7.3.27) Bestäm $\int \sqrt{5+4x-x^2}\,dx$

Lösn: $5+4x-x^2 = \{\text{Kvadratkompl.}\} = -(x-2)^2+9$

$$\Rightarrow \int \sqrt{5+4x-x^2}\,dx = \int \sqrt{3^2-(x-2)^2}\,dx$$

$$= \left\{ \begin{array}{l} x-2 = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{array}, \quad \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \right\}$$

$$= \int \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta\,d\theta$$

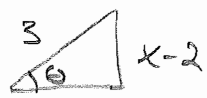
$$= 9\int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = 9\int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= \left\{\theta \in \left[-\tfrac{\pi}{2},\tfrac{\pi}{2}\right] \Rightarrow \cos\theta \geq 0\right\} = 9\int \cos^2\theta\,d\theta$$

$$= \left\{\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)\right\} = \frac{9}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \{\sin 2\theta = 2\sin\theta\cos\theta\}$$

$$= \frac{9}{2}\left(\theta + \sin\theta\cos\theta\right) + C = \Bigg\{$$



$$\sqrt{9-(x-2)^2} = \sqrt{5+4x-x^2} \Bigg\}$$

$$= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{9}{2}\,\frac{(x-2)}{3}\,\frac{\sqrt{5+4x-x^2}}{3} + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C$$

7.4.47) Uttryck integranden som rationell funktion genom variabelbyte, och utvärdera integralen

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$$

<u>Lösn</u>: Använder substitutionen $u=e^x$

$$I = \int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx = \left\{\begin{matrix} u=e^x, & x=\ln u \\ du = e^x dx = u\,dx \end{matrix}\right\}$$

$$= \int \frac{u^2}{u^2+3u+2}\cdot\frac{1}{u}\,du = \int \frac{u}{u^2+3u+2}\,du$$

$$= \int \frac{u}{(u+1)(u+2)}\,du \qquad \text{Partialbråksuppdela!}$$

$$\frac{u}{(u+1)(u+2)} = \frac{A}{(u+1)} + \frac{B}{(u+2)} = \frac{A(u+2)+B(u+1)}{(u+1)(u+2)}$$

$$\Rightarrow \begin{cases} A+B=1 \\ 2A+B=0 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=2 \end{cases}$$

$$\Rightarrow I = \int \left(-\frac{1}{u+1} + \frac{2}{u+2}\right) du$$

$$= -\ln|u+1| + 2\ln|u+2| + C = \ln\frac{(u+2)^2}{|u+1|} + C$$

$$= \ln\left(\frac{(e^x+2)^2}{e^x+1}\right) + C$$

7.5.49) Bestäm $\int \frac{1}{x\sqrt{4x+1}}\,dx$

<u>Lösn</u>: $\int \frac{1}{x\sqrt{4x+1}}\,dx = \left\{ \begin{array}{l} u = \sqrt{4x+1} \Rightarrow x = \frac{u^2-1}{4} \\ du = \frac{1}{2}\frac{1}{\sqrt{4x+1}}\cdot 4\,dx = \frac{2}{u}\,dx \end{array} \right\}$

$= \int \frac{4}{u^2-1}\,\frac{1}{u}\cdot\frac{u}{2}\,du = 2\int \frac{1}{u^2-1}\,du$

$= 2\int \frac{1}{(u+1)(u-1)}\,du = \{\text{Partialbråk}\}$

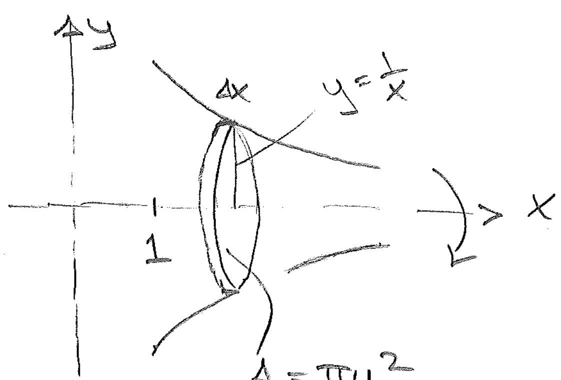
$= 2\int \frac{1}{2}\left(-\frac{1}{u+1} + \frac{1}{u-1}\right)du = \ln\left|\frac{u-1}{u+1}\right| + C$

$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$

7.8.63) Vi har sett att arean under $y = \frac{1}{x}$ för $x \geq 1$ är oändlig (dvs $\int_1^\infty \frac{1}{x}\,dx = \infty$)

Visa att volymen då området roteras kring x-axeln är ändlig.

<u>Lösn</u>:



$\Delta V = \pi y^2(x)\,\Delta x$
$= \pi\left(\frac{1}{x}\right)^2 \Delta x$

$\Rightarrow V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$

$$\Rightarrow V = \lim_{t \to \infty} \pi \int_1^t \frac{1}{x^2} dx = \pi \lim_{t \to \infty} \left[-\frac{1}{x} \right]_1^t$$

$$= \pi \lim_{t \to \infty} \left(-\frac{1}{t} + 1 \right) = \pi$$

9.2.21) Använd Eulers metod med $h = 0{,}2$ för att uppskatt $y(1)$ där y ges av $y' = 1 - xy$, $y(0) = 0$

Lösn: Eulers metod: $y' = F(x,y) = 1 - xy$

$x_n = x_{n-1} + h$, $y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$

$x_0 = 0$; $y_0 = 0$; $F(x_0, y_0) = 1$

$x_1 = 0{,}2$; $y_1 = y_0 + hF(x_0, y_0) = 0.2$; $F(x_1, y_1) = 0{,}96$

$x_2 = 0{,}4$; $y_2 = y_1 + hF(x_1, y_1) = 0{,}392$; $F(x_2, y_2) \simeq 0{,}8432$

$x_3 = 0{,}6$; $y_3 = y_2 + hF(x_2, y_2) \simeq 0{,}5606$; $F(x_3, y_3) \simeq 0{,}6636$

$x_4 = 0{,}8$; $y_4 = y_3 + hF(x_3 + y_3) \simeq 0{,}6934$; $F(x_4, y_4) \simeq 0{,}4453$

$x_5 = 1$; $y_5 = y_4 + hF(x_4, y_4) \simeq 0{,}7824$

9.3.11) Lös BVP $\frac{dy}{dx} = xe^y$, $y(0) = 0$

<u>Lösn</u>: Separabel DE

$$\int e^{-y}\,dy = \int x\,dx$$

$$\Rightarrow \quad -e^{-y} = \frac{1}{2}x^2 + C$$

$$\Rightarrow \quad y = -\ln\left(-\frac{1}{2}x^2 - C\right)$$

$$y(0) = -\ln(-C) = 0 \quad \Rightarrow \quad C = -1$$

$$\Rightarrow \quad y(x) = -\ln\left(1 - \frac{1}{2}x^2\right)$$