Anonym kod	LMA019 Algebra 2018-08-20	Poäng

1. Till nedanstående uppgifter skall korta lösningar redovisas, samt svar anges, på anvisad plats (endast lösningar och svar på detta blad, och på anvisad plats, beaktas).

   (a) Bestäm konstanten $a \in \mathbb{R}$ så att de två vektorerna $\mathbf{u} = (1, 2, a)$ och $\mathbf{v} = (5, 10, -1)$ blir parallella. (3 p)

   Lösning: $|u \times v| = |u||v|\sin\theta$ så $u \parallel v \Leftrightarrow u \times v = 0$

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & a \\ 5 & 10 & -1 \end{vmatrix} = \begin{pmatrix} -2-10a \\ 1+5a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a = -\frac{1}{5}$$

   Svar: $a = -\frac{1}{5}$

   (b) Låt $T : \mathbb{R}^2 \to \mathbb{R}^2$ vara en linjär avbildning som uppfyller $T(1, 0) = (1, -1)$ och $T(0, 1) = (2, 1)$. Finn ett $\mathbf{x} \in \mathbb{R}^2$ sådan att $T(\mathbf{x}) = (1, 1)$. (3 p)

   Lösning: $T(x) = Ax$ där $A = [T(e_1)\ T(e_2)] = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$

$$T(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 1 & 1 \end{array}\right) \begin{array}{c} 1 \\ \end{array} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \end{pmatrix} \begin{array}{l} \cdot 3 \\ \cdot(-2) \end{array} \sim \begin{pmatrix} 3 & 6 & 3 \\ 0 & -6 & -4 \end{pmatrix} \begin{array}{c} \\ 1 \end{array} \sim \begin{pmatrix} 3 & 0 & -1 \\ 0 & -6 & -4 \end{pmatrix} \begin{array}{l} \cdot \frac{1}{3} \\ \cdot(-\frac{1}{6}) \end{array} \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \end{array}\right)$$

   Svar: $x = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

   (c) Låt $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ där $\mathbf{v}_1 = (1, 1, 1)$ och $\mathbf{v}_2 = (1, -1, 5)$. Ligger vektorn $\mathbf{u} = (3, 4, 5)$ i W. (3 p)

   Lösning: $u \in W$ om det finns $x_1, x_2 \in \mathbb{R}$ s.a. $x_1 v_1 + x_2 v_2 = u$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 4 \\ 1 & 5 & 5 \end{array}\right) \begin{array}{c} -1 \\ \\ \\ \end{array} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & 1 \\ 0 & 4 & 2 \end{pmatrix} \begin{array}{c} \\ 2 \\ \\ \end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & 1 \\ 0 & 0 & 4 \end{array}\right) \leftarrow 0 = 4 \ \text{↯ Går ej}$$

   Svar: Nej! $u \notin W$

(d) Beräkna determinanten av A^3 då (3 p)

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Lösning: $\det(A) = \begin{vmatrix} \overset{+}{1} & \overset{-}{2} & \overset{+}{2} \\ 0 & 3 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 1\cdot(6-2) - 2(0-2) + 2(0-6) =$

$= 4 + 4 - 12 = -4$

$\det(A^3) = \det(A^2 \cdot A) = \det(A^2)\cdot\det(A) = -4\cdot\det(A\cdot A) =$

$= -4\cdot\det(A)\cdot\det(A) = (-4)^3 = -64$

Svar: -64

(e) Beräkna inversen till matrisen (3 p)

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 3 & 3 & -1 \end{pmatrix}.$$

Lösning: $(B \,\vdots\, I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 3 & 3 & -1 & 0 & 0 & 1 \end{array}\right) (-2)\ (-3) \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -4 & -3 & 0 & 1 \end{pmatrix} \cdot 4 \sim$

$\sim \begin{pmatrix} 4 & 4 & 4 & 4 & 0 & 0 \\ 0 & -3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -4 & -3 & 0 & 1 \end{pmatrix} (1) \sim \begin{pmatrix} 4 & 4 & 0 & 1 & 0 & 1 \\ 0 & -3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -4 & -3 & 0 & 1 \end{pmatrix} \begin{matrix} \cdot 3 \\ \cdot 4 \\ \end{matrix} \sim$

$\sim \begin{pmatrix} 12 & 12 & 0 & 3 & 0 & 3 \\ 0 & -12 & 0 & -8 & 4 & 0 \\ 0 & 0 & -4 & -3 & 0 & 1 \end{pmatrix} \begin{matrix} \\ (1) \\ \cdot(-3) \end{matrix} \sim \begin{pmatrix} 12 & 0 & 0 & -5 & 4 & 3 \\ 0 & -12 & 0 & -8 & 4 & 0 \\ 0 & 0 & 12 & 9 & 0 & -3 \end{pmatrix} \Rightarrow B^{-1} = \frac{1}{12}\begin{pmatrix} -5 & 4 & 3 \\ 8 & -4 & 0 \\ 9 & 0 & -3 \end{pmatrix}$

Test: $B^{-1}B = \frac{1}{12}\begin{pmatrix} -5 & 4 & 3 \\ 8 & -4 & 0 \\ 9 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 3 & 3 & -1 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}$ ok!

Svar: $B^{-1} = \frac{1}{12}\begin{pmatrix} -5 & 4 & 3 \\ 8 & -4 & 0 \\ 9 & 0 & -3 \end{pmatrix}$

2. $\ell_1: \frac{x+3}{2} = \frac{y-5}{1} = \frac{z-1}{3} \Rightarrow \begin{matrix} \mathbf{x}_1 = (-3, 5, 1) \\ \mathbf{v}_1 = (2, 1, 3) \end{matrix}$

$\Rightarrow \ell_1: \mathbf{x} = (-3, 5, 1) + t(2, 1, 3), \quad t \in \mathbb{R}$

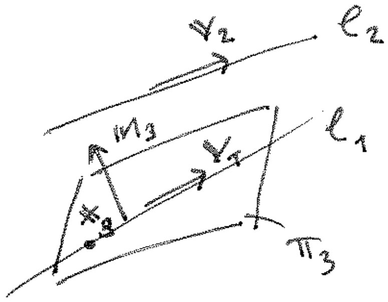
$\pi_1: x + y + 2z = 5 \Rightarrow \mathbf{n}_1 = (1, 1, 2)$

$\pi_2: x + y = 3 \Rightarrow \mathbf{n}_2 = (1, 1, 0)$

$$\Rightarrow \mathbf{v}_2 = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix} \end{array} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

$\begin{cases} x + y + 2z = 5 \\ x + y = 3 \end{cases} \Rightarrow \mathbf{x}_2 = (3, 0, 1)$ uppfyller båda dessa ekvationer

$\Rightarrow \ell_2: \mathbf{x} = (3, 0, 1) + t(-2, 2, 0)$



$$\mathbf{n}_3 = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ -2 & 2 & 0 \end{vmatrix} \end{array} = \begin{pmatrix} -6 \\ -6 \\ 6 \end{pmatrix}$$

$\Rightarrow \mathbf{n}_3 = (1, 1, -1)$ (längden av $\mathbf{n}_3$ irrelevant!)

$\Rightarrow \pi_3: x + y - z = D$

$\mathbf{x}_3 = \mathbf{x}_1 \in \pi_3: \quad D = -3 + 5 - 1 = 1$

$\pi_3: x + y - z = 1$

3 (a) <u>Definition</u>: $T : \mathbb{R}^n \to \mathbb{R}^m$ är <u>surjektiv</u> om det för varje $\mathbf{y} \in \mathbb{R}^m$ finns (åtminstone) ett $\mathbf{x} \in \mathbb{R}^n$ s.a. $T(\mathbf{x}) = \mathbf{y}$.

(b) $$T\begin{pmatrix}1\\0\\0\end{pmatrix} = T\left(\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \{T \text{ linjär}\} = T\begin{pmatrix}1\\0\\1\end{pmatrix} - T\begin{pmatrix}0\\0\\1\end{pmatrix} =$$

$$= \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = T\left(\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \{T \text{ linjär}\} = T\begin{pmatrix}1\\1\\0\end{pmatrix} - T\begin{pmatrix}1\\0\\0\end{pmatrix} =$$

$$= \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$$

$T(\mathbf{x}) = A\mathbf{x}$ där $A = \begin{pmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3)\end{pmatrix} =$

$$= \underline{\underline{\begin{pmatrix}2 & -2 & -1\\0 & 1 & 1\end{pmatrix}}}$$

4. $\begin{cases} k + m = 1 \\ 0 + m = 0 \\ 2k + m = -1 \\ -k + m = -2 \end{cases} \iff \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} k \\ m \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}}_{b}$

$$A^T A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$A^T A x = A^T b \implies \left(\begin{array}{cc|c} 6 & 2 & 1 \\ 2 & 4 & -2 \end{array}\right) \sim \begin{pmatrix} 2 & 4 & -2 \\ 6 & 2 & 1 \end{pmatrix} \begin{matrix} -3 \\ \end{matrix} \sim$$

$$\sim \begin{pmatrix} 2 & 4 & -2 \\ 0 & -10 & 7 \end{pmatrix} \cdot 5 \sim \begin{pmatrix} 10 & 20 & -10 \\ 0 & -10 & 7 \end{pmatrix} \begin{matrix} \\ 2 \end{matrix} \sim \begin{pmatrix} 10 & 0 & 4 \\ 0 & 10 & 7 \end{pmatrix} \begin{matrix} \cdot \frac{1}{10} \\ \cdot \frac{1}{10} \end{matrix} \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 4/10 \\ 0 & 1 & 7/10 \end{array}\right)$$

$$\therefore \quad y = \frac{4}{10} x + \frac{7}{10}$$

5. $AXB = C + AX \iff AXB - AX = C \iff$

$\iff AX(B - I) = C \iff X(B-I) = A^{-1}C \iff$

$\iff X = A^{-1}C(B-I)^{-1}$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{3-2}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

$B - I = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

$\Rightarrow (B-I)^{-1} = \frac{1}{6-5}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$\Rightarrow X = A^{-1}C(B-I)^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} =$

$= \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 7 & -10 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 19 & -28 \\ -6 & 9 \end{pmatrix}$

$\therefore \quad X = \begin{pmatrix} 19 & -28 \\ -6 & 9 \end{pmatrix}$

6. $$\underbrace{\begin{pmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 3 & 11 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{b}}$$

$$A_3(\mathbf{b}) = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 11 & 1 \end{pmatrix} \Rightarrow \det(A_3(\mathbf{b})) = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 11 & 1 \end{vmatrix} \end{array} =$$

$$= 1 \cdot (5-0) - 3(2-0) + 1 \cdot (22-15) =$$

$$= 5 - 6 + 7 = 6$$

$$\det(A) = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 3 & 11 & 1 \end{vmatrix} \end{array} = 1 \cdot (5-11) - 3(2-3) + 1 \cdot (22-15) =$$

$$= -6 + 3 + 7 = 4$$

Cramers regel: $x_3 = \dfrac{\det(A_3(\mathbf{b}))}{\det(A)} = \dfrac{6}{4} = \underline{\underline{\dfrac{3}{2}}}$

7\. (a) Sant $\quad \sin(x)\cos(x) = \frac{1}{2}\sin(2x) \leq \frac{1}{2}$

(b) Sant $\quad |e^{-it}| = |\cos(t) - i\sin(t)| = \sqrt{\cos^2(t) + (-\sin(t))^2} = 1$

(c) Sant

(d) Sant $\quad A^T A = I \Rightarrow \det(A^T A) = \det(I) \Leftrightarrow$

$\Leftrightarrow \det(A^T)\det(A) = 1 \Leftrightarrow (\det(A))^2 = 1$

$\Rightarrow \det(A) = \pm 1$

(e) Sant $\quad T(\mathbf{x}) = A\mathbf{x}$ där $A = \begin{pmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{pmatrix} =$

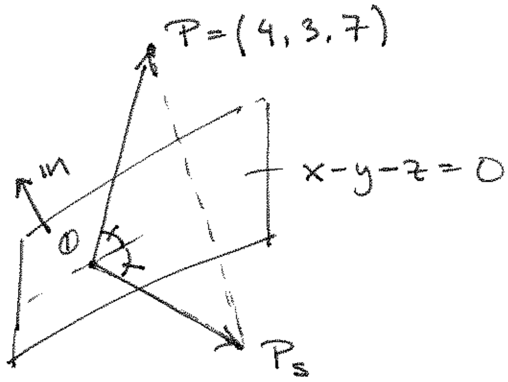
$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

A har en pivotposition i varje rad/kolumn.

8. Givet: $P = (4, 3, 7)$

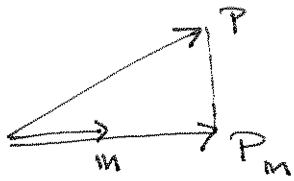
$n = (1, -1, -1)$

Sökt: P_s



Låt P_n vektorproj. av $\overrightarrow{OP}$

längs n:



$$P_n = \frac{\overrightarrow{OP} \cdot n}{|n|^2}\, n = \frac{-6}{3}(1, -1, -1) =$$

$$= (-2, 2, 2)$$

$$P_s = P - 2P_n = (4, 3, 7) - 2 \cdot (-2, 2, 2) =$$

$$= (4, 3, 7) - (-4, 4, 4) = \underline{\underline{(8, -1, 3)}}$$

8. $$\underbrace{\begin{pmatrix} 1 & 2 & 2a \\ 0 & 1 & a \\ 1 & 1 & 2a-1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbb{x}} = \underbrace{\begin{pmatrix} 1 \\ b \\ 3 \end{pmatrix}}_{\mathbb{b}}$$

$$\det(A) = \begin{vmatrix} 1 & 2 & 2a \\ 0 & 1 & a \\ 1 & 1 & 2a-1 \end{vmatrix} = 1\cdot(2a-1-a) - 2\cdot(0-a) + 2a\cdot(0-1) =$$

$$= a - 1 + 2a - 2a = a - 1$$

$$\det(A) = 0 \iff a = 1$$

$\therefore$ Unik lösning då $a \neq 1$, för alla $b \in \mathbb{R}$.

<u>$a = 1$:</u> $$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & b \\ 1 & 1 & 1 & 3 \end{array}\right) \overset{-1}{\sim} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & b \\ 0 & -1 & -1 & 2 \end{pmatrix} \overset{1}{\sim}$$

$$\sim \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & b+2 \end{pmatrix}$$

$\therefore$ Saknas lösning då $a = 1$, $b \neq -2$

Oändligt många lösningar då $a = 1$, $b = -2$

10. Bevis: $0 \leq |\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) =$

$= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 - 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$

$\Leftrightarrow \quad 0 \leq |\mathbf{u}|^2 - 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 \quad \Leftrightarrow$

$\Leftrightarrow \quad 2\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u}|^2 + |\mathbf{v}|^2$

$$\therefore \quad \mathbf{u} \cdot \mathbf{v} \leq \frac{|\mathbf{u}|^2 + |\mathbf{v}|^2}{2}$$

$\blacksquare$