Matematisk analys LMA400

4.5.13) Skissa grafen till $y = \frac{x}{x^2-4}$

Lösn: $y = f(x) = \frac{x}{x^2-4} = \frac{x}{(x+2)(x-2)}$

A) $D_f = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

B) $y = 0 \iff x = 0$

C) Symmetri: $f(-x) = -f(x) \Rightarrow f(x)$ udda fkn.

D) Asymptoter:

H: $\lim_{x \to \pm\infty} \frac{x}{x^2-4} = \frac{1}{x - \frac{4}{x}} \overset{\to 0}{=} 0$

$y = 0$ horisontell asymptot då $x \to \pm\infty$

V: $\lim_{x \to -2^-} \frac{x}{x^2-4} = -\infty \qquad \lim_{x \to -2^+} \frac{x}{x^2-4} = \infty$

$\lim_{x \to 2^-} \frac{x}{x^2-4} = -\infty \qquad \lim_{x \to 2^+} \frac{x}{x^2-4} = \infty$

$\Rightarrow x = \pm 2$ vertikala asymptoter

E) $f'(x) = \frac{(x^2-4) - x(2x)}{(x^2-4)^2} = -\frac{\overbrace{x^2+4}^{>0}}{\underbrace{(x+2)^2(x-2)^2}_{>0}} < 0 \quad \forall x \in D_f$

$f'(x) = 0 \Rightarrow$ saknar lösn

$\nexists f'(x) \Rightarrow x = \pm 2$

} kritiska punkter

F) $f''(x) = -\left[\frac{2x(x^2-4)^{\cancel{2}1} - (x^2+4)\cdot 2(\cancel{x^2-4})2x}{(x^2-4)^{\cancel{4}3}} \right]$

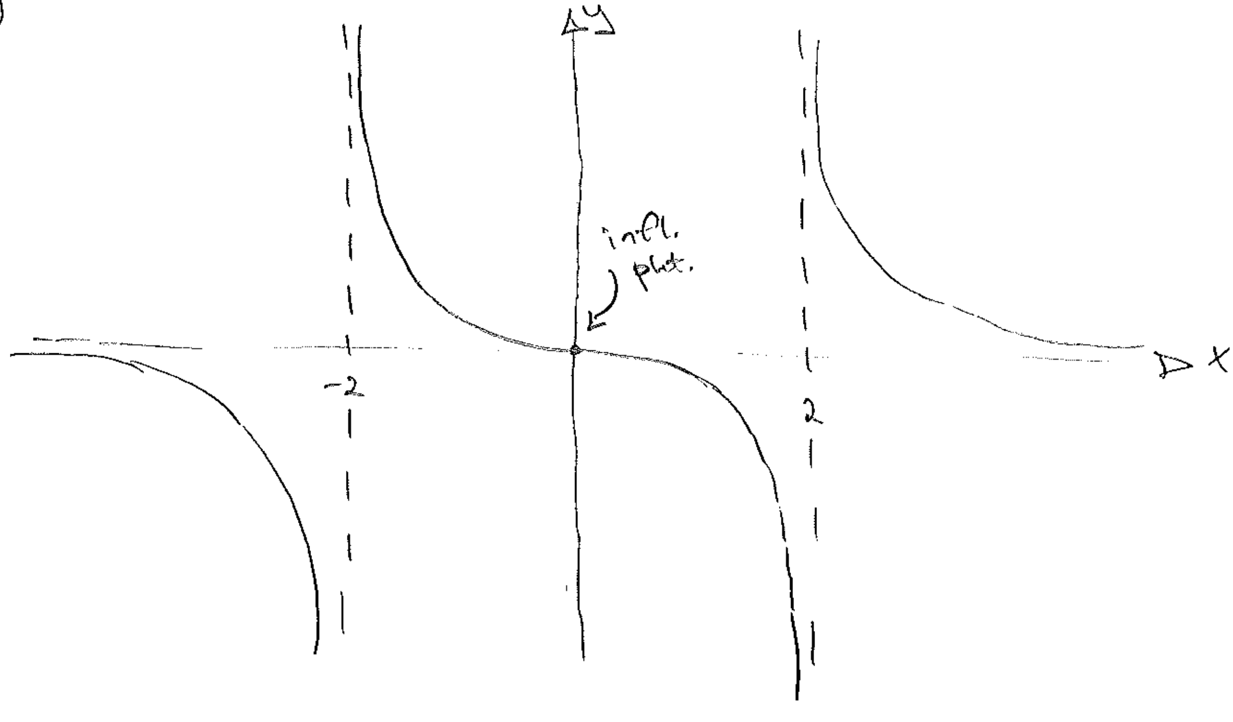
$$= -\frac{2x^3-8x-4x^3-16x}{(x^2-4)^3} = \frac{2x^3+24x}{(x^2-4)^3} = \frac{2x(x^2+12)}{(x+2)^3(x-2)^3}$$

$f''(x)=0 \Rightarrow x=0$ pot. infl. pkt.

G) Tabell över f, f', f''

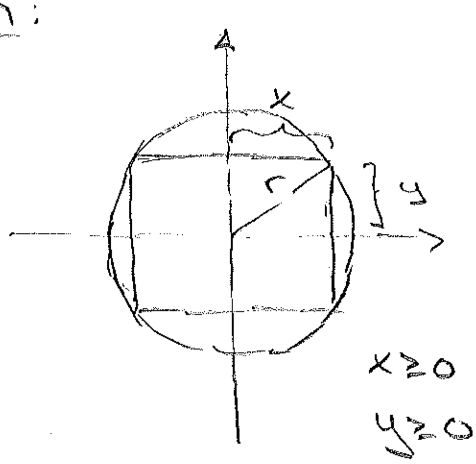
x		-2		0		2	
$x+2$	$-$	0	$+$		$+$		$+$
x	$-$		$-$	0	$+$		$+$
$x-2$	$-$		$-$		$-$	0	$+$
f'	$-$	ej def.	$-$		$-$	ej def.	$-$
f''	$-$	"	$+$	0	$-$	"	$+$
f	↘	"	↘		↘	"	↘
	$\cap$		$\cup$	infl. pkt!	$\cap$		$\cup$

H)



4.7.25) Bestäm dimensionen för rektangeln med största möjliga area inskriven i cirkel med radie r.

Lösn:



$$x^2+y^2=r^2$$

$$\Rightarrow \quad y = \overset{+}{(-)}\sqrt{r^2-x^2}$$

$$\Rightarrow \quad A(x) = 4xy(x) = 4x\sqrt{r^2-x^2}$$

$x \geq 0$

$y \geq 0$

$$A'(x) = 4\left(\sqrt{r^2-x^2} + x \cdot \frac{1}{2}\frac{1}{\sqrt{r^2-x^2}} \cdot (-2x)\right)$$

$$= 4\left(\frac{r^2-x^2-x^2}{\sqrt{r^2-x^2}}\right) = 4\,\frac{r^2-2x^2}{\sqrt{r^2-x^2}}$$

$$A'(x) = 0 \Rightarrow r^2-2x^2=0 \Rightarrow x = \overset{+}{(-)}\frac{r}{\sqrt{2}}$$

A kont. på $[0,r]$ $\Rightarrow$ Globalt max i kritisk punkt eller ändpunkt

$$A(0) = 0 \quad , \quad A(r) = 0$$

$$A\left(\frac{r}{\sqrt{2}}\right) = 4\frac{r}{\sqrt{2}}\sqrt{r^2-\frac{r^2}{2}} = 2r^2 > \max\left(A(0), A(r)\right)$$

$\Rightarrow \quad x = \frac{r}{\sqrt{2}}$ globalt max.

4.9.13) Bestäm allmän primitiv till $f(x) = \frac{1}{5} - \frac{2}{x}$

Lösn: $D_f = (-\infty, 0) \cup (0, \infty)$

$x > 0$: $F_1(x) = \frac{1}{5}x - 2\ln|x| + C_1$

$x < 0$: $F_2(x) = \frac{1}{5}x - 2\ln|x| + C_2$

Ser att $F_1(x)$ och $F_2(x)$ är primitiva

ty $\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad x \neq 0$!

$$\Rightarrow F(x) = \begin{cases} \frac{1}{5}x - 2\ln|x| + C_1 & x > 0 \\ \frac{1}{5}x - 2\ln|x| + C_2 & x < 0 \end{cases}$$

5.2.21) Använd integralens definition för att utvärdera $\int_{-1}^{5} (1+3x)\,dx$.

Lösn: Vi har $\Delta x = \frac{5-(-1)}{n} = \frac{6}{n}$ $\quad x_i = -1 + i\Delta x = -1 + \frac{6i}{n}$

$$\int_{-1}^{5} (1+3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + 3\left(-1 + \frac{6i}{n}\right)\right)\frac{6}{n}$$

$$= \lim_{n\to\infty} \frac{6}{n}\left[\sum_{i=1}^{n}(1-3) + \frac{18}{n}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{\cancel{n}}\frac{\cancel{n}(n+1)}{2}\right] = \lim_{n\to\infty}\left(-12 + 54\left(1 + \frac{1}{n}\right)\right)$$

0

$$= -12 + 54 = 42.$$

5.3.29) Utvärdera $\displaystyle\int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx$

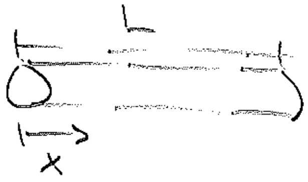
Lösn: $\displaystyle\int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx = \int_1^4 \left(2x^{-1/2} + x^{3/2}\right)dx$

$$= \left[4x^{1/2} + \frac{2}{5}x^{5/2}\right]_1^4 = 4\cdot 2 + \frac{2}{5}\cdot 32 - 4 - \frac{2}{5} =$$

$$= 4 + \frac{62}{5} = \frac{82}{5}$$

5.4.63) Linjär densitet hos stav med $L=4\,m$ ges av $\rho(x) = 9 + 2\sqrt{x}$ kg/m där $x=0$ i stavens ena ände. Bestäm stavens massa.

Lösn:



$$m'(x) = \rho(x) = 9 + 2\sqrt{x}$$

$$m = \int_0^L m'(x)\,dx = \int_0^L \rho(x)\,dx$$

$$= \int_0^4 (9 + 2\sqrt{x})\,dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4$$

$$= 36 + \frac{32}{3} = \frac{140}{3} \text{ kg.}$$