Idag: * Area mellan kurvor

* Volymer

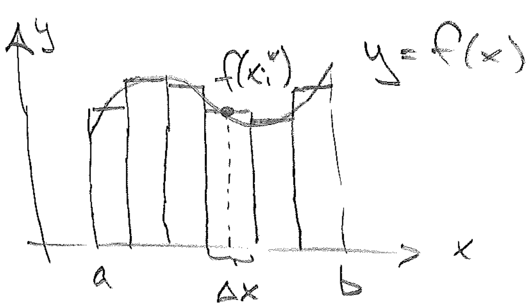
* Cylindriska skal

Area mellan kurvor (6.1)

För f kont. på $[a,b]$ s.a. $f(x) \geq 0 \ \forall x \in [a,b]$

har vi sett arean under kurvan:

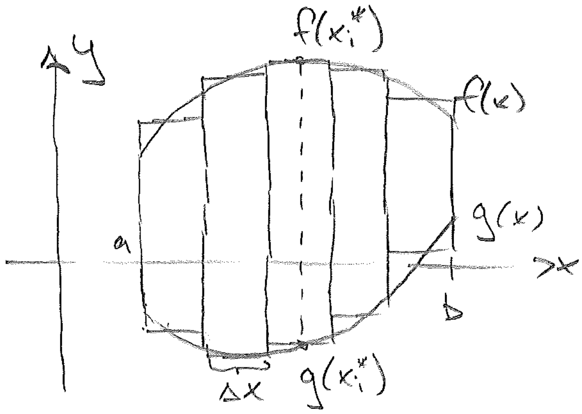
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$



För f, g kont på $[a,b]$ s.a. $f(x) \geq g(x) \ \forall x \in [a,b]$

får vi pss arean mellan kurvorna

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x = \int_a^b \left(f(x) - g(x)\right)dx$$

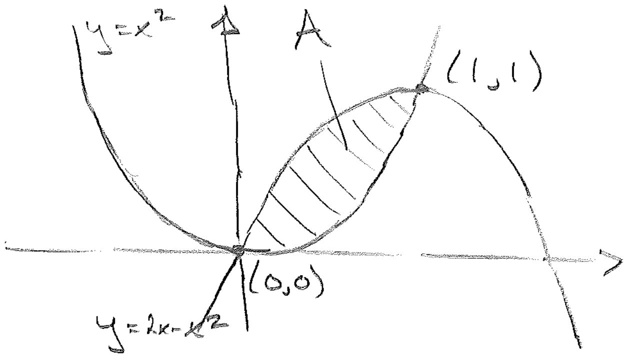


Om a, b okända avses den ändliga

arean som innesluts av f och g.

Ex Bestäm arean mellan $y = x^2$ och $y = 2x - x^2$

Lösn:



Skärningspkt:

$x^2 = 2x - 2x^2$

$\Rightarrow x(x-1) = 0$

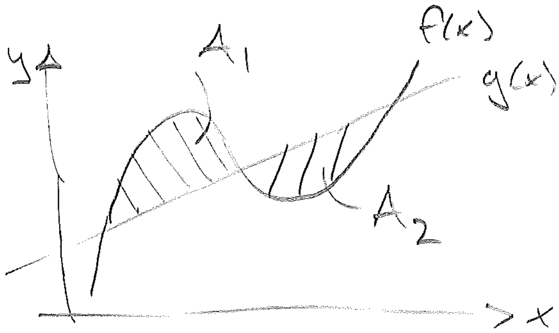
$\Rightarrow x = 0,\ x = 1$

$$A = \int_0^1 (y_T - y_B)\,dx = \int_0^1 (2x - x^2 - x^2)\,dx$$

$$= 2\int_0^1 (x - x^2)\,dx = 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = 2 \cdot \tfrac{1}{6} = \tfrac{1}{3}.$$

För f, g kont. på $[a,b]$ generellt

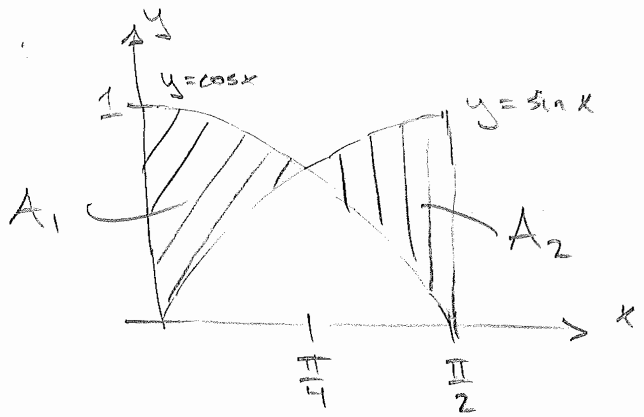
$$A = \int_a^b |f(x) - g(x)|\,dx$$



$\Rightarrow$ Dela upp integralen beroende på tecknet för $f(x) - g(x)$

Ex Bestäm arean mellan $y = \sin x$, $y = \cos x$, $x = 0$ och $x = \frac{\pi}{2}$.

Lösn.



Skärningspkt:

$\cos x = \sin x \quad , \quad x \in [0, \frac{\pi}{2}]$

$\Rightarrow \quad x = \frac{\pi}{4}$

$$A = \int_0^{\pi/2} |\cos x - \sin x| \, dx = A_1 + A_2$$

$$= \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx$$

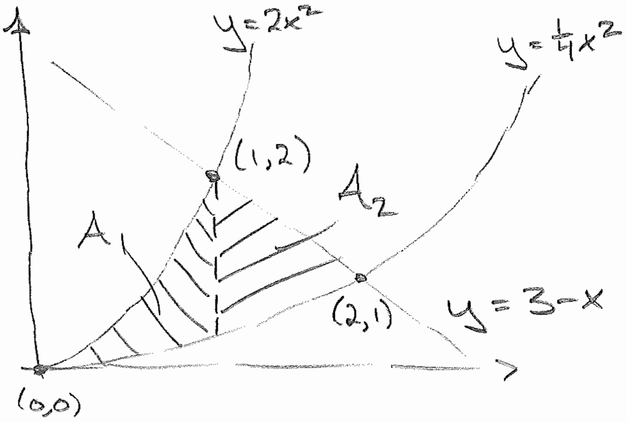
$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (1+0)\right] + \left[-0 - 1 - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= 4\frac{1}{\sqrt{2}} - 2 = 2\sqrt{2} - 2 .$$

Ex Bestäm arean mellan $y = \frac{1}{4}x^2$ $y = 2x^2$ och $y = 3 - x$ för $x \geq 0$.

Lösn.



Skärningspkt:

i) $\frac{1}{4}x^2 = 3 - x$

$\Rightarrow x^2 + 4x - 12 = 0$

$\Rightarrow x = -2 \overset{+}{(-)} 4 = 2$

ii) $2x^2 = 3 - x$

$\Rightarrow x^2 + \frac{1}{2}x - \frac{3}{2} = 0$

$\Rightarrow x = -\frac{1}{4} \overset{+}{(-)} \frac{5}{4} = 1$

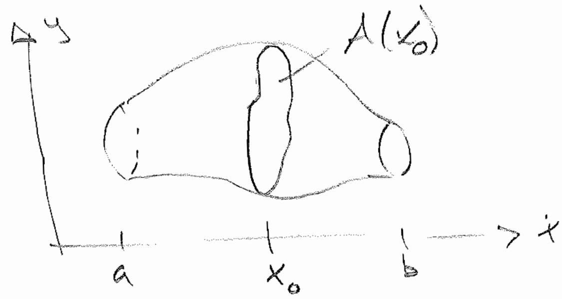
$$A_1 = \int_0^1 \left(2x^2 - \tfrac{1}{4}x^2\right) dx = \ldots = \frac{7}{12}$$

$$A_2 = \int_1^2 \left(3 - x - 2x^2\right) dx = \ldots = \frac{11}{12}$$

$$\Rightarrow A = A_1 + A_2 = \frac{18}{12} = \frac{3}{2}$$

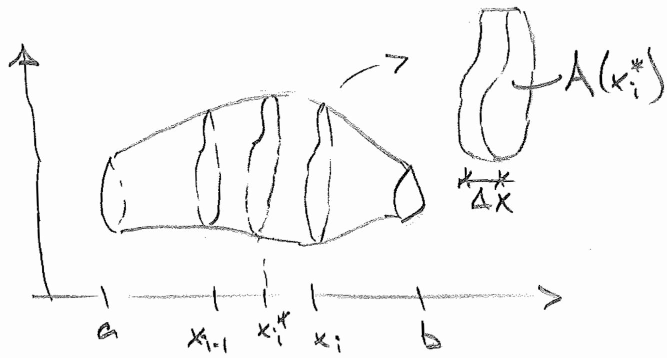
<u>Volymer</u> (6.2)

Betrakta kropp S med tvärsnitt $A(x_0)$ för $x = x_0$



Dela in i n skivor med bredd Δx och volym

$$V_i = A(x_i^*)\Delta x$$



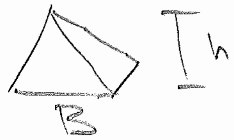
Volymen av S approximeras av Riemannsumma

$$\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} A(x_i^*)\Delta x$$

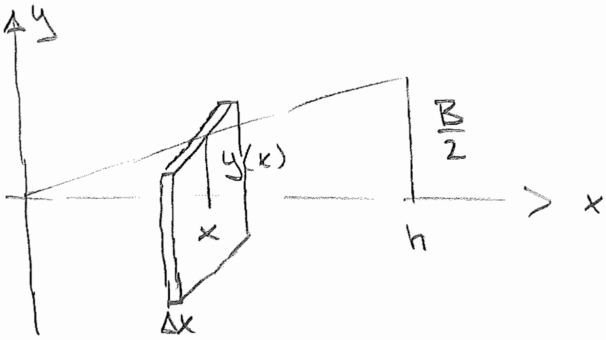
Volymen av S ges alltså av

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$

Ex Bestäm volymen av pyramid



Lösn:



$$A(x) = (2y(x))^2$$

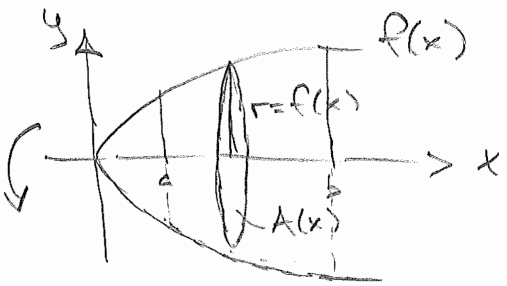
Likformiga trianglar: $\frac{y}{x} = \frac{B/2}{h} \Rightarrow y = \frac{Bx}{2h}$

$$A(x) = (2y)^2 = \left(\frac{Bx}{h}\right)^2 = \frac{B^2}{h^2}x^2$$

$$V = \int_0^h A(x)\,dx = \frac{B^2}{h^2}\int_0^h x^2\,dx = \frac{B^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h = \frac{1}{3}B^2h$$

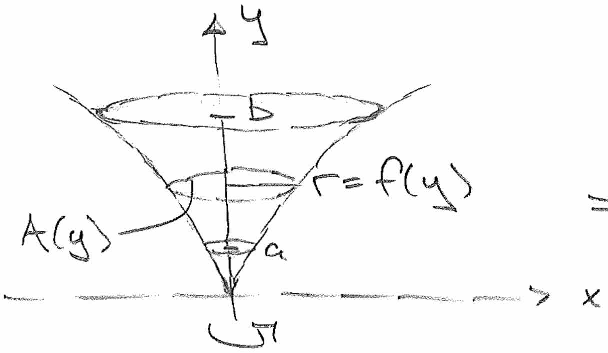
Rotationsvolymer:

När vi roterar kurva runt axel sveper arean under kurvan ut volym



$$A(x) = \pi r^2 = \pi f(x)^2$$

$$\Rightarrow V = \int_a^b \pi f(x)^2\,dx$$

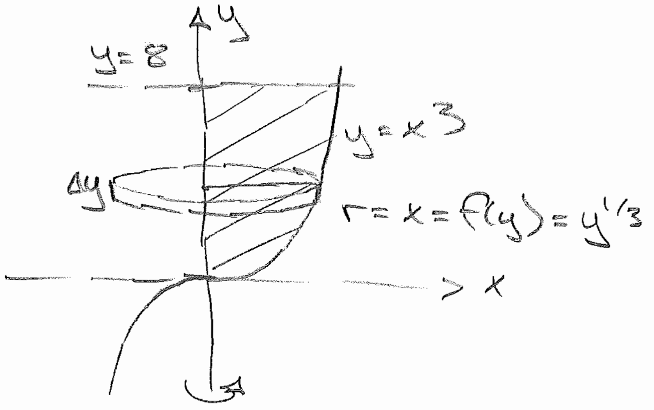


$$A(y) = \pi r^2 = \pi f(y)^2$$

$$\Rightarrow V = \int_a^b \pi f(y)^2\,dy$$

Ex Bestäm volymen som erhålls genom rotation av arean mellan $y = x^3$, $y = 8$ och $x = 0$ kring y-axeln.

Lösn:

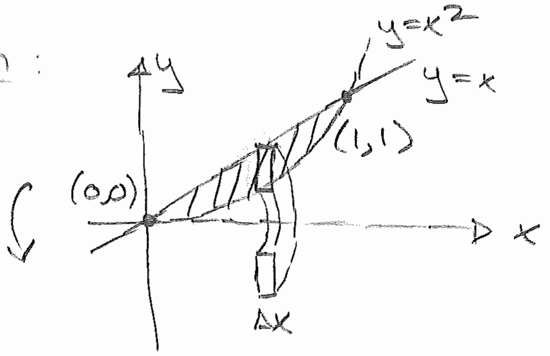


$$A(y) = \pi f(y)^2 = \pi y^{2/3}$$

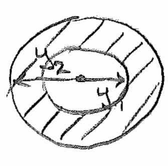
$$V = \int_0^8 A(y)\,dy = \pi \int_0^8 y^{2/3}\,dy = \pi \left[\frac{3}{5} x^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

Ex Bestäm volymen då arean mellan $y = x$ och $y = x^2$ roteras kring x-axeln.

Lösn:



Rotationskroppens tvärsnitt:



$$A(x) = \pi y_2^2 - \pi y_1^2 = \pi x^2 - \pi (x^2)^2$$

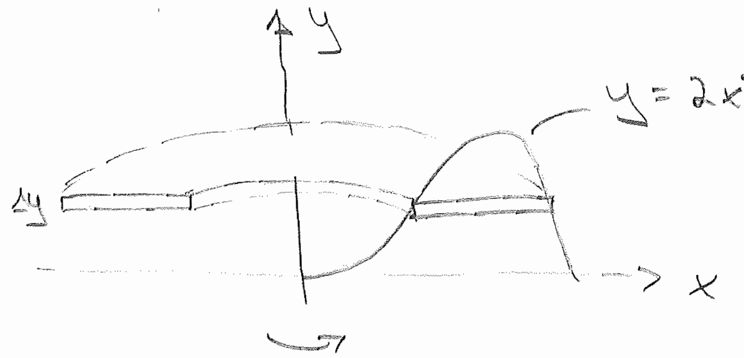
$$\Rightarrow V = \int_0^1 \pi (x^2 - x^4)\,dx = \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

Cylindriska skal (6.3)

Iblend fördelaktigt att beräkna rotationsvolymer med cylindriska skal.

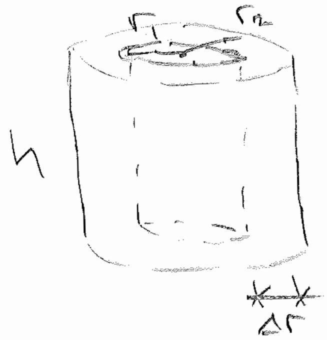
Ex



$\Rightarrow$ Cylinder med tjocklek Δy!

Men, svårt att lösa $y = 2x^2 - x^3$ för radier!

Cylindriska skal:

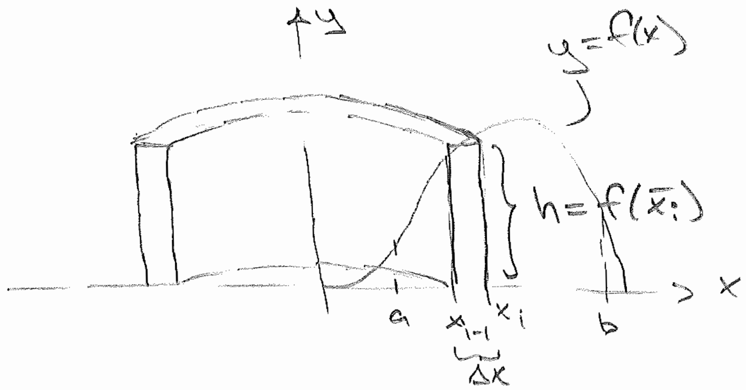


$$V_1 = \pi r_1^2 h \qquad V_2 = \pi r_2^2 h$$

$$\Rightarrow V = V_2 - V_1 = \pi h (r_2^2 - r_1^2)$$

$$= 2\pi h \underbrace{\frac{r_2 + r_1}{2}}_{\bar{r}} \underbrace{(r_2 - r_1)}_{\Delta r}$$

Låt $f(x) \geq 0$ på $[a,b]$ och $a \geq 0$



$$V_i = 2\pi f(\bar{x}_i)\, \bar{x}_i\, \Delta x$$

$$= \underbrace{2\pi \bar{x}_i}_{\text{omkrets}} \underbrace{f(\bar{x}_i)}_{\text{höjd}} \underbrace{\Delta x}_{\text{tjocklek}}$$

Volymen approximeras med Riemannsumman

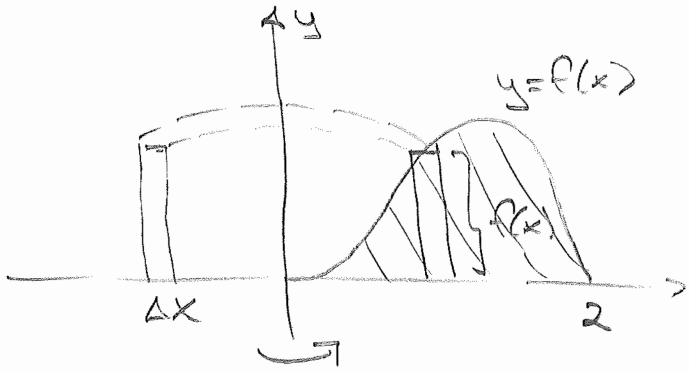
$$\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \bar{x}_i \, f(\bar{x}_i) \Delta x$$

Volymen ges av

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i) \Delta x = \int_a^b 2\pi x f(x) \, dx$$

Ex Bestäm volymen som sveps ut då arean under $y = 2x^2 - x^3$, $x \geq 0$ roteras kring y-axeln

Lösn:



$f(x) = 2x^2 - x^3$

$f(x) = 0 \Rightarrow x = 0, x = 2$

$$V = \int_0^2 2\pi x \, (2x^2 - x^3) \, dx = 2\pi \int_0^2 (2x^3 - x^4) \, dx$$

$$= 2\pi \left[\frac{1}{2} x^4 - \frac{1}{5} x^5 \right]_0^2 = 2\pi \left(8 - \frac{32}{5} \right) = \frac{16\pi}{5}$$