Uppgift 1. Se laboration 1.

Uppgift 2(a). Med $u_1 = x$, $u_2 = x'$, $u_3 = y$, $u_4 = y'$, har vi

$$\begin{cases} u_1' = u_2 \\ u_2' = -s(r)\frac{u_1}{m\,r} \\ u_3' = u_4 \\ u_4' = -s(r)\frac{u_3}{m\,r} - g \end{cases}$$

där $r = \sqrt{u_1^2 + u_3^2}$ och

$$s(r) = \begin{cases} k(r-L), & r \geq L \\ 0, & r < L \end{cases}$$

Med vektorbeteckningar har vi standardformen

$$\begin{cases} \boldsymbol{u}' = \boldsymbol{f}(t, \boldsymbol{u}) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \end{cases}$$

med

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \boldsymbol{f}(t, \boldsymbol{u}) = \begin{bmatrix} u_2 \\ -s(r)\frac{u_1}{m\,r} \\ u_4 \\ -s(r)\frac{u_3}{m\,r} - g \end{bmatrix}, \quad \boldsymbol{u}_0 = \begin{bmatrix} x(0) \\ x'(0) \\ y(0) \\ y'(0) \end{bmatrix}$$

och r samt $s(r)$ som tidigare.

(b).

```
function udot=elas(t,u,k,m,L,g)
% x=u(1), x'=u(2), y=u(3), y'=u(4)
r=sqrt(u(1)^2+u(3)^2);
if r>=L
    s=k*(r-L);
else
    s=0;
end
udot=[u(2);-s*u(1)/(m*r);u(4);-s*u(3)/(m*r)-g];
```

(c).

```
m=0.12; L=0.4; k=10; g=9.81;
x0=0.3; y0=-0.4; vx0=0; vy0=0;
tspan=[0,30]; u0=[x0;vx0;y0;vy0];
[t,U]=ode45(@(t,u)elas(t,u,k,m,L,g),tspan,u0);
plot(U(:,1),U(:,3)), axis equal
```

Uppgift 3(a). Inför $x_i = ih, i = 0, 1, \cdots, n+1$, med $h = \frac{1}{n+1}$.

$$-u''(x_i) + 15x_i^2 u(x_i) = \exp(-x_i^2), \; i = 1, 2, \cdots, n$$

$$-\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+15x_i^2u_i=\exp(-x_i^2),\ i=1,2,\cdots,n$$
$$-u_{i-1}+(2+h^2 15x_i^2)u_i-u_{i+1}=h^2\exp(-x_i^2),\ i=1,2,\cdots,n$$

Randvillkoret $u'(0)+u(0)=1$ approximeras av $\frac{u_1-u_0}{h}+u_0=1$, dvs. $u_0=\frac{1}{1-h}u_1-\frac{h}{1-h}$, och randvillkoret $u(1)=5$ ger $u_{n+1}=5$.

Matrisformulering med $c_1=2-\frac{1}{1-h}+h^2 15x_1^2,\ c_i=2+h^2 15x_i^2,\ i=2,\cdots,n$, lyder

$$\begin{bmatrix} c_1 & -1 & & & \\ -1 & c_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c_{n-1} & -1 \\ & & & -1 & c_n \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}=\begin{bmatrix} h^2\exp(-x_1^2)-\frac{h}{1-h} \\ h^2\exp(-x_2^2) \\ \vdots \\ h^2\exp(-x_{n-1}^2) \\ h^2\exp(-x_n^2)+5 \end{bmatrix}$$

(b). Se laboration 1.

(c).

```
n=30; L=1; uL=5;
h=L/(n+1); xi=h*[1:n]'; e=ones(n,1);
A=spdiags([-e (2+h^2*15*xi.^2) -e],[-1 0 1],n,n); A(1,1)=A(1,1)-1/(1-h);
b=h^2*exp(-xi.^2); b(1)=b(1)-h/(1-h); b(n)=b(n)+uL;
u=A\b;
x=[0;xi;L]; u=[(u(1)-h)/(1-h);u;uL];
plot(x,u)
```

Uppgift 4(a). Egenvärdesproblemet:

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I})=\left|\begin{bmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{bmatrix}\right|=(1-\lambda)(2-\lambda)-6=$$
$$=\lambda^2-3\lambda-4=(\lambda-4)(\lambda+1)=0$$

Egenvektorn som hör ihop med $\lambda_1=-1$:

$$(\boldsymbol{A}-\lambda_1\boldsymbol{I})\boldsymbol{v}=(\boldsymbol{A}+\boldsymbol{I})\boldsymbol{v}=\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}\ \text{ger}\ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Egenvektorn som hör ihop med $\lambda_2=4$:

$$(\boldsymbol{A}-\lambda_2\boldsymbol{I})\boldsymbol{v}=(\boldsymbol{A}-4\boldsymbol{I})\boldsymbol{v}=\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}\ \text{ger}\ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Lösningen till ODE-systemet:

$$\boldsymbol{u}(t)=c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix}\exp(-t)+c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\exp(4t)$$

$$\boldsymbol{u}(0)=c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix}+c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 5 \\ 0 \end{bmatrix}\Rightarrow\boldsymbol{c}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{u}(t)=\begin{bmatrix} 3 \\ 2 \end{bmatrix}\exp(-t)+2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\exp(4t)$$

(b). Se laboration 4 eller Lay kapitel 5.

(c). Euler framåt: $\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + h\boldsymbol{A}\boldsymbol{u}_n = (\boldsymbol{I} + h\boldsymbol{A})\boldsymbol{u}_n, n = 0, 1, \cdots, \boldsymbol{u}_0 = [\,5\;\;0\,]^T$

$$\boldsymbol{u}_1 = (\boldsymbol{I} + 0.1\boldsymbol{A})\boldsymbol{u}_0 = \begin{bmatrix} 1.1 & -0.3 \\ -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5.5 \\ -1 \end{bmatrix}$$

Uppgift 5(a).

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_2 - x_1 x_2^2 + 1 \\ x_1^2 + 4x_2^2 - x_2 - 10 \end{bmatrix}, \quad \boldsymbol{Df}(\boldsymbol{x}) = \begin{bmatrix} -x_2^2 & 1 - 2x_1 x_2 \\ 2x_1 & 8x_2 - 1 \end{bmatrix}$$

(b). Se laboration 5.

Newtons metod:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{d}_k, \quad \text{där} \quad \boldsymbol{Df}(\boldsymbol{x}_k)\,\boldsymbol{d}_k = -\boldsymbol{f}(\boldsymbol{x}_k)$$

(c).

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}: \quad \boldsymbol{Df}(\boldsymbol{x}_0)\,\boldsymbol{d}_0 = -\boldsymbol{f}(\boldsymbol{x}_0) \Leftrightarrow \begin{bmatrix} -1 & -1 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} d_{01} \\ d_{02} \end{bmatrix} = -\begin{bmatrix} 1 \\ -6 \end{bmatrix} \Leftrightarrow \boldsymbol{d}_0 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{d}_0 = \begin{bmatrix} 1.2 \\ 1.8 \end{bmatrix}$$

(d).

```
f1=@(x1,x2)x2-x1.*x2.^2+1; f2=@(x1,x2)x1.^2+4*x2.^2-x2-10;
f=@(x)[f1(x(1),x(2));f2(x(1),x(2))];
Df=@(x)[-x(2)^2 1-2*x(1)*x(2); 2*x(1) 8*x(2)-1];
x=[1;1];
kmax=10; tol=0.5e-8;
for k=1:kmax
    d=-Df(x)\f(x);
    x=x+d;
    disp([x' norm(d)])
    if norm(d)<tol, break, end
end
```

Uppgift 6(a). Inför nät med steglängden h i rumsled och ersätt u''_{xx} med D_+D_- och u'_x med D_+. Låt $u_i(t)$ beteckna approximationen av $u(x_i, t)$. För differentialekvationen får vi

$$u_i'(t) = \kappa \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{h^2} + C(u_{omg} - u_i(t)) + f(x_i), \quad i = 1, \cdots, n, \quad 0 < t < T$$

och randvillkoren ger

$$\frac{u_1(t) - u_0(t)}{h} = 0, \text{ dvs. } u_0(t) = u_1(t), \text{ samt } u_{n+1}(t) = g_L, \quad 0 \le t \le T$$

Begynnelsevillkoren blir

$$u_i(0) = u_{beg}, \quad i = 1, \cdots, n$$

Begynnelsevärdesproblemet för ODE blir

$$\begin{cases} \boldsymbol{U}'(t) = \frac{\kappa}{h^2}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{U}(t)) + C(u_{omg} - \boldsymbol{U}(t)) + \boldsymbol{f}, \quad 0 < t < T \\ \boldsymbol{U}(0) = \boldsymbol{U}_0 \end{cases}$$

där

$$\boldsymbol{U}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n-1}(t) \\ u_n(t) \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_L \end{bmatrix}$$

och

$$\boldsymbol{f} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{bmatrix} \quad \boldsymbol{U}_0 = \begin{bmatrix} u_{beg} \\ u_{beg} \\ \vdots \\ u_{beg} \\ u_{beg} \end{bmatrix}$$

(b).

```
n=30;
kappa=2; C=3; uomg=15; f=@(x)exp(-(x-L/2).^2); L=1; gL=60;
T=1; ubeg=15;
h=L/(n+1); xi=h*(1:n)'; e=ones(n,1);
A=spdiags([-e 2*e -e],[-1 0 1],n,n); A(1,1)=1;
b=[0;zeros(n-2,1);gL];
F=@(t,u)kappa/h^2*(b-A*u)+C*(uomg-u)+f(xi);
tspan=linspace(0,T,n+1); U0=ubeg*e;
[t,U]=ode45(F,tspan,U0);
x=[0;xi;L]; U=[U(:,1),U,gL*ones(size(t))];
subplot(1,2,1)
surf(x,t,U), xlabel('x'), ylabel('t')
subplot(1,2,2)
contourf(x,t,U,20), xlabel('x'), ylabel('t')
```