Multiple access protocols and LANs

Broadcast link: multiple sending and receiving nodes connected to shared broadcast channel

Scheduled access systems: TDM, Radio, etc

Random access systems: Ethernet, Satellite, Wireless Internet, etc

1. Slotted Aloha model
2. The slotted Aloha Markov chain
3. ODE approximation method
4. Remark on SDE approximation method
5. CSMA, CSMA/CD
6. Collision resolution, binary splitting method

The slotted Aloha Markov chain

- Slotted time,
- m users contend for a single transmission channel,
- A packet appears at each user input with probability p, each slot,
- If two or more users attempt to send packets in same slot, the channel remains idle and packets are returned for retransmission,
- Backlogged users attempt to retransmit with probability q, each slot,
- Usual independence assumptions,
- No new arrivals from backlogged users.

A user is called *backlogged* from the time slot it suffered a collision until the slot the packet has been successfully sent, at which time the user is *free* to contend. Introduce

$$\begin{aligned} K_n &= \text{\# backlogged users in slot } n \\ Z_n &= \text{\# attemps to retransmit packets in slot } n \\ N_n &= \text{\# new packets arriving to the link from free users in slot } n \end{aligned}$$

which are seen to satisfy the relation

$$K_{n+1} = K_n + N_{n+1} - 1_{\{Z_{n+1}+N_{n+1}=1\}}.$$

The problem is to find

$$\text{Throughput} = \text{expected fraction of successfully transmitted packets per slot},$$

which we interpret as the limit

$$\text{Throughput} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{\{Z_k+N_k=1\}} = P(Z+N=1).$$

The key observation is that, given K_n, we have

$$\begin{aligned} Z_{n+1} &\in \text{Bin}(K_n, q) \\ N_{n+1} &\in \text{Bin}(m-K_n, p) \end{aligned}$$

and, still conditional on K_n, these two random variables are independent. Now we study

$$\{K_n, n\geq 1\} \qquad \text{the slotted Aloha Markov Chain.}$$

Consider the *diffusion approximation scaling*

$$x_t^{(m)} = \frac{1}{m} K_{[mt]}, \qquad p \mapsto \frac{\lambda}{m} \quad q \mapsto \frac{\alpha}{m}.$$

Here $0 \leq K_n/m \leq 1$ is the proportion of backlogged nodes, i.e. a *space rescaling* of the original sequence and $x_t^{(m)}$ is the result after an additional *time rescaling.* If a limit function

$$x_t = \lim_{m\to\infty} x_t^{(m)}, \qquad x_\infty = \lim_{t\to\infty} x_t$$

can be found, then, in equilibrium,

$$\text{throughput } \sim P(Z+N=1) \sim E(N) \sim p(m - E(K)) \sim \lambda(1-x_t) \sim \lambda(1-x_\infty).$$

To investigate $\{x_t^{(m)}, t \geq 0\}$, compute *drift* and *quadratic variation*:

$$\mu_m(x) = \frac{1}{h} E(x_{t+h}^{(m)} - x_t^{(m)} | x_t^{(m)} = x) = E(K_{n+1} - K_n | K_n = mx), \quad n = [mt] \quad h = \frac{1}{m}$$

and

$$\sigma_m^2(x) = \frac{1}{h} E((x_{t+h}^{(m)} - x_t^{(m)})^2 | x_t^{(m)} = x) = \frac{1}{m} E((K_{n+1} - K_n)^2 | K_n = mx), \quad n = [mt] \quad h = \frac{1}{m}.$$

The previous Lindley type equation implies

$$\begin{aligned} E(K_{n+1} - K_n | K_n) &= E(N_{n+1} | K_n) - P(Z_{n+1} = 0, N_{n+1} = 1 | K_n) \\ &\quad - P(Z_{n+1} = 1, N_{n+1} = 0 | K_n) \\ &= (m - K_n)p - (1-q)^{K_n}(m - K_n)p(1-p)^{m-K_n-1} \\ &\quad - K_n q(1-q)^{K_n - 1}(1-p)^{m-K_n}. \end{aligned}$$

Therefore, changing to scaled parameters $\lambda = mp$, $\alpha = mq$ and state variable $x = k/m$ for some integer $0 \leq k \leq m$,

$$\begin{aligned} E(K_{n+1} - K_n | K_n = mx) &= \lambda(1-x) - (1 - \frac{\alpha}{m})^{mx} \lambda(1-x)(1 - \frac{\lambda}{m})^{m(1-x)-1} \\ &\quad - \alpha x(1 - \frac{\alpha}{m})^{mx-1}(1 - \frac{\lambda}{m})^{m(1-x)}, \end{aligned}$$

and we obtain the limiting drift function $\mu(x)$ defined on $0 \leq x \leq 1$ by taking the limit $m \to \infty$,

$$\mu(x) = \lim_{m\to\infty} \mu_m(x) = \lambda(1-x) - (\lambda(1-x) + \alpha x)\, e^{-\lambda(1-x)-\alpha x}. \tag{1}$$

Similarly, a quadratic variation function $\sigma^2(x)$ defined on the unit interval $[0, 1]$ is obtained by taking the limit

$$\begin{aligned} \sigma^2(x) &= \lim_{m\to\infty} m\, \sigma_m^2(x) = \lim_{m\to\infty} E((K_{n+1} - K_n)^2 | K_n = mx) \\ &= \lambda^2(1-x)^2 + \lambda(1-x) - (\lambda(1-x) - \alpha x)\, e^{-\lambda(1-x)-\alpha x}. \end{aligned}$$

The sequence $\{x_t^{(m)}, t \geq o\}$ converges as $m \to \infty$ to a (deterministic) function x_t, namely the solution of the ordinary differential equation

$$\frac{d}{dt} x_t = \mu(x_t), \qquad \text{given initial condition } x_0$$

where $\mu(\cdot)$ is the limit drift function in Equation 1 and x_0 is an initial value of the asymptotic fraction of backlogged users.

For $\alpha < \alpha_{\text{crit}}$ there is for any λ a unique limit x_∞ regardless of x_0, the unique stationary point such that $\mu(x_\infty) = 0$. This value can be interpreted as the *equilibrium backlog* and we obtain the *equilibrium throughput* of cells as $\lambda(1 - x_\infty)$. For $\alpha > \alpha_{\text{crit}}$ there are two stable stationary points (and one unstable), backlog and throughput depends on initial state; *Hysteresis effect.*

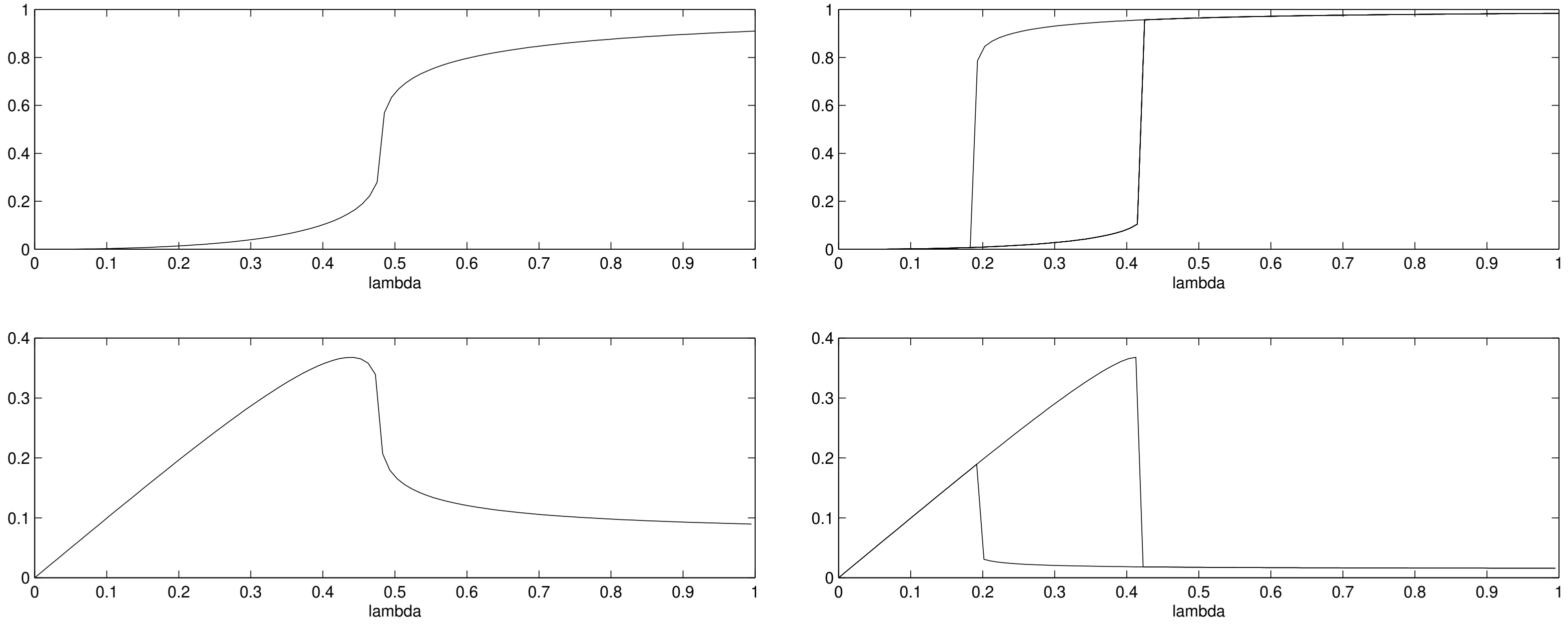

Figure 1: Backlog and througput for scaled Aloha, $\alpha = 4$ $\alpha = 6$

The *throughput thrashing* phenomena can be studied using the SDE approximation:

$$dx_t^{(m)} = \mu(x_t^{(m)})\,dt + \frac{1}{\sqrt{m}}\sigma(x_t^{(m)})\,dW_t.$$

with stationary probability density function

$$f(x) = \text{const}\frac{m}{\sigma^2(x)}\,e^{2m\int_0^x \mu(y)/\sigma^2(y)\,dy} \qquad 0 \le x \le 1.$$

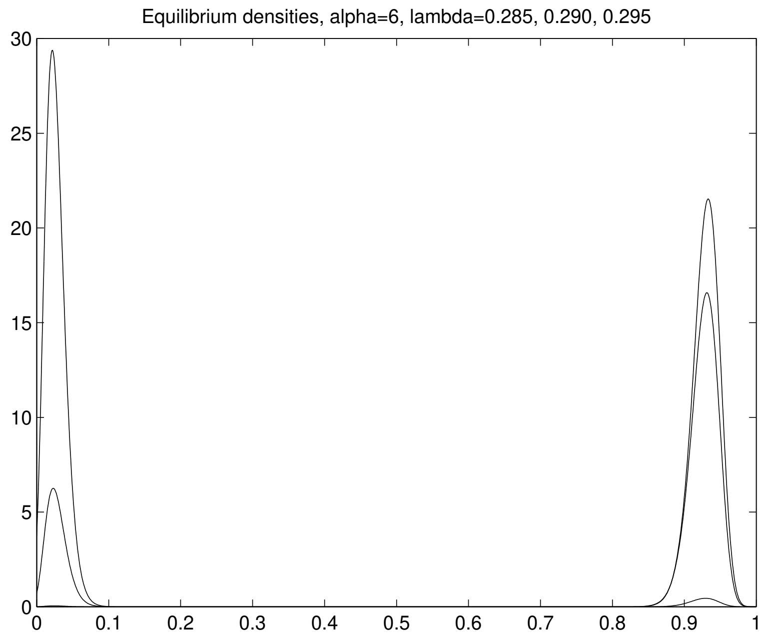

Figure 2: Equilibrium backlog densities, $\alpha = 6$, $\lambda = 0.285, 0.290, 0.295$

Performance analysis of slotted, nonpersistent CSMA and CSMA/CD

- Time is measured in minislots;
- Packets are of fixed length $T+1 \geq 1$ minislots;
- Users are either free or backlogged (blocked);
- In any given minislot each free user receives and transmits a packet with probability p;
- A minislot during which transmission is initiated is called a *firing* minislot;
- The channel remains busy T minislots following a firing minislot;
- In any minislot backlogged users retransmit with probability q;
- Free users sensing the channel busy become blocked, backlogged users remain blocked;
- Collision detection (CD): If collision occurs in firing minislot, the channel is cleared after R minislots rather than T, $0 \leq R \leq T$;

Consider the backlog K_n at firing minislot n. This is embedded Markov chain such that

$$K_{n+1} = m - X_{n+1} - 1_{\{N_{n+1}+Z_{n+1}=1\}},$$

where

$$\begin{aligned} N_n &= \text{\# free users transmitting, firing minislot nr } n \\ Z_n &= \text{\# retransmission attempts, firing minislot nr } n \\ X_n &= \text{\# free users not sensing the channel during transmission period } n, \end{aligned}$$

and

$$X_{n+1} \in \mathrm{Bin}\left(m - K_n - N_{n+1}, r(K_n)(1-p)^T + (1 - r(K_n))(1-p)^R\right)$$

with

$$r_m(k) = P(N_{n+1} + Z_{n+1} = 1 | N_{n+1} + Z_{n+1} \geq 1, K_n = k).$$

Diffusion approximation scaling:

$$x_t^{(m)} = \frac{1}{m} K_{[mt]} \qquad p \mapsto \frac{\lambda}{m(1+T)} \quad q \mapsto \frac{\alpha}{m(1+T)}.$$

Limiting drift function:

$$\begin{aligned}\mu(x) &= \lambda(1-x)\left(\frac{1+R}{1+T} + \frac{T-R}{1+T}\, r(x)\right) \\ &\quad - \frac{\lambda(1-x)+\alpha x}{1+T}\, \exp\{-\frac{\lambda(1-x)+\alpha x}{1+T}\}.\end{aligned}$$

where

$$r(x) = \frac{(\lambda(1-x)+\alpha x)\exp\{-\frac{\lambda(1-x)+\alpha x}{1+T}\}}{(1+T)(1-\exp\{-(\lambda(1-x)+\alpha x)/(1+T)\})}.$$

As before solve $\mu(x) = 0$ to find backlog x_∞ and throughput $\lambda(1-x_\infty)$, see Figures 6.11-13.

Collision resolution algorithms

Designed to avoid the instability in contention protocols based on retransmission.

Binary splitting algorithm with blocked access and binary feedback information:

Slotted time. $\mathrm{Bin}(m,p)$ active users per slot. Suppose slot n is *idle* (no undelivered cells remaining at end of slot $k-1$).

- No request in slot n: bandwidth wasted
- $A_n = 1$ request: Successful transmission
- $A_2 \geq 2$ requests: Collision to be resolved. This takes

$$Y_n = 1_{\{A_n=0\}} + A_n 1_{\{A_n \geq 1\}} + L_n,$$

slots, where L_n is the *collision resolution interval* (CRI). The CRI is determined in a binary splitting contest. Let

$$L_{K(m,p)} = \text{length of CRI required to resolve } K \in \mathrm{Bin}(m,p) \text{ stations}$$

(cycle numbering n suppressed in notation). Then

$$L_{A(m,p)} = 1_{\{A\geq 2\}} + L_{K'} + L_{K''}, \quad K' \in \mathrm{Bin}(A,1/2), \quad K'' = A - K' \in \mathrm{Bin}(A,1/2).$$

This gives

$$E(L_{A(m,p)}) = P(A(m,p) \geq 2) + 2E(L_{A(m,p/2)})$$

hence

$$\begin{aligned} E(L) = E(L_{A(m,p)}) &= \sum_{k=0}^{\infty} 2^k P(A(m, p/2^k) \geq 2) \\ &= \sum_{k=0}^{\infty} 2^k \left(1 - (1 - \frac{p}{2^k})^m - \frac{mp}{2^k}(1 - \frac{p}{2^k})^{m-1}\right) \\ &= \sum_{j=1}^{\infty} j \binom{m}{j+1} (-p)^{j+1} \frac{2^j}{2^j - 1}. \end{aligned}$$

The throughput is therefore

$$\text{Throughput}_{\text{CR}} = \frac{mp}{(1-p)^m + mp + \sum_{j=1}^{\infty} j \binom{m}{j+1} (-p)^{j+1} \frac{2^j}{2^j - 1}}.$$

Scaling $\lambda = mp$, $m \to \infty$, yields

$$\text{Throughput}_{\text{CR}} \to \frac{\lambda}{e^{-\lambda} + \lambda + \sum_{j=1}^{\infty} j \frac{(-\lambda)^{j+1}}{(j+1)!} \frac{2^j}{2^j - 1}}.$$

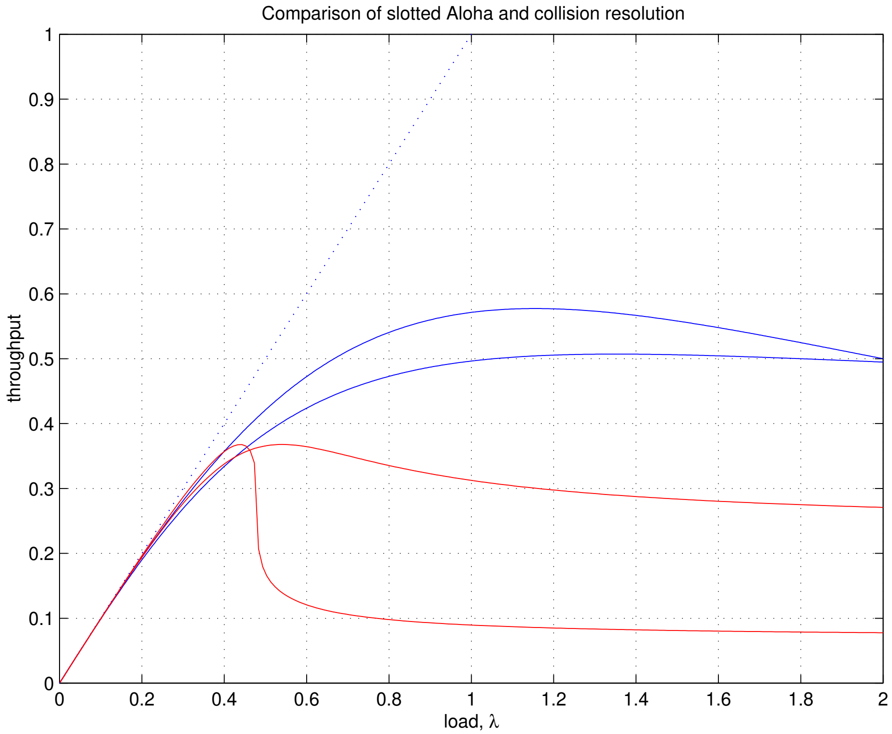

Figure 3: Comparison Aloha and collision resolution