

$$E_n(231) - O_n(231) = (-1)^{\binom{n}{2}} C_{(n-1)/2};$$

$$E_n(312) - O_n(312) = (-1)^{\binom{n}{2}} C_{(n-1)/2};$$

$$E_n(213) - O_n(213) = C_{(n-1)/2}.$$

Theorem 3. *[RSFS] For every* $n \geq 1$,

$$E_n(123) - O_n(123) = (-1)^{\binom{n}{2}} C_{(n-1)/2}.$$

Theorem 4. *[RSFS] For every $n \geq 1$, the number of involutions in $S_n(123)$ is:*

$$I_n(123) = \binom{n}{[n/2]};$$

among these, the number of even permutations is

$$EI_n(123) = \begin{cases} \frac{1}{2}\binom{n}{[n/2]}, & \textit{if } \; n \equiv 0 \; or \; 2 \pmod 4, \\ \binom{n}{[n/2]}, & \textit{if } \; n \equiv 1 \pmod 4, \\ 0, & \textit{if } \; n \equiv 3 \pmod 4. \end{cases}$$

Theorem 5. *[RSFS] For every $n \geq 1$, and for each $p \in \{132, 321, 213\}$,*

$$I_n(p) = \binom{n}{[n/2]},$$

and

$$EI_n(p) - OI_n(p) = (-1)^{\binom{n}{2}} C_{(n-1)/2},$$

or, equivalently,

$$EI_n(p) = \begin{cases} \frac{1}{2} I_n(p), & \textit{if } n \textit{ is even;} \\ \binom{n-1}{[n/2]}, & \textit{if } n \equiv 1 \pmod 4; \\ \binom{n-1}{[n/2]-1}, & \textit{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 6. *[RSFS] For every $n \geq 1$, and $p \in \{231, 312\}$,*

$$I_n(p) = 2^{n-1},$$

and

$$EI_n(p) - OI_n(p) = \frac{7 - 3\sqrt{-7}}{28}\alpha^n + \frac{7 + 3\sqrt{-7}}{28}\beta^n,$$

where

$$\alpha = \frac{1 + \sqrt{-7}}{2}, \quad \beta = \frac{1 - \sqrt{-7}}{2}.$$

pattern p	$N_n(p) < c^n$	**formula for** $N_n(p)$	**P-recursive**
1234 4321	yes (Regev)	$(\star)$ (Gessel)	yes (Zeilberger)
1342 2431 3124 4213	yes (Bóna)	$(\star\star)$ (Bóna)	yes (Bóna)
1324 4231	yes (Bóna)	open	open

$$(\star) = 2\sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2+2k+1-n-2kn}{(k+1)^2(k+2)(n-k+1)};$$

$$(\star\star) = \tfrac{7n^2-3n-2}{2} \cdot (-1)^{n-1} + 3\sum_{i=2}^{n} 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \binom{n-i+2}{2} (-1)^{n-i}.$$

Theorem 7. *[Regev] For all n, $N_n(12\ldots k)$ asymptotically equals*

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}.$$

Here

$$\lambda_k = {\gamma_k}^2 \int_{x_1 \geq} \int_{x_2 \geq} \cdots \int_{\geq x_k} [D(x_1, x_2, \ldots, x_k) \cdot e^{-(k/2)x^2}]^2 \, dx_1 dx_2 \ldots dx_k,$$

where $D(x_1, x_2, \ldots, x_k) = \prod_{i<j} (x_i - x_j)$, *and* $\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}$.

$N_{n,r}(p)$ – the number of n-permutations with exactly r occurrences of a pattern p.

Theorem 8. *[Bóna] For all n we have*

$$N_{n,1}(132) = \binom{2n-3}{n-3}.$$

Theorem 9. *[Noonan] For all n we have*

$$N_{n,1}(123) = \frac{3}{n}\binom{2n}{n+3}.$$

Conjecture 1. *[Noonan and Zeilberger]*

For all patterns p and for all r,

$N_{n,r}(p)$ *is a P-recursive function of n.*

Theorem 10. *[Bóna]* $N_{n,r}(132)$ *is a P-recursive function of n.*

2. Permutations with Double Restrictions

Lemma 2. *[RSFS] For every n,*

- $N_n(123, 132) = N_n(123, 213) = N_n(231, 321) = N_n(312, 321)$;
- $N_n(132, 213) = N_n(231, 312)$;
- $N_n(132, 231) = N_n(213, 312)$;
- $N_n(132, 312) = N_n(213, 231)$;
- $N_n(132, 321) = N_n(123, 231) = N_n(123, 312) = N_n(213, 321)$.

Theorem 11. *[RSFS] For all $n \geq 1$,*

$$N_n(123, 132) = N_n(132, 213) = N_n(132, 231) = N_n(132, 312) = 2^{n-1}.$$

Theorem 12. *[RSFS] For all $n \geq 1$,*

$$N_n(132, 321) = \binom{n}{2} + 1.$$

Theorem 13. *[RSFS]*

$$N_n(123, 321) = \begin{cases} 0, & \textit{if } n \geq 5; \\ n, & \textit{if } n = 1 \textit{ or } 2; \\ 4, & \textit{if } n = 3 \textit{ or } 4. \end{cases}$$

Recursive definition of a *generating tree* consists of

1. the label of the root
2. a set of *succession rules* explaining how to derive, given the label of a parent, the number of children and their labels

The complete binary tree:

Root: (2)

Rule: (2) $\rightarrow$ (2)(2)

The complete k-ary tree:

Root: (k)

Rule: (k) $\rightarrow$ $(\mathsf{k})^k$

The Fibonacci tree:

Root: (1)

Rule: (1) $\rightarrow$ (2)
(2) $\rightarrow$ (1)(2)

The Catalan tree:

Root: (2)

Rule: (k) $\rightarrow$ (2)(3)...(k+1)

restrictions	**formula**	**author**
$S_n(123, 4321)$	0	West
$S_n(123, 3421)$	$\binom{n}{4} + 2\binom{n}{3} + n$	West
$S_n(132, 4321)$	$\binom{n}{4} + \binom{n+1}{4} + \binom{n}{2} + 1$	West
$S_n(123, 4231)$	$\binom{n}{5} + 2\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + 1$	West
$S_n(123, 3241)$	$3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$	West
$S_n(123, 3412)$	$2^{n+1} - \binom{n+1}{3} - 2n - 1$	Stanley
$S_n(132, 4231)$	$1 + (n-1)2^{n-2}$	Guibert
$S_n(132, 3421)$	$1 + (n-1)2^{n-2}$	West
$S_n(132, 3214)$	g.f.: $\frac{(1-x)^3}{1-4x+5x^2-3x^3}$	West

The following were given by West:

restrictions	**restrictions**	**formula**
$S_n(123, 2143)$	$S_n(312, 1342)$	
$S_n(123, 2413)$	$S_n(312, 3241)$	
$S_n(132, 2314)$	$S_n(312, 3214)$	
$S_n(132, 2341)$	$S_n(123, 3214)$	F_{2n}
$S_n(312, 2314)$	$S_n(312, 4321)$	(Fibonacci number)
$S_n(132, 3412)$	$S_n(312, 3421)$	
$S_n(312, 1432)$	$S_n(132, 3241)$	
$S_n(3142, 2413)$	$S_n(4132, 4231)$	g.f.: $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$

3. Permutations with >2 Restrictions from S_3

Lemma 3. *[RSFS] For every n,*

- $N_n(123, 132, 213) = N_n(231, 312, 321)$;
- $N_n(123, 132, 231) = N_n(123, 213, 312) = N_n(132, 231, 321) = N_n(213, 312, 321)$;
- $N_n(132, 213, 231) = N_n(132, 213, 312) = N_n(132, 231, 312) = N_n(213, 231, 312)$;
- $N_n(123, 132, 312) = N_n(123, 213, 231) = N_n(132, 312, 321) = N_n(213, 231, 321)$;
- $N_n(123, 231, 312) = N_n(132, 213, 321)$;
- $N_n(R) = 0$ *for all* $R \supset \{123, 321\}$ *if* $n \geq 5$; $N_n(R) = n$ *for* $1 \leq n \leq 3$ *and* $N_4(R) = 1$.

Theorem 14. *[RSFS] For every $n \geq 1$,*

$$N_n(123, 132, 213) = F_{n+1},$$

where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence, with $F_0 = 0$, $F_1 = 1$.

Theorem 15. *[RSFS] For every $n \geq 1$,*

$$N_n(123, 132, 231) = N_n(132, 213, 231) =$$
$$N_n(123, 132, 312) = N_n(123, 231, 312) = n$$

Theorem 16. *[RSFS] If $R \subset S_3$ and $|R| = 4$, then*

- $N_n(R) = 0$ *if* $R \supset \{123, 321\}$ *and* $n \geq 5$*;*
- $N_n(R) = 2$ *if* $R \not\supset \{123, 321\}$, $n \geq 2$*;*

For $|R| = 5$ *and* $n \geq 4$*,*

- $N_n(R) = 0$ *except for*

$$N_n(S_3 - \{123\}) = N_n(S_3 - \{321\}) = 1.$$

The Generalisation of Pattern Avoidance (the Babson-Steingrímsson Patterns)

A *generalised pattern* of length r is a word $p = x_1 p_1 x_2 p_2 \ldots x_r p_r x_{r+1}$, where $[p_1, \ldots, p_r]$ is a permutation on $\{1, 2, \ldots, r\}$ and x_j is either the empty string ϵ or a dash "$-$".

If $x_j =$ "$-$" then

in the classical definition of pattern avoidance we allow to be

$i_j > i_{j-1} + 1$; also

$i_1 > 1$ if $x_1 =$ "$-$", and

$i_r < n$ if $x_{r+1} =$ "$-$";

else $i_j = i_{j-1} + 1$; in particular

$i_1 = 1$ if $x_1 = \epsilon$, and

$i_r = n$ if $x_{r+1} = \epsilon$.

Claesson has found a combinatorial interpretation for permutations avoiding all the patterns of type $(x-yz)$ and $(xy-z)$, where $xyz \in S_3$. For instance, he has shown that $N_n(1-32)$ is the nth Bell number.

Theorem 17. *[Kitaev]* $N_{n,r}(1-32) = \sum_{i=1}^{n} T[n,r,i]$, *where*

$$T[n,r,i] = \sum_{j=1}^{i-1} T[n-1,r,j] + \sum_{j=i}^{n-1} T[n-1,r-i+1,j].$$