Uppgift 1(a). Med $u_1 = u$, $u_2 = u'$, har vi

$$\begin{cases} u_1' = u_2 \\ u_2' = \mu(1-u_1^2)u_2 - u_1 \end{cases}$$

Med vektorbeteckningar har vi standardformen

$$\begin{cases} \boldsymbol{u}' = \boldsymbol{f}(t,\boldsymbol{u}) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \end{cases} \quad \text{med} \;\; \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \;\; \boldsymbol{f}(t,\boldsymbol{u}) = \begin{bmatrix} u_2 \\ \mu(1-u_1^2)u_2 - u_1 \end{bmatrix}, \;\; \boldsymbol{u}_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

(b).

```
f=@(t,u,mu)[u(2)
            mu*(1-u(1)^2)*u(2)-u(1)];
T=20; mu=0.5; u0=[-2;4];
[t,U]=ode45(@(t,u)f(t,u,mu),[0,T],u0);
plot(t,U(:,1),t,U(:,2))
```

Uppgift 2(a). Inför $x_i = ih, i = 0, 1, \cdots, n+1$, med $h = \frac{L}{n+1}$.

$$-u''(x_i) + u(x_i) = \lambda u(x_i), \; i = 1, 2, \cdots, n$$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + u_i = \lambda u_i, \; i = 1, 2, \cdots, n$$

$$-u_{i-1} + (2+h^2)u_i - u_{i+1} = h^2\lambda u_i, \; i = 1, 2, \cdots, n$$

Randvillkor: $u(0) = 0$ ger $u_0 = 0$, $u'(L) + u(L) = 0$ ger $\frac{u_{n+1}-u_n}{h} + u_{n+1} = 0$, dvs. $u_{n+1} = \frac{1}{1+h}u_n$.
Matrisformulering med $c_i = 2 + h^2$, $i = 1, \cdots, n-1$, $c_n = 2 + h^2 - \frac{1}{1+h}$ lyder

$$\begin{bmatrix} c_1 & -1 & & & \\ -1 & c_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c_{n-1} & -1 \\ & & & -1 & c_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = h^2\lambda \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$$

(b). Se laboration 1.

(c).

```
n=50; L=1; k=5;
h=L/(n+1); x=h*[0:n+1]'; e=ones(n,1);
A=spdiags([-e (2+h^2)*e -e],[-1 0 1],n,n); A(n,n)=A(n,n)-1/(1+h);
[V,D]=eigs(A,k,'SM');
V=[zeros(1,k);V;V(n,:)/(1+h)]; lambda=diag(D)/h^2;
for j=1:k
    subplot(5,1,j), plot(x,V(:,j))
    title(['\lambda = ',num2str(lambda(j))])
end
```

Uppgift 3. Se laboration 4 eller Lay kapitel 5.

Uppgift 4 (a).

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_1\sin(x_1+x_2)+x_1^2+2x_2-7 \\ \cos(x_1)+x_1+0.2x_1x_2-0.4 \end{bmatrix}$$

$$\boldsymbol{Df}(\boldsymbol{x}) = \begin{bmatrix} \sin(x_1+x_2)+x_1\cos(x_1+x_2)+2x_1 & x_1\cos(x_1+x_2)+2 \\ -\sin(x_1)+1+0.2x_2 & 0.2x_1 \end{bmatrix}$$

(b). Se laboration 5.

(c). Newtons metod: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{d}_k$, där $\boldsymbol{Df}(\boldsymbol{x}_k)\,\boldsymbol{d}_k = -\boldsymbol{f}(\boldsymbol{x}_k)$

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}: \quad \boldsymbol{Df}(\boldsymbol{x}_0)\,\boldsymbol{d}_0 = -\boldsymbol{f}(\boldsymbol{x}_0) \Leftrightarrow \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} d_{01} \\ d_{02} \end{bmatrix} = -\begin{bmatrix} -7 \\ 0.6 \end{bmatrix} \Leftrightarrow \boldsymbol{d}_0 = \begin{bmatrix} -0.6 \\ 3.5 \end{bmatrix}$$

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{d}_0 = \begin{bmatrix} -0.6 \\ 3.5 \end{bmatrix}$$

(d).

```
f1=@(x1,x2)x1.*sin(x1+x2)+x1.^2+2*x2-7;
f2=@(x1,x2)cos(x1)+x1+0.2*x1.*x2-0.4;
f=@(x)[f1(x(1),x(2));f2(x(1),x(2))];
Df=@(x)[ sin(x(1)+x(2))+x(1)*cos(x(1)+x(2))+2*x(1)  x(1)*cos(x(1)+x(2))+2
        -sin(x(1))+1+0.2*x(2)                        0.2*x(1)              ];
x=[0;0];
kmax=10; tol=0.5e-8;
for k=1:kmax
    d=-Df(x)\f(x);
    x=x+d;
    disp([x' norm(d)])
    if norm(d)<tol, break, end
end
```

Uppgift 5(a). Vi har $f(\boldsymbol{x}) = -x_1^3 + 5x_1x_2 - 3x_2^3 - 1$ och

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = \begin{bmatrix} -3x_1^2+5x_2 \\ 5x_1-9x_2^2 \end{bmatrix} \qquad \boldsymbol{H}(\boldsymbol{x}) = \begin{bmatrix} -6x_1 & 5 \\ 5 & -18x_2 \end{bmatrix}$$

(b).

```
f=@(x1,x2)-x1.^3+5*x1.*x2-3*x2.^3-1;
dfdx1=@(x1,x2)-3*x1.^2+5*x2;
dfdx2=@(x1,x2)5*x1-9*x2.^2;
gradf=@(x)[dfdx1(x(1),x(2))
           dfdx2(x(1),x(2))];
H=@(x)[-6*x(1)  5
        5      -18*x(2)];
%%
x1min=-4; x1max=4; x2min=-4; x2max=4;
x1=linspace(x1min,x1max,50); x2=linspace(x2min,x2max,50);
[X1,X2]=meshgrid(x1,x2); Z=f(X1,X2);
figure(1), clf
surf(x1,x2,Z), xlabel('x'), ylabel('y')
figure(2), clf
```

```
contour(x1,x2,Z,linspace(-40,40,50)), xlabel('x'), ylabel('y')
hold on
contour(x1,x2,dfdx1(X1,X2),[0 0],'k')
contour(x1,x2,dfdx2(X1,X2),[0 0],'m')
axis equal, grid on
```

(c).

```
x0=ginput(1)';
x=fsolve(gradf,x0)
plot(x(1),x(2),'ro')
eg=eig(H(x))
```

Uppgift 6(a). Inför nät med steglängden h i rumsled och ersätt u''_{xx} med D_+D_-. Låt $u_i(t)$ beteckna approximationen av $u(x_i, t)$. För differentialekvationen får vi

$$u_i'(t) = \kappa \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{h^2}, \quad i = 1, \cdots, n, \quad 0 < t < T$$

och för randvillkoren

$$-K\frac{u_1(t) - u_0(t)}{h} + H(u_0(t) - u_{omg}) = Q(t), \quad \frac{u_{n+1}(t) - u_n(t)}{h} = 0, \quad 0 \le t \le T$$

eller

$$u_0(t) = (K/h + H)^{-1}(K/h\, u_1(t) + Hu_{omg} + Q(t)), \quad u_{n+1}(t) = u_n(t), \quad 0 \le t \le T$$

Begynnelsevillkoren blir

$$u_i(0) = u_{omg}, \quad i = 1, \cdots, n$$

Begynnelsevärdesproblemet för ODE blir

$$\begin{cases} \boldsymbol{U}'(t) = \frac{\kappa}{h^2}(\boldsymbol{b}(t) - \boldsymbol{A}\boldsymbol{U}(t)), & 0 < t < T \\ \boldsymbol{U}(0) = \boldsymbol{U}_0 \end{cases}$$

där

$$\boldsymbol{A} = \begin{bmatrix} a_1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & a_n \end{bmatrix}$$

med $a_1 = 1 + (K/h + H)^{-1}H$, $a_n = 1$ och

$$\boldsymbol{U}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n-1}(t) \\ u_n(t) \end{bmatrix} \quad \boldsymbol{b}(t) = \begin{bmatrix} (K/h + H)^{-1}(Hu_{omg} + Q(t)) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{U}_0 = \begin{bmatrix} u_{omg} \\ u_{omg} \\ \vdots \\ u_{omg} \\ u_{omg} \end{bmatrix}$$

(b).

```
function dUdt=broms(t,U,kappa,h,A,K,H,uomg,n)
Q=(1-t/3)*1e7;
b=[(H*uomg+Q)/(K/h+H);zeros(n-1,1)];
dUdt=(b-A*U)*kappa/h^2;
```

(c).

```
kappa=8e-6; L=6e-3; K=100; H=1e4; uomg=20;
Q=@(t)(1-t/3)*1e7;
n=30; h=L/(n+1); x=h*[0:n+1]; tspan=linspace(0,3,n+1); e=ones(n,1);
U0=uomg*e;
A=spdiags([-e 2*e -e],[-1 0 1],n,n); A(1,1)=1+H/(K/h+H); A(n,n)=1;
[t,U]=ode45(@(t,U)broms(t,U,kappa,h,A,K,H,uomg,n),tspan,U0);
U=[(K/h*U(:,1)+H*uomg+Q(t))/(K/h+H),U,U(:,n)];  % Kantar lösningen med randvärden
surf(x,t,U)
xlabel('x'), ylabel('t')
```