Jump Modelling for Financial Asset Prices

Yueming Zhang

August 28, 2005
Abstract

In this master thesis we study the character of jumps occurring in a financial asset pricing process with IBM stock. We present a formal definition of jumps and find that based on this definition, the durations between upward jumps and the sizes of downward jumps have a heavy-tail distribution, but the durations between downward jumps and the sizes of upward jumps have light-tail distribution. By the BDS test, it is shown that the durations between jump are correlated. Here the AR(1) model is suggested to be the proper model for the logarithm of jump duration. The occurrence of jumps can be treated as a counting process. In our counting process model, we discuss the stochastic intensity, the distribution of jump duration, the distribution of arrival times and the properties of the counting process.
Acknowledgement  I would like to express my great appreciation to my supervisor Patrik Albin for his comments and suggestions. And I also like to thank the following people: Oscar Hammar, Jan Lennartsson, Min Shu, Viktor Olsbo, Johan Tykesson and Alexander Herbertsson, for their help during my thesis work.
5 Counting Process

5.1 Stochastic Intensity .............................................. 24
5.2 Properties of the Counting Process .............................. 28
  5.2.1 Distribution Function for the Interarrival Time .......... 28
  5.2.2 Distribution Function for the Arrival Time ............... 30
  5.2.3 Probability Distribution of N(t) ............................ 31
5.3 Conclusions .......................................................... 32

6 Conclusions ............................................................. 33
Chapter 1

Introduction

The issue to build a model beyond Black-Scholes is a classical topic in the field of mathematical finance. And in last thirty years many models have been suggested; most of them can be classified into two kinds: stochastic-volatility models and jump-diffusion models. Stochastic-volatility models use a stochastic volatility instead of the constant volatility in the Black-Scholes model with or without jumps, while Jump-diffusion models add a jump part to the Black-Scholes model (see e.g. [21]). Generally speaking, compared with stochastic-volatility models, jump-diffusion models are simpler and can qualitatively catch the financial market phenomena, like the large fluctuations in the asset prices (see e.g. [21]).

In jump-diffusion models for financial asset pricing, jumps are usually defined as all the discontinuities in a sample path of a Brownian motion; however as in the real world the stock prices are never continuous, such a definition will be quite difficult to put into practice. To solve this problem, people usually argue that when the difference between two consecutive stock prices is larger than a certain threshold, then a jump occurs. Unfortunately this threshold always varies to with different stocks. Moreover people usually assume that the arrival time of jumps can be a Poisson process. Is this assumption really true? Clearly a formal definition of jumps is needed.

In this master thesis we study the issue of defining jumps for stock prices. Further, we will study the properties of jumps.

Chapter 2 gives a formal definition of the jumps, the concepts of jump size and jump duration.

In Chapter 3 we investigate the probability distribution properties of jump size and jump duration.

Chapter 4 is in part devoted to the study of dependence structures of jump size and jump duration. Furthermore we discuss the models for jump duration.

In Chapter 5 we provide an idea about the possibility to establish a new jump model for asset pricing.

And finally in Chapter 6, we make some concluding remarks, as well as discuss the future research.
Chapter 2

Exploring Jump-Diffusions

In this work we use IBM stock prices, from 3-Jan-1984 to 22-Oct-2004, as our price process. We remove the sample mean from the corresponding log returns and then divide them with the sample standard deviation. Compared with the standard normal distribution, the devolatilized log returns are more heavy-tailed, as can be seen in Figure 2.2 and 2.3.

\[ \mu = 5.476 - 4 \quad \text{and} \quad \sigma = 0.0191 \]

2.1 What are Jumps?

In jump-diffusion models, asset prices are modeled as Levy processes with a nonzero Gaussian component and a jump part (see [9], pp 103); and jumps are represented as the rare events - crashes or sudden upsurges (see [14]). Such models can explain why heavy tails will appear in the marginal distribution for some stochastic price processes. In other words, the jumps reflect the heavy tailed part on a distribution.

---

The sample mean \( \bar{\mu} = 5.476 - 4 \) and the sample standard deviation \( \bar{\sigma} = 0.0191 \)
Definition 1 (Jump) For an asset price process $X_t$, a threshold $\alpha$ ($\alpha > 0$) and $-\alpha$ are set on both the positive and the negative side of the devolatilized log returns of $X_t$. Once the devolatilized log return exceeds the threshold on either side at time $t$, i.e.

$$\left| \frac{\log X_t - \log X_{t-1} \hat{\mu}}{\hat{\sigma}} \right| \geq \alpha,$$

we say that a jump occurs at time $t$.

Following this definition, we set natural definitions of jump size and duration between jumps.

Definition 2 (Jump size) When a jump occurs at time $t$, the jump size is the difference of the log prices, i.e.

$$\{\text{jump size}\}_t = \log X_t - \log X_{t-1}.$$

Obviously the jump size at time $t$ equals the value of the log return at time $t$.

Definition 3 (Duration between jumps) When a jump occurs at time $t$ and the next jump occurs at time $s$, where $t > s$, we say that the duration between these two consecutive jumps is $t - s$.

The following investigation of jumps are based on the above three definitions.

2.2 Jumps of IBM Stock Prices

Since jumps are rare events, it is reasonable to set the threshold $\alpha = 2.00$. According to the definition of jump (Definition 1), we register the time when jumps occurs. The
corresponding log returns\(^2\) are the size of jumps.

**Definition 4 (Jump–up & Jump–down)** If the size of a jump \(> 0\), then this jump is upward; and if the size \(< 0\), then this jump is downward.

Considering that the size of downward jumps is always negative, for simplicity, we will take the absolute value for the size of downward jumps in following parts.

It is well known that the size of downward jumps tend to be larger than that of upward jumps for many asset prices (see e.g. [14]). Hence it is motivated to investigate whether upward jumps and downward jumps really have different properties.

![Histograms of Jump Size](image)

**Figure 2.4:** The histograms of the jump size

<table>
<thead>
<tr>
<th></th>
<th>Number</th>
<th>Max Value</th>
<th>Min Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of jump-up</td>
<td>138</td>
<td>0.1236</td>
<td>0.0388</td>
</tr>
<tr>
<td>Size of jump-down</td>
<td>116</td>
<td>0.2682</td>
<td>0.0379</td>
</tr>
</tbody>
</table>

**Table 2.1:** Information about the jump size

In this thesis, we only consider the duration between two consecutive jumps. And as argued above, we would like to separate the durations for upward jumps and downward jumps, as well as their jump sizes.

<table>
<thead>
<tr>
<th></th>
<th>Number</th>
<th>Max Value</th>
<th>Min Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration for jump-up</td>
<td>138</td>
<td>620</td>
<td>1</td>
</tr>
<tr>
<td>Duration for jump-down</td>
<td>116</td>
<td>365</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 2.2:** Information about the duration between jumps

\(^2\)According to the definition of height of jump (Definition 2), these log returns are non-devolatilized. In the following parts, if the log returns are not specifically indicated to be devolatilized, then they are non-devolatilized.
2.3 Distribution Test

F. B. Hanson and J. J. Westman [14] consider that the sizes of jumps are uniformly distributed and the jump process is a space-time Poisson process. However, through observing the histograms of the jump size (Figure 2.4), we see that the heights of neither upward jumps nor downward jumps are uniformly distributed. And in the following part, we will investigate whether the increments of jump process, i.e. the duration between jumps, are exponentially distributed.

2.3.1 Goodness of Fit

We choose Kolmogorov-Smirnov (KS) distance as the statistical test for assumptions about distribution. The test statistic (see e.g. [16]) is given by

\[
KS = \max_{x \in \mathbb{R}} \left| F_{\text{emp}}(x) - F_{\text{fit}}(x) \right|
\]

where \(F_{\text{fit}}\) is the fitted distribution function and \(F_{\text{emp}}\) is the empirical distribution function, which is given by

\[
F_{\text{emp}}(x) = \frac{\#(X_i \leq x, i = 1, \ldots, n)}{n}, \quad n = \text{the number of total observations}
\]

In practice, we can calculate KS distance with the following formula (see e.g. [16]):

\[
KS = \max_{1 \leq i \leq n} \left( \max \left\{ \left| \frac{i - 0.5}{n} - F(X_{(i)}) \right|, \left| \frac{i + 0.5}{n} - F(X_{(i)}) \right| \right\} \right),
\]

where \(X_{(1)} \leq \ldots \leq X_{(n)}\) is the ordered data set.

Figure 2.5: The histograms of the duration between jumps
2.3.2 Testing Results

First we use the Maximum Likelihood (ML) method to estimate the parameter $\lambda$ of the exponential probability density function,

$$f_{\text{Exp}}(x) = \lambda e^{-\lambda x}, \quad x > 0,$$

which is the distribution of durations between jumps in Poisson models. Then we test goodness-of-fit with the KS-distance.

<table>
<thead>
<tr>
<th></th>
<th>$\lambda$</th>
<th>KS test</th>
</tr>
</thead>
<tbody>
<tr>
<td>durations for jump-up</td>
<td>0.0263</td>
<td>0.1481</td>
</tr>
<tr>
<td>durations for jump-down</td>
<td>0.0232</td>
<td>0.2529</td>
</tr>
</tbody>
</table>

Table 2.3: The estimated parameters and KS-distance, with MLE method

![Graph](image1.png)  
(a) The duration between upward jumps

![Graph](image2.png)  
(b) The duration between downward jump

Figure 2.6: The comparison between empirical distribution of duration and exponential distribution with MLE method

Observing Figure 2.6, we can notice that the empirical distribution of duration between upward jumps fits the exponential distribution quite well, though its KS-distance 0.1481 is a little too large to indicate that we really have exponential distribution; however for the durations between downward jumps, the empirical does not fit Exponential distribution well, and its KS-distance is also very large, 0.2529. Hence we conclude that Exponential distribution is not the best distribution for the jump duration.
Chapter 3

Distribution Fitting

In the previous chapter, we saw that the standard distribution assumptions about the exponential distribution and the uniform distribution can not fit the data (jump size and duration between jumps) well. In this chapter, we will try to find proper distributions for these data.

3.1 Tail Behavior

3.1.1 Mean Excess Plot

With the mean excess plot, we can test the tail properties of jump size and jump duration graphically. And this will be helpful to choose the proper distribution for the data of jump size and jump duration, as it is usually in the tails problems with fitted distributions occur.

The main idea is that with the help of mean excess plot, we choose the threshold \( u \) such that the sample mean excess function is nearly linear above \( u \), (see e.g. [15]). We then consider the sample data, which exceed the threshold, to belong to the tail. Then we can decide whether the tail is heavy or light.

Definition 5 (Mean excess function, see [10]) Let \( X \) be a random variable, then
\[
e(u) = E(X - u|X > u), \quad u \geq 0,
\]
is called the mean excess function of \( X \).

The sample mean excess function is given by
\[
e_n(u) = \frac{1}{N_u} \sum_{i=1}^{n} (X_i - u)\mathbb{1}\{X_i > u\},
\]
where \( X_1 \ldots X_n \), are iid random variables, \( u \) is the threshold, \( N_u = \#\{i : 1 \leq i \leq n, X_i > u\} \) and \( \mathbb{1}\{X > u\} \) is an indicator function. The mean excess plot consists of the points (see e.g. [15])
\[
\{(X_{k,n}, e_n(X_{k,n})): k = 1, \ldots, n - 1\}.
\]
Using the mean excess plot, we find the threshold for duration to be 60 and the threshold for size to be 0.06.

### 3.1.2 Log Transformation

One example of a heavy-tail distribution given by the probability density function

\[
f(x) = \frac{(1 + (x/\delta)^\rho)^{-\beta}}{C(\delta, \rho, \beta)}, \quad x > 0,
\]

where \( \delta > 0, \rho > 0 \) and \( \beta > 0 \). Heavy-tail distributions are also called polynomial distributions, since when \( x \) is large,

\[
1 - F(x) \approx Cx^{-\rho},
\]

Now for some \( C > 0 \) and \( \rho > 0 \), take logarithms on both sides and use the empirical distribution \( F_{\text{Emp}}(X_{(i)}) = \frac{i}{n} \) instead of the distribution function \( F(x) \), i.e.

\[
1 - F(x) \approx Cx^{-\rho},
\]

\[
\log(1 - F(x)) \approx \log C - \rho \log x,
\]

\[
\log(1 - \frac{i}{n}) \approx \log C - \rho \log x_i. \tag{3.1}
\]
Using the least square (LS) method to estimate \( \log C \) and \( \rho \), then we may observe whether the linear relation (3.1) really holds, and decide whether data are heavy-tail distributed.

One example of a light-tail distribution has a probability density function,

\[
f(x) = \frac{x^{\beta-1}e^{-\lambda x^\alpha}}{C(\beta, \lambda)}, \quad x > 0,
\]

where \( \alpha > 0, \beta > 0 \) and \( \lambda > 0 \). And light-tail distributions are also called exponential distributions, since when \( x \) is large,

\[
1 - F(x) \approx C_1 x^{-\rho} \exp\{-C_2 x^\alpha\}.
\]

We can use the similar method as above to decide whether data are light-tail distributed; however here we need to take logarithm twice, i.e.

\[
1 - F(x) \approx C_1 x^{-\rho} \exp\{-C_2 x^\alpha\},
\]

\[
\log(1 - F(x)) \approx \log C_1 - \rho \log x - C_2 x^\alpha
\]

\[
\approx -C_2 x^\alpha,
\]

\[
\log(-\log(1 - F(x))) \approx \log C_2 + \alpha \log x,
\]

\[
\log(-\log(1 - \frac{i}{n})) \approx \log C_2 + \alpha \log x_i. \tag{3.2}
\]

Now with the data of jump durations and jumps sizes, which exceed the thresholds, we may estimate the coefficients \( \rho \) and \( \log C \) for heavy-tail distribution and check for linearity.

<table>
<thead>
<tr>
<th></th>
<th>( \rho )</th>
<th>( \log C )</th>
<th>No. of data</th>
</tr>
</thead>
<tbody>
<tr>
<td>upward jump duration</td>
<td>1.5146</td>
<td>6.1787</td>
<td>25</td>
</tr>
<tr>
<td>downward jump duration</td>
<td>1.5650</td>
<td>6.6709</td>
<td>26</td>
</tr>
</tbody>
</table>

Table 3.1: coefficients for heavy-tail distribution, with the threshold \( u = 60 \)

And similarly we estimate the coefficients \( \alpha \) and \( \log C_2 \) for light-tail distribution and check the fit by looking for linearity in a plot of the transformed data (3.2).
<table>
<thead>
<tr>
<th></th>
<th>( \rho )</th>
<th>( \log C^2 )</th>
<th>No. of data</th>
</tr>
</thead>
<tbody>
<tr>
<td>upward jump size</td>
<td>3.5089</td>
<td>-9.7521</td>
<td>34</td>
</tr>
<tr>
<td>downward jump size</td>
<td>2.3132</td>
<td>-6.5922</td>
<td>26</td>
</tr>
</tbody>
</table>

Table 3.2: coefficients for heavy-tail distribution, with the threshold \( u = 0.06 \)

<table>
<thead>
<tr>
<th></th>
<th>( \alpha )</th>
<th>( \log C^2 )</th>
<th>No. of data</th>
</tr>
</thead>
<tbody>
<tr>
<td>upward jump duration</td>
<td>1.9031</td>
<td>-9.4589</td>
<td>25</td>
</tr>
<tr>
<td>downward jump duration</td>
<td>2.3477</td>
<td>-11.9262</td>
<td>26</td>
</tr>
</tbody>
</table>

Table 3.3: coefficients for light-tail distribution, with the threshold \( u = 60 \)

<table>
<thead>
<tr>
<th></th>
<th>( \alpha )</th>
<th>( \log C^2 )</th>
<th>No. of data</th>
</tr>
</thead>
<tbody>
<tr>
<td>upward jump size</td>
<td>4.7874</td>
<td>11.4798</td>
<td>34</td>
</tr>
<tr>
<td>downward jump size</td>
<td>2.7809</td>
<td>6.2745</td>
<td>26</td>
</tr>
</tbody>
</table>

Table 3.4: coefficients for light-tail distribution, with the threshold \( u = 0.06 \)
Observing the figures above, we conclude that the upward jump duration and the downward jump size are heavy-tail distributed, and the downward jump duration and the upward jump size are light-tail distributed.

3.2 Distributions

Here we choose four distributions to test. They are the Generalized Pareto Distribution, the Pearson VII Distribution, the Generalized Hyperbolic Distribution and the Gamma Distribution. The first two distributions have the heavy-tailed property; and the last two have the light-tailed property.

3.2.1 Generalized Pareto Distribution

The density function of the Generalized Pareto (GP) distribution is given by

$$f_{\text{GP}}(x; u, \sigma, \xi) = \frac{1}{\sigma}(1 + \frac{x - u}{\sigma})^{-1/\xi - 1} \text{ for } x > u,$$

where $u \in \mathbb{R}$ is a threshold, $\xi \geq 0$ is a shape parameter and $\sigma > 0$ is a scale parameter (see e.g. [16]). The distribution function is given by

$$F_{\text{GP}}(x; u, \sigma, \xi) = 1 - (1 + \frac{x - u}{\sigma})^{-1/\xi} \text{ for } x > u.$$

3.2.2 Pearson VII Distribution

The Pearson VII distribution has the density function,

$$f_{\text{Pearson}}(x; m, c) = \frac{2 \Gamma(m) \ell}{c \Gamma(m - \frac{\ell}{\pi})}(1 + \frac{x}{c})^{-m} \text{ for } x > 0,$$

1In fact when $\alpha = 1$, this distribution is called semi-heavy-tail distribution.
2Actually they are semi-heavy tailed and have $\alpha = 1$. 

Figure 3.5: Exponential Curves for Jump Size (Light-tail)
with the distribution function,
\[ F_{\text{Pearson}}(x; m, c) = \frac{2\Gamma(m) x_2 F_1\left(\frac{1}{2}, m, \frac{3}{2}, -\frac{x^2}{\sigma^2}\right)}{\sqrt{\pi} \sigma \Gamma(m - \frac{1}{2})} \quad \text{for} \quad x > 0, \]
where \( m > 1/2 \) is a shape parameter and \( c > 0 \) is a scale parameter (see e.g. [16]).

### 3.2.3 Generalized Hyperbolic Distribution

The probability density function for the Generalized Hyperbolic (GH) distribution is given by
\[ f_{\text{GH}}(x; \lambda, \alpha, \beta, \delta, \mu) = \frac{(\alpha^2 - \beta^2)\lambda/2}{\sqrt{2\pi\alpha^{\lambda-1/2}}} K_\lambda(\delta \sqrt{\alpha^2 - \beta^2}) \times K_{\lambda-1/2}(\alpha \sqrt{\beta^2 + (x - \mu)^2}) e^{\beta(x - \mu)} \quad \text{for} \quad x \in \mathbb{R}, \]
where \( K_\lambda \) is the modified Bessel function of the third kind
\[ K_\lambda = \frac{1}{\pi} \int_0^\infty y^{\lambda-1} e^{-x/y + (y+1)/y} dy \quad \text{for} \quad x \in \mathbb{R}, \]
(see e.g. [16]). The permitted values of the parameters are as follows:
\[ \lambda, \beta, \mu \in \mathbb{R} \quad \text{with} \quad \begin{cases} \delta \geq 0 \quad \text{and} \quad |\beta| < \alpha & \text{if} \quad \lambda > 0, \\ \delta > 0 \quad \text{and} \quad |\beta| < \alpha & \text{if} \quad \lambda = 0, \\ \delta > 0 \quad \text{and} \quad |\beta| \leq \alpha & \text{if} \quad \lambda < 0. \end{cases} \]

### 3.2.4 Gamma Distribution

The probability density function for the Gamma distribution is given by
\[ f_{\text{Gamma}} = \frac{x^{\alpha-1} e^{-\lambda x} \lambda^\alpha}{\Gamma(\alpha)} \quad \text{for} \quad x \in [0, \infty), \]
and the corresponding distribution function is given by
\[ F_{\text{Gamma}} = P(\alpha, \lambda x) \quad \text{for} \quad x \in [0, \infty), \]
where \( P(a, z) \) is a regularized gamma function. (see e.g. [22])

### 3.3 Test Results

#### 3.3.1 Heavy-tail Part

For the data of the upward jump duration and the downward jump size, which are considered as heavy-tail distributed, we use the GP distribution and the Pearson VII distribution to fit them with ML method. Notice that when estimating the parameters for the downward jump size, we shift the data set, i.e. let the data set minus the minimum value of the data. The fitting results are as follows.
<table>
<thead>
<tr>
<th>Duration for jump-up</th>
<th>Size of jump-down</th>
</tr>
</thead>
<tbody>
<tr>
<td>ξ 0.375665</td>
<td>σ 23.5202</td>
</tr>
<tr>
<td>ξ 0.369871</td>
<td>σ 0.0114926</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>General Pareto</th>
<th>Pearson VII</th>
</tr>
</thead>
<tbody>
<tr>
<td>m 1.20422</td>
<td>m 1.26635</td>
</tr>
<tr>
<td>c 25.022</td>
<td>c 0.0132479</td>
</tr>
<tr>
<td>KS Distance</td>
<td>KS Distance 0.0556826</td>
</tr>
<tr>
<td>KS Distance</td>
<td>KS Distance 0.0681976</td>
</tr>
</tbody>
</table>

Table 3.5: Parameters for GP and Pearson VII estimated by ML and KS distance

Figure 3.6: Comparison of empirical distribution and fitted distribution for the data of upward jump duration

Figure 3.7: Comparison of empirical distribution and fitted distribution for the data of downward jump size

3.3.2 Light-tail Part

Similarly we use the GH distribution and the Gamma distribution to fit the data of the downward jump duration and the upward jump size, which were found to be light-tailed. Note that our data are all positive; however for the GH distribution, data can be both positive and negative. Hence it is necessary to set a cutoff point to
keep the GH distribution just working on the positive side. Also another thing that need to be mentioned here is that when estimating the parameters of the Gamma distribution, we shift the data set; the idea is that we consider the data set minus the minimum value of the data, so that the whole data set starts out close to the zero point.\footnote{In practice, we also have to plus a very small number, e.g. 0.000000001, to keep the data greater than zero such that the computer code can work correctly.}

The fitting results are as follows.

<table>
<thead>
<tr>
<th>Generalized Hyperbolical Distribution</th>
<th>cutoff</th>
<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration for jump-down</td>
<td>−24.155</td>
<td>0.139297</td>
<td>4.31309</td>
<td>4.30681</td>
</tr>
<tr>
<td>Size of jump-up</td>
<td>1.94047</td>
<td>−4.80819</td>
<td>5.37966</td>
<td>4.55007</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\mu$</th>
<th>KS Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration for jump-down</td>
<td>−0.845762</td>
<td>0.79237</td>
</tr>
<tr>
<td>Size of jump-up</td>
<td>0.0387968</td>
<td>−11.2119</td>
</tr>
</tbody>
</table>

Table 3.6: Parameters for GH distribution estimated by ML and KS distance

<table>
<thead>
<tr>
<th>Gamma distribution</th>
<th>$\alpha$</th>
<th>$\lambda$</th>
<th>KS Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration for jump-down</td>
<td>0.576991</td>
<td>74.6291</td>
<td>0.132548</td>
</tr>
<tr>
<td>Size of jump-up</td>
<td>0.697845</td>
<td>0.0234174</td>
<td>0.0640025</td>
</tr>
</tbody>
</table>

Table 3.7: Parameters for Gamma distribution estimated by ML and KS distance

![Comparison of empirical distribution and fitted distribution for the data of downward jump duration](a) GH distribution (b) Gamma distribution

Figure 3.8: Comparison of empirical distribution and fitted distribution for the data of downward jump duration

### 3.3.3 Conclusions

Observing the comparing plots of empirical distribution and fitted distribution, we find that for the upward jump duration and the downward jump size, both the GP distribution and the Pearson VII distribution fit data quite well; and according to
Further according to the comparing plots and the KS distances for the light-tail part, we consider that the GH distribution can fit the data of the downward jump duration well; and for the upward jump size, the Gamma distribution can fit well.

In conclusion, these test results show that the upward jump duration and the downward jump size are heavy-tail distributed, and the downward jump duration and the upward jump size are light-tail distributed.
Chapter 4

Dependence Structure Test

As discussed in Chapter 2, the jump process for asset pricing is usually modelled as a Poison process. This implies two assumptions. The first assumption that the duration between jumps are exponentially distributed fails, as we have shown in previous chapters. Another assumption, that the duration between jumps are independent random variables, also need to be checked. In this chapter, we will test dependence structure for both jump duration and jump size.

4.1 BDS Test

Here we use the BDS statistic (see [7]) to test dependence structure of the time series for both jump duration and jump size. The BDS statistic is defined as

\[ w_{m,n}(\epsilon) = \sqrt{n - m + 1} \frac{c_{m,n}(\epsilon) - c_{1,n-m+1}(\epsilon)}{\sigma_{m,n}(\epsilon)}. \]  

(4.1)

Here \( n \) is the sample size, \( m \) is the embedding dimension, and \( c_{m,n}(\epsilon) \) is defined as

\[ c_{m,n}(\epsilon) = \frac{2}{(n - m + 1)(n - m)} \sum_{s=m+1}^{n} \sum_{t=s+1}^{n} \prod_{j=0}^{m-1} I_\epsilon(X_{s-j}, X_{t-j}), \]

where

\[ I_\epsilon(X_{s-j}, X_{t-j}) = \begin{cases} 1 & \text{if } |X_{s-j} - X_{t-j}| < \epsilon, \\ 0 & \text{otherwise.} \end{cases} \]

And the consistent estimator \( \sigma_{m,n}^2(\epsilon) \) is

\[ \sigma_{m,n}^2(\epsilon) = 4 \left[ k^m + 2 \sum_{j=1}^{m-1} k^{m-j} \epsilon^{2j} + (m - 1)^2 \epsilon^{2m} - m^2 k \epsilon^{2m-2} \right]. \]
where
\[ c = c_{1,n}(\epsilon) , \]
\[ k = k_n(\epsilon) = \frac{6}{n(n-1)(n-2)} \sum_{t=1}^{n} \sum_{s=t+1}^{n} \sum_{r=s+1}^{n} h_r(X_t, X_s, X_r) , \]
\[ h_r(i,j,k) = \frac{1}{3} [ I_r(i,j)I_r(j,k) + I_r(i,k)I_r(k,j) + I_r(j,i)I_r(i,k) ] . \]

Since the BDS statistic is asymptotically \( N(0,1) \) distributed and is two-sided, the null of independence and identical distribution will be rejected at 5% level when \( |w_{m,n}(\epsilon)| > 1.96 \). (See [3])

### 4.2 Dependence Test Results

This test is carried out with the Matlab code of Ludwig Kanzler, (see [17]). Here the testing objectives are non-normally distributed and with a small sample size, therefore, besides the \texttt{bds} function, we also need the \texttt{bdssig} function. The return value of the \texttt{bdssig} function will be 0.005, 0.01, 0.025, 0.05 or 1. For example, when the return value is 0.01, the equivalent two-sided significance level will approximately be 0.02, (the detailed description for these two functions can also be found in [4]). The test results are as follows.

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>0.5985</td>
<td>0.9521</td>
<td>0.7380</td>
<td>0.3449</td>
<td>0.0172</td>
<td>-0.5839</td>
<td>-0.8183</td>
<td>-0.8076</td>
<td>-0.7000</td>
</tr>
<tr>
<td>return</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 4.1: Size of upward jump**

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>1.3159</td>
<td>1.4190</td>
<td>0.9864</td>
<td>0.8537</td>
<td>0.0529</td>
<td>-0.6276</td>
<td>-1.1339</td>
<td>-0.8389</td>
<td>-0.7461</td>
</tr>
<tr>
<td>return</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 4.2: Size of downward jump**

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>return</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
</tr>
</tbody>
</table>

**Table 4.3: Duration for upward jump**

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>1.2599</td>
<td>2.1148</td>
<td>2.6047</td>
<td>3.2856</td>
<td>3.2031</td>
<td>3.7322</td>
<td>4.4233</td>
<td>5.1866</td>
<td>5.8934</td>
</tr>
<tr>
<td>return</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0500</td>
<td>0.0250</td>
<td>0.0250</td>
<td>0.0100</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
</tr>
</tbody>
</table>

**Table 4.4: Duration for downward jump**

20
According to the test results in the above tables, we conclude that the data of jump sizes for both upward jumps and downward jumps are independent. However, for durations for upward jumps and downward jumps, the data are dependent. In the following figures, we also see that the durations of jump are clustered but the jump sizes are not.

\[ \text{(a) Duration for jump-up} \quad \text{(b) Duration for jump-down} \]

**Figure 4.1: Stems of jump duration**

\[ \text{(a) Size of jump-up} \quad \text{(b) Size of jump-down} \]

**Figure 4.2: Stems of jump size**

### 4.3 Modelling for Jump Duration

In the previous section, the BDS test results show that the jump durations for both jump-up and jump-down are correlated. In this section we will try to find a proper model for jump duration.
4.3.1 AR(1) Model

The AR(1) model is given by

\[ y_n = a_0 + a_1 y_{n-1} + \sigma \epsilon_n, \]  

(4.2)

where \( a_0 \) and \( a_1 \) are constants, and \( \epsilon_n \sim N(0,1) \).

Here, taking the logarithm for data of jump-up duration and jump-down duration, we estimate the parameters \( a_0 \) and \( a_1 \) by LS method, and estimate the parameter \( \sigma \) of the residual by ML method. The estimated parameters are listed in the following table.

<table>
<thead>
<tr>
<th>Duration for jump-up</th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration for jump-up</td>
<td>2.17197</td>
<td>0.223581</td>
<td>1.31999</td>
</tr>
<tr>
<td>Duration for jump-down</td>
<td>2.34962</td>
<td>0.115215</td>
<td>1.53022</td>
</tr>
</tbody>
</table>

Table 4.5: Estimated parameters for jump duration

With the estimated parameters for the jump-up duration, we simulate one sample path and compare the empirical distribution of simulated sample data with the original data. And same to the jump-down duration.

![Graphs showing simulated sample paths and empirical distribution comparisons](image)

Figure 4.3: Simulated Sample Paths and Empirical Distribution Comparing
Observing the comparing plots of empirical distributions, we find that the empirical distributions of the data simulated by the AR(1) model can fit the empirical distributions of the original data very well. For further verification of the AR(1) model, we simulated 1000 sample paths for jump-up and jump-down respectively and compare their empirical distributions with the empirical distributions of the original logarithmic data with the KS distance test\(^1\).

\[
\begin{array}{cccc}
\text{KS distance} & 0.0993 & 8.1053e-4 & 0.0435 \\
\text{p value} & 0.535 & 0.0776 & 0.2319
\end{array}
\]

Table 4.6: KS distance and \(p\) value for 1000 simulations of jump-up duration

\[
\begin{array}{cccc}
\text{KS distance} & 0.1322 & 0.0014 & 0.0507 \\
\text{p value} & 0.2706 & 0.0651 & 0.9993
\end{array}
\]

Table 4.7: KS distance and \(p\) value for 1000 simulated paths of jump-down duration

According to the results of KS distance test, there are 33 simulations for jump-up duration rejected by the KS test in 1000 simulations; and for jump-down duration there are 198 simulations rejected in 1000 simulations. Hence we can consider that the AR(1) model performs very well for jump-up duration. And though in the jump-down duration case, the performance of the AR(1) model is not as good as that in jump-up duration case, we still consider the AR(1) model as a reasonably proper model for jump-down duration.

\(1\)Here we use the Matlab built-in function, “kstest2”.

Figure 4.4: Histogram of KS distance
Chapter 5

Counting Process

Letting $N(t)$ be a counting process for counting the jump times, we consider the jump model for asset pricing can be

$$S_t = S_0 \exp \left\{ \mu t + \sigma W_t + \sum_{i=1}^{N^1(t)} \xi^1_i - \sum_{i=1}^{N^2(t)} \xi^2_i \right\},$$

(5.1)

where $N^1(t)$ and $N^2(t)$ count the number of upward jumps and downward jumps separately; $\xi^1_i$ and $\xi^2_i$ are the upward and downward jump size. In the last chapter, we find that duration for jumps are correlated and the logarithm of the data can be an AR(1) process. With these results, we will investigate the relation between the jump duration model and the intensity process, $\lambda(t)$, of the counting process. Moreover we will also discuss the properties of the counting process $N(t)$.

5.1 Stochastic Intensity

Definition 6 (Stochastic Intensity, (see C. G. Bowsher [6] Definition 3)) Let $N(t)$ be a simple point process on $[0, \infty]$ that is adapted to some filtration $\{\mathcal{F}_t\}$, and let $\lambda(t)$ be a positive, $\mathcal{F}_t$-predictable process. If

$$E[N(s) - N(t)|\mathcal{F}_t] = E[\int_t^s \lambda(u)du|\mathcal{F}_t] \quad P\text{-a.s.,}$$

for all $t, s$ such that $0 \leq t \leq s$, then $\lambda(t)$ is the $(P, \mathcal{F}_t)$-intensity of $N(t)$.

The relation between the stochastic intensity $\lambda(t)$ and the compensator of a counting process $A(t)$ is given by

$$A(t) = \int_0^t \lambda(s)ds.$$  

If $A(t)$ is differentiable, then

$$\lambda(t) = A'(t-).$$
Further the compensator $A(t)$ of $N(t)$ exists uniquely. Hence the stochastic intensity $\lambda(t)$ also exists uniquely. In other words, one predictable process $\lambda(t)$ can characterize one counting process $N(t)$.

**Theorem 1 (F. C. Klebaner [18] Theorem 9.6)** Let $N$ be a counting process generated by the sequence $T_n$, and denote by $U_{n+1} = T_{n+1} - T_n$ the interarrival times, $T_0 = 0$. Let $F_n(t) = P(U_{n+1} \leq t | T_1, \ldots, T_n)$ denote the regular conditional distributions, and $F_0 = P(T_1 \leq t)$. Then the compensator $A(t)$ is given by

$$A(t) = \sum_{i=0}^{\infty} \int_{0}^{t \wedge T_{i+1} - t \wedge T_i} \frac{dF_i(s)}{1 - F_i(s^{-})}$$

(5.2)

For a counting process, we have

$$N(t) = \sum_{n=1}^{\infty} I(T_n \leq t), \quad N(0) = 0,$$

where $T_1, T_2, \ldots$ denotes the arrival time of the events\(^1\). Applying Theorem 1, we can get the compensator $A(t)$ of the counting process, and if the conditional distribution $F_n$ are continues with $F_0 = 0$, we can simplify equation (5.2) and get

$$A(t) = -\sum_{n=0}^{\infty} \log(1 - F_n(t \wedge T_{n+1} - t \wedge T_n)),$$

(5.3)

(see [18]). We have

$$t \wedge T_{n+1} = \begin{cases} t & \text{if } t < T_{n+1}, \\ T_{n+1} & \text{if } t \geq T_{n+1}, \end{cases}$$

(5.4)

and

$$t \wedge T_n = \begin{cases} t & \text{if } t < T_n, \\ T_n & \text{if } t \geq T_n. \end{cases}$$

(5.5)

Since $\{T_n\}$ are the sequence of arrival times, we have $T_{n+1} > T_n$. Hence with equation (5.4) and equation (5.5), we have

$$t \wedge T_{n+1} - t \wedge T_n = \begin{cases} 0 & \text{if } t < T_n < T_{n+1}, \\ t - T_n & \text{if } T_n \leq t < T_{n+1}, \\ T_{n+1} - T_n & \text{if } T_n < T_{n+1} \leq t. \end{cases}$$

(5.6)

Letting $g(t, T_n, T_{n+1})$ denote $t \wedge T_{n+1} - t \wedge T_n$, we can assume that $g(t, T_n, T_{n+1})$ is a continuous function of time $t$ with “parameters” $T_n$ and $T_{n+1}$. Note that $T_n$ and $T_{n+1}$ are not the parameters in normal sense, they are random variables (stopping times). In other words, $T_n$ is distributed with a certain distribution, so is $T_{n+1}$.

The sequence of stopping time, $T_1, T_2, \ldots, T_n, \ldots$, divide $[0, \infty)$ into many small intervals, i.e. $[T_n, T_{n+1})$, $n = 1, 2, 3, \ldots$. For any certain time $t$, it must locate in one of these small intervals and must locate in only one interval, (since there is

\(^1\)In this report, the events are the jumps of the stock prices.
no intersection part in these small intervals). Now let $t$ locate in $[T_k, T_{k+1})$, i.e. $T_k \leq t < T_{k+1}$. With equation (5.6), we know that for this interval, $[T_k, T_{k+1})$,

$$t \land T_{n+1} - t \land T_n = t - T_k.$$ 

And for the small intervals on the left side of $[T_k, T_{k+1})$, all of $T_1, T_2, \ldots, T_{k-1}$ are less than $t$. Hence for these intervals, $[T_j, T_{j+1}), j = 0, 1, \ldots, k-1$, with equation (5.6), we know that

$$t \land T_{n+1} - t \land T_n = T_{j+1} - T_j, \quad j = 0, 1, \ldots, k-1.$$ 

Similarly for the intervals on the right side of $[T_k, T_{k+1})$, all the stopping time, $T_{k+1}, T_{k+2}, \ldots$, are greater than $t$. Hence for these intervals, $[T_i, T_{i+1}), i = k + 1, k + 2, \ldots$, according to equation (5.6), we can get that

$$t \land T_{n+1} - t \land T_n = 0, \quad i = k + 1, k + 2, \ldots.$$ 

Through the analysis above, we can expand the series on right-hand side of equation (5.3) in the following form,

$$A(t) = -\sum_{n=0}^{\infty} \log(1 - F_n(t \land T_{n+1} - t \land T_n))$$

$$= -\left[ \log(1 - F_1(T_2 - T_1)) + \log(1 - F_2(T_3 - T_2)) + \cdots + \right.$$

$$+ \log(1 - F_{k-1}(T_k - T_{k-1})) + \log(1 - F_k(t - T_k))+$$

$$+ \log(1 - F_{k+1}(0)) + \log(1 - F_{k+2}(0)) + \cdots \right].$$  

(5.7)

Using equation (5.7), we find that there is only one term containing the time $t$, that is the term $\log(1 - F_k(t - T_k))$. Hence differentiation of both sides of equation (5.7) gives

$$A'(t) = -\frac{d}{dt} \log(1 - F_k(t \land T_{k+1} - t \land T_k))$$

$$= -\frac{d}{dt} \log(1 - F_k(t - T_k))$$

$$= \frac{F'_k(t - T_k) \frac{d}{dt}(t - T_k)}{1 - F_k(t - T_k)}$$

$$= \frac{f_k(t - T_k)}{1 - F_k(t - T_k)}.$$  

(5.8)

Here we assume the conditional distribution function $F_k(s)$ is differentiable and let $f_k(s) = \frac{d}{ds} F_k(s)$ denote the corresponding conditional density function.

However we need to notice that $T_1, T_2, \ldots$ are a sequence of random variables, so for a certain time $t$, the small interval $[T_k, T_{k+1})$ in which $t$ is located is also in random, i.e. $k$ is a random variable. Hence to improve the expression of $A'(t)$ in equation (5.8), we introduce two random variables, $T_{N(t)}$ and $T_{N(t)+1}$, in which the subscripts are random. Let $T_{N(t)}$ denote the time when the last event happens just
before time $t$ (or exactly at time $t$) and $T_{N(t)+1}$ denote the time when the first event happens after time $t$. With these two random variables, we have

$$T_{N(t)} \leq t < T_{N(t)+1},$$

in other words, for any certain time $t$, it is located in the random interval $[T_{N(t)}, T_{N(t)+1})$. Hence we use $N(t)$ instead of $k$ in equation (5.8) and get a new expression of $A'(t)$,

$$A'(t) = \frac{f_{N(t)}(t - T_{N(t)})}{1 - F_{N(t)}(t - T_{N(t)})}.$$  \hspace{1cm} (5.9)

If we assume that all the conditional density function $f_n(s)$, $n = 1, 2, \ldots$ are continuous functions, then equation (5.9) is also the expression of the stochastic intensity $\lambda(t) = A'(t-)$, i.e.

$$\lambda(t) = \frac{f_{N(t)}(t - T_{N(t)})}{1 - F_{N(t)}(t - T_{N(t)})}.$$  \hspace{1cm} (5.10)

Based on our result in the previous chapter, i.e. the logarithm of the jump duration can be an AR(1) process, we can use the expression (5.10) to compute $\lambda(t)$. Recall that the AR(1) model is given by

$$y_n = a_0 + a_1 y_{n-1} + \sigma \epsilon_n,$$

where $a_0$ and $a_1$ are constant, and $\epsilon_n \sim N(0, 1)$. And in Theorem 1, we let $U_n$ denote the interarrival time between the $n$th event and the $n - 1$th event\(^2\). Then we have

$$y_n = \log U_n$$

and

$$U_n = T_n - T_{n-1}.$$

Since

$$U_{n+1} \leq t \iff e^{y_{n+1}} \leq t,$$

we can get

$$F_n(t) = P(U_{n+1} \leq t|T_1, \ldots , T_n) = P(e^{y_{n+1}} \leq t|T_1, \ldots , T_n) = P(e^{a_0 + a_1 y_n + \sigma \epsilon_{n+1}} \leq t|y_n) = P(e^{\sigma \epsilon_{n+1}} \leq t e^{-a_0 - a_1 y_n}|y_n) = \psi(t e^{-a_0 - a_1 y_n}),$$  \hspace{1cm} (5.11)

here $\psi(x) = \frac{1}{\sqrt{2\pi} \sigma} \int_0^x \frac{1}{u} e^{-\frac{(\ln u)^2}{2\sigma^2}} du$ is the lognormal distribution function with parameters $(0, \sigma)$. This is result quite obvious, since that in $e^{a_0 + a_1 y_n + \sigma \epsilon_{n+1}}$, $y_n$ just depends

\(^2\)In this report, $U_n$ is also called as the duration between the jumps.
on $T_n - T_{n-1}$ and $\epsilon_{n+1}$ is independent from $T_1, \ldots, T_n$; hence $e^{\epsilon_0 + a_1 y_0 + \sigma \epsilon_{n+1}}$ just depends on $T_n - T_{n-1}$, i.e. just depends on $U_n$; or we can say it just depends on $y_n$.

From (5.11), we have
\[
F_n(t - T_{N(t)}) = \psi((t - T_{N(t)})e^{-(\epsilon_0 + a_1 y_n)})
= \psi((t - T_{N(t)})e^{-\epsilon_0 U_n^{-a_1}})
= \psi((t - T_{N(t)})e^{-\epsilon_0 (T_n - T_{n-1})^{-a_1}}).
\] (5.12)

Since $F_{N(t)}(s)$ must be one of $F_0(s), F_1(s), \ldots, F_n(s)$, we therefore get
\[
F_{N(t)}(t - T_{N(t)}) = \psi((t - T_{N(t)})e^{-\epsilon_0 (T_{N(t)} - T_{N(t)-1})^{-a_1}}),
\] (5.13)

and
\[
f_{N(t)}(t - T_{N(t)}) = F'_{N(t)}(t - T_{N(t)})
= \frac{1}{\sqrt{2\pi}\sigma} e^{-(\epsilon_0 + a_1 y_n)} \frac{1}{(t - T_{N(t)})e^{-(\epsilon_0 + a_1 y_n)}}
= \exp\left\{ -\frac{1}{2\sigma^2} \left[ -\epsilon_0 - \log(t - T_{N(t)}) - a_1 \log(T_{N(t)} - T_{N(t)-1}) \right] \right\}
\cdot \frac{1}{\sqrt{2\pi}\sigma(t - T_{N(t)})}.
\] (5.14)

Now the stochastic intensity $\lambda(t)$ of the counting process $N(t)$ in equation (5.1) is given by
\[
\lambda(t) = \exp\left\{ -\frac{1}{2\sigma^2} \left[ -\epsilon_0 - a_1 \log(T_{N(t)} - T_{N(t)-1}) + \log(t - T_{N(t)}) \right] \right\}
\cdot \frac{\sqrt{2\pi}\sigma(t - T_{N(t)})}{1 - \psi(e^{-\epsilon_0 (t - T_{N(t)})(T_{N(t)} - T_{N(t)-1})^{-a_1})}},
\] (5.15)

where $\psi(x)$ is the lognormal distribution function with parameter $(0, \sigma)$.

### 5.2 Properties of the Counting Process

In the previous section, we discussed the counting process $N(t)$ with the stochastic intensity given by equation (5.15). Now we will further discuss the properties of this counting process model.

#### 5.2.1 Distribution Function for the Interarrival Time

Here we will compute $F_{U_n}(t)$, the distribution function for the interarrival time $U_n, (n = 1, 2, \ldots)$.

We have $y_1 = a_0 + a_1 y_0 + \sigma \epsilon_1$, $y_0 = \frac{a_0}{1 - a_1}$ and get
\[
y_1 = a_0 + \frac{a_1 a_0}{1 - a_1} + \sigma \epsilon_1
= \frac{a_0}{1 - a_1} + \sigma \epsilon_1,
\]
where \( \epsilon_1 \sim N(0, 1) \) and \( y_1 \sim N\left(\frac{a_0}{a_1}, \sigma^2\right) \). Further \( U_1 = e^{y_1} \), so that \( U_1 \) is lognormal distributed.

For \( F_{U_1}(t) \), we have

\[
F_{U_1}(t) = P(U_1 \leq t) = F(e^{y_1} \leq t)
= \frac{1}{\sqrt{2\pi}a} \int_0^t \frac{1}{u} e^{-\frac{(\log u - \frac{a_0}{a_1})^2}{2\sigma^2}} du
= \frac{1}{\sqrt{2\pi}a} \int_0^t \frac{1}{u} e^{-\frac{(\log \tau - \frac{a_0}{a_1})^2}{2\sigma^2}} d\tau.
\] (5.16)

Let \( e^{-\frac{a_0}{a_1}u} = \tau \), then \( \frac{1}{u} du = e^{-\frac{a_0}{a_1}} \frac{1}{\tau} e^{-\frac{a_0}{a_1}} d\tau = \frac{1}{\tau} d\tau \). And equation (5.16) can be rewritten as

\[
F_{U_1}(t) = \frac{1}{\sqrt{2\pi}a} \int_0^{te^{-\frac{a_0}{a_1}}} \frac{1}{\tau} e^{-\frac{(\log \tau - \frac{a_0}{a_1})^2}{2\sigma^2}} d\tau
= \psi(te^{-\frac{a_0}{a_1}}),
\] (5.17)

where

\[
\psi(x) = \frac{1}{\sqrt{2\pi}a} \int_0^x \frac{1}{u} e^{-\frac{(\log u - \frac{a_0}{a_1})^2}{2\sigma^2}} du
\]
is the lognormal distribution function with parameters \((0, \sigma)\).

For \( F_{U_n}(t) \), \( n = 2, 3, \ldots \), we have

\[
F_{U_n}(t) = P(U_n \leq t) = \int_0^\infty P(U_n \leq t|U_{n-1} = \tau) f_{U_{n-1}}(\tau) d\tau,
\] (5.18)

where \( f_{U_{n-1}} \) is the density function of the interarrival time \( U_{n-1} \). And since \( y_n = a_0 + a_1 y_{n-1} + \sigma \epsilon_n \) and \( y_n = \log U_n \), we get

\[
P(U_n \leq t|U_{n-1} = \tau) = P(e^{y_n} \leq t|e^{y_{n-1}} = \tau)
= P(e^{a_0 + a_1 y_{n-1} + \sigma \epsilon_n} \leq t|y_{n-1} = \log \tau)
= P( e^{\sigma \epsilon_n} \leq te^{-a_0 - a_1 \log \tau}|y_{n-1} = \log \tau)
= \psi(te^{-a_0 - a_1 \log \tau}).
\] (5.19)

Letting \( F(t, \tau) = \psi(te^{-a_0 - a_1 \log \tau}) \), with equation (5.18), we get the recursive formula for \( F_{U_n}(t) \),

\[
F_{U_n}(t) = \int_0^\infty F(t, \tau) f_{U_{n-1}}(\tau) d\tau.
\] (5.20)

Putting equation (5.16) and (5.20) together, we have the distribution function for the interarrival time

\[
F_{U_n}(t) = \begin{cases} 
\psi(te^{-\frac{a_0}{a_1}}) & \text{if } n = 1, \\
\int_0^\infty F(t, \tau) f_{U_{n-1}}(\tau) d\tau & \text{if } n > 1.
\end{cases}
\] (5.21)

29
5.2.2 Distribution Function for the Arrival Time

Let $F_{T_n}(t)$ denote the distribution function for the arrival time $T_n, (n = 1, 2, \ldots)$. With the result of equation (5.16), we get

$$F_{T_1}(t) = P(T_1 \leq t) = P(U_1 \leq t) = \psi(\text{e}^{-\frac{t}{1-a_1}}).$$  \hspace{1cm} (5.22)

For $F_{T_n}(t), n = 2, 3, \ldots$, we have

$$F_{T_n}(t) = P(T_n \leq t) = \int_0^t P(T_n \leq t|T_{n-1} = \tau)f_{T_{n-1}}(\tau)d\tau$$
$$= \int_0^t P(\sum_{j=1}^n U_j \leq t|\sum_{j=1}^{n-1} U_j = \tau)f_{T_{n-1}}(\tau)d\tau$$
$$= \int_0^t P(U_n \leq t - \tau|\sum_{j=1}^{n-1} U_j = \tau)f_{T_{n-1}}(\tau)d\tau. \hspace{1cm} (5.23)$$

Let $R_n(t, \tau) = P(U_n \leq t|\sum_{j=1}^{n-1} U_j = \tau)$. Then equation (5.23) can be written as

$$F_{T_n}(t) = \int_0^t R_n(t - \tau, \tau)f_{T_{n-1}}(\tau)d\tau. \hspace{1cm} (5.24)$$

Now the function $R_n(t, \tau), (n \geq 2)$ can be computed recursively. Because $R_2(t, \tau) = P(U_2 \leq t|U_1 = \tau)$, with the definition of $F(t, \tau)$ in equation (5.19), we get

$$R_2(t, \tau) = F(t, \tau). \hspace{1cm} (5.25)$$

For $R_n(t, \tau), (n > 2)$, we have

$$R_n(t, \tau) = \int_0^t F(t, s)R'_{n-1}(s, \tau - s)ds, \quad n > 2, \hspace{1cm} (5.26)$$

where $R'_{n-1}(s, u) = \frac{\partial}{\partial u}R_{n-1}(s, u)$. We prove this by mathematical induction:

First when $n = 3,$

$$R_3(t, \tau) = P(U_3 \leq t|U_1 + U_2 = \tau)$$
$$= \int_0^\tau P(U_3 \leq t|U_2 = s, U_1 + U_2 = \tau)f_{U_2}(s|U_1 + U_2 = \tau)ds$$
$$= \int_0^\tau P(U_3 \leq t|U_2 = s)f_{U_2}(s|U_1 = \tau - s)ds \quad \text{(since } U_3 \text{ just depends on } U_2)$$
$$= \int_0^\tau F(t, s)R'_2(s, \tau - s)ds. \hspace{1cm} (5.27)$$
If the formula for $R_n(t, \tau)$ holds, then

$$R_{n+1}(t, \tau) = P(U_{n+1} \leq t | \sum_{j=1}^{n} U_j = \tau)$$

$$= \int_0^\tau P(U_{n+1} \leq s, \sum_{j=1}^{n} U_j = \tau) f_{U_n}(s) \sum_{j=1}^{n} U_j = \tau) ds$$

$$= \int_0^\tau P(U_{n+1} \leq s) f_{U_n}(s) \sum_{j=1}^{n-1} U_j = \tau - s) ds$$

$$= \int_0^\tau F(t, \tau) R'_n(s, \tau - s) ds.$$

(5.28)

Hence equation (5.26) holds by induction.

Now we can write the distribution function for arrival time

$$F_{T_n}(t) = \begin{cases} \psi(te^{-a_0-a_1 \log \tau}) & \text{if } n = 1, \\ \int_0^t R_n(t-\tau, \tau) d\tau & \text{if } n \geq 2, \end{cases}$$

(5.29)

where

$$R_n(t, \tau) = \begin{cases} F(t, \tau) & \text{if } n = 2, \\ \int_0^\tau F(t, \tau) R'_n(s, \tau - s) ds & \text{if } n \geq 3, \end{cases}$$

(5.30)

and $F(t, \tau) = \psi(te^{-a_0-a_1 \log \tau})$.

5.2.3 Probability Distribution of $N(t)$

The probability of $n$ jumps occurring up till time $t$ is

$$P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n + 1\}$$

$$= P\{T_n \leq t\} - P\{T_{n+1} \leq t\}$$

$$= F_{T_n}(t) - F_{T_{n+1}}(t)$$

$$= \int_0^t dF_{T_n}(t) - \int_0^t R_{n+1}(t-\tau, \tau) f_{T_n}(\tau) d\tau$$

$$= \int_0^t (1 - R_{n+1}(t-\tau, \tau)) dF_{T_n}(\tau).$$

(5.31)

Since $N(t) = \sum_{n=1}^{\infty} I(T_n \leq t)$, the mean value of the jump times in $[0, t]$ is

$$E[N(t)] = \sum_{n=1}^{\infty} E[I(T_n \leq t)]$$

$$= \sum_{n=1}^{\infty} P(T_n \leq t)$$

$$= \sum_{n=1}^{\infty} F_{T_n}(t).$$

(5.32)
5.3 Conclusions

Using the assumption that the logarithm of the jump duration is an AR(1) process, we derive the analytical expression of the stochastic intensity $\lambda(t)$. Moreover through analysis of the properties of the counting process, we find the distributions for inter-arrival times or arrival times.
Chapter 6

Conclusions

In this project, we focus on jump characteristics for financial asset pricing. We give a formal definition of jumps, based on which we analyze the properties of jump sizes and jump durations for upward and downward jumps.

We have shown that the durations for jump-up and the sizes of jump-down are heavy-tailed distributed, while the durations for jump-down and the sizes of jump-up are light-tailed distributed. We also investigate the dependence structure for both jump size sequence and jump duration sequence. Moreover one very interesting result is that the duration between the jumps are correlated and the logarithm of the data can be modelled as an AR(1) process. Though our test result is based on one stock, we notice that it is a common phenomenon that in financial market, jumps are clustered. Hence we think that for most stocks, jump durations are dependent and can not simply be modelled as a Poisson process as is assumed in the literatures (see e.g. [14]).

Finally we discuss some properties of the new counting process model which we can derive form our empirical findings.

In the future work, we need to test more stocks, and verify whether other stocks have the similar tail properties of jumps as IBM stock and whether the jumps are also correlated. Further in the previous chapters, we saw that the AR(1) model for jump-down duration still can be improved: to find a better model for jump duration is also one important part in future research.
Bibliography


Department of Mathematical Statistics, Chalmers University of Technology and Gothenburg University, 2004.

http://www2.gol.com/users/kanzler/index.htm

[18] Fima C. Klebaner Introduction to Stochastic Calculus with Applications
Imperial College Press, 1998.

Department of Mathematical Statistics, Chalmers University of Technology and Gothenburg University, 2005.

[20] Steven E. Shreve Stochastic Calculus for Finance II: Continuous-Time Models


[22] Website of mathworld.
http://mathworld.wolfram.com

[23] Yahoo Finance
http://finance.yahoo.com/