Idag: * Trigonometriska substitutioner
* Rationella funktioner (partialbråksuppdelning)

Trigonometriska substitutioner (7.3)

Om $g(t)$ är inverterbar (injektiv) kan vi def. invers substitution genom $x = g(t)$

$$\Rightarrow \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$$

Användbart för vissa rotuttryck med $g(\theta)$ trigonometrisk funktion:

$\Rightarrow$ Begränsning av D_g för att göra $g(\theta)$ inverterbar

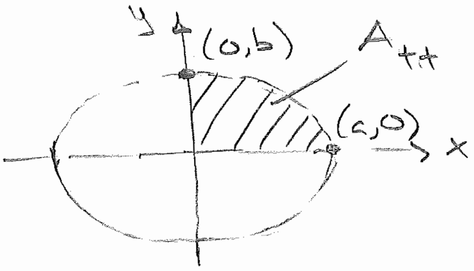
$\Rightarrow$ Trigonometrisk identitet för förenkling

$$f(x) = \sqrt{a^2 - x^2} \Rightarrow g(\theta) = a \sin x \,, \quad x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

$$f(x) = \sqrt{a^2 + x^2} \Rightarrow g(\theta) = a \tan x \,, \quad x \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

Ex Bestäm arean av ellipsen $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad a,b>0$

Lösn:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

symmetri ger $A=4\cdot A_{++}$

Betrakta $0\le x\le a,\ 0\le y\le b \Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}$

$$A_{++}=\int_0^a y(x)\,dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx=$$

$$=\left\{\begin{array}{ll} x=a\sin\theta,\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}] & x=0\Rightarrow\theta=0 \\ dx=a\cos\theta\,d\theta \quad , & x=a\Rightarrow\theta=\frac{\pi}{2}\end{array}\right\}$$

$$=\frac{b}{a}\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta=ab\int_0^{\pi/2}\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$=\{\sin^2\theta+\cos^2\theta=1\}=ab\int_0^{\pi/2}\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$=ab\int_0^{\pi/2}|\cos\theta|\cos\theta\,d\theta=\{\theta\in[0,\tfrac{\pi}{2}]\}$$

$$=ab\int_0^{\pi/2}\cos^2\theta\,d\theta=\{\cos2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1\}$$

$$=ab\int_0^{\pi/2}\frac{1}{2}(1+\cos2\theta)=\frac{1}{2}ab\left[\theta+\frac{1}{2}\sin2\theta\right]_0^{\pi/2}$$

$$=\frac{\pi}{4}ab \Rightarrow A=4A_{++}=\pi ab.$$

Ex Bestäm $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Lösn.
$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \left\{\begin{array}{l} x = 2\tan\theta, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ dx = 2\,\frac{1}{\cos^2\theta}\,d\theta \end{array}\right\}$$

$$= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\,\frac{2}{\cos^2\theta}\,d\theta$$

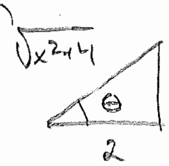
$$= \frac{1}{4}\int \frac{1}{\tan^2\theta\,\cos^2\theta\sqrt{\tan^2\theta+1}}\,d\theta = \left\{1+\tan^2\theta = \frac{1}{\cos^2\theta}\right\}$$

$$= \frac{1}{4}\int \frac{|\cos\theta|}{\cos^2\theta\,\tan^2\theta}\,d\theta = \left\{\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \left\{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}\right\}$$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du = -\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C$$

$$= \left\{ \quad x \Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}} \right\} = -\frac{1}{4}\,\frac{\sqrt{x^2+4}}{x} + C$$



Rationella funktioner (7.4)

För att integrera rationella funktioner $f(x) = \frac{P(x)}{Q(x)}$,
P, Q polynom, använder vi följande metod:

i) Om $\deg(P) \geq \deg(Q)$ polynomdivision

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \qquad \deg(R) < \deg(Q)$$

ii) Faktorisera $Q(x)$

$(ax+b)^m$: linjär faktor

$(ax^2+bx+c)^n$: irreducibel kvadratisk faktor $b^2-4ac<0$

iii) Uttryck $\frac{R(x)}{Q(x)}$ som summa av partialbråk

$$\sum_{i=1}^{m} \frac{A_i}{(ax+b)^i} \quad , \quad \sum_{j=1}^{n} \frac{A_j x + B_j}{(ax^2+bx+c)^j}$$

iv) Integrera!

Ex Beräkna $\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

Lösn: Partialbråksuppdelning

$$\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-2)(x-3)} \underset{\text{Ansatz}}{=} \frac{A}{x-2} + \frac{B}{x-3}$$

$$\Rightarrow \quad x-4 = A(x-3) + B(x-2) = (A+B)x + (-3A-2B)$$

Likheten uppfylld för godtyckligt x

$$\Rightarrow \begin{cases} A+B=1 \\ -3A-2B=-4 \end{cases} \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ -3 & -2 & -4 \end{array}\right) \begin{array}{c} (3) \end{array}$$

$$\sim \left(\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right) (-1) \sim \left(\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right) \Rightarrow \begin{cases} A=2 \\ B=-1 \end{cases}$$

$$\Rightarrow \int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \int_0^1 \left(\frac{2}{x-2} - \frac{1}{x-3}\right) dx$$

$$= \Big[2\ln|x-2| - \ln|x-3| \Big]_0^1 = \left[\ln \frac{(x-2)^2}{|x-3|} \right]_0^1$$

$$= \ln\frac{1}{2} - \ln\frac{4}{3} = \ln\frac{3}{8}$$

Ex Bestäm $I = \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

Lösn: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = \ldots = x+1+\frac{4x}{x^3-x^2-x+1}$

Faktorisera: $x^3-x^2-x+1 = (x-1)(x^2-1) = (x-1)^2(x+1)$

Partialbråk: $\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$

$\Rightarrow \quad 4x = A(x-1)(x+1)+B(x+1)+C(x-1)^2$

$x=1 \Rightarrow 4 = 2B \Rightarrow B=2$

$x=-1 \Rightarrow -4 = 4C \Rightarrow C=-1$

$x^2: \quad 0 = A+C \Rightarrow A=-C=1$

$$\Rightarrow I = \int\left(x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}x^2+x+\ln|x-1|-\frac{2}{(x-1)}-\ln|x+1|+C$$

$$= \frac{1}{2}x^2+x-\frac{2}{(x-1)}+\ln\left|\frac{x-1}{x+1}\right|+C$$

Ex Bestäm $I = \int \frac{x^2 - 2x - 1}{(x-1)^2 (x^2+1)} \, dx$

Lösn: $\frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{(x^2+1)}$

$\Rightarrow x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

$x = 1 \Rightarrow -2 = 2B \Rightarrow B = -1$

$x^3: \quad 0 = A + C \Rightarrow A = -C$

$x^0: \quad -1 = -A + B + D = -A - 1 + D \Rightarrow D = A$

$x=2 \Rightarrow -1 = 5A + 5B + 2C + D = 5A - 5 - A \Rightarrow A = 1$

$\Rightarrow I = \int \left(\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x}{x^2+1} + \frac{1}{x^2+1} \right) dx$

$= \ln|x-1| + \frac{1}{(x-1)} - \frac{1}{2} \ln(x^2+1) + \arctan(x) + C$