1) Se beviset av "Rolles theorem"

s. 287 i kap 4.2 i

"Calculus" av Stewart.

2) Se beviset av "The fundamental

Theorem of Calculus, Part 2"

s. 396 i kap. 5.3 i

"Calculus" av Stewart.

3a) $\lim_{x\to\infty} \sqrt{9x^2+7x} - 3x = \{\text{förläng med konjugat}\}$

$$= \lim_{x\to\infty} \frac{9x^2+7x-\cancel{9x^2}}{\sqrt{9x^2+7x}+3x} = \lim_{x\to\infty} \frac{7x}{\sqrt{9x^2+7x}+3x}$$

$$= \{|x| = x \text{ då } x\to\infty\} = \lim_{x\to\infty} \frac{7\cancel{x}}{\cancel{x}\sqrt{9+\frac{7}{x}}+3\cancel{x}}$$

$$= \lim_{x\to\infty} \frac{7}{\sqrt{9+\underbrace{\frac{7}{x}}_{\xrightarrow[x\to\infty]{}0}}+3} = \frac{7}{\sqrt{9}+3} = \frac{7}{6}$$

<u>Svar:</u> $\lim_{x\to\infty}\left(\sqrt{9x^2+7x} - 3x\right) = \frac{7}{6}$

b) $\lim_{x\to 0^+} \sin x \ln x \leftarrow [0\cdot\infty]$

$$= \lim_{x\to 0^+} \frac{\sin x}{x} x\ln x = \underbrace{\lim_{x\to 0^+} \frac{\sin x}{x}}_{=1} \cdot \lim_{x\to 0^+} x\ln x$$

$$= \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \leftarrow \left[\frac{\infty}{\infty}\right] = \{\text{l'Hôpital}\}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} x = 0$$

<u>Svar:</u> $\lim_{x\to 0^+} \sin x \ln x = 0$

4a) $\displaystyle\int \frac{\sin(\sqrt{1-x})}{\sqrt{1-x}}\,dx = \left\{\begin{array}{l} u=\sqrt{1-x} \\ du=-\frac{1}{2}\frac{1}{\sqrt{1-x}}\,dx \end{array}\right\}$

$= -2\displaystyle\int \sin u \, du = -2(-\cos u) + C$

$= 2\cos u + C = 2\cos(\sqrt{1-x}) + C$

Svar: $\displaystyle\int \frac{\sin(\sqrt{1-x})}{\sqrt{1-x}}\,dx = 2\cos(\sqrt{1-x}) + C$

b) Integranden $\frac{x^4}{x^2-4}$ rationell funktion

=> Polynomdivision

$$\begin{array}{l|l} \multicolumn{1}{c}{x^2+4} & \\ \hline x^4 & x^2-4 \\ -(x^4-4x^2) & \\ \hline 4x^2 & \\ -(4x^2-16) & \\ \hline 16 & \end{array}$$

$$\Rightarrow \quad \frac{x^4}{x^2-4} = x^2+4+\frac{16}{x^2-4}$$

=> Partialbråksuppdelning:

$$\frac{16}{x^2-4} = \frac{16}{(x+2)(x-2)} = \frac{A}{x+2}+\frac{B}{x-2} = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$

$$\Rightarrow \quad 16 = A(x-2)+B(x+2)$$

$$\Rightarrow \begin{cases} A+B=0 \\ -2A+2B=0 \end{cases} \Rightarrow \begin{cases} A=-B \\ 4B=16 \end{cases} \Rightarrow \begin{cases} A=-4 \\ B=4 \end{cases}$$

$$\Rightarrow \int_0^1 \frac{x^4}{x^2-4}\,dx = \int_0^1 \left(x^2+4-\frac{4}{x+2}+\frac{4}{x-2}\right)dx$$

$$= \left[\frac{1}{3}x^3+4x-4\ln\frac{|x+2|}{|x-2|}\right]_0^1 = \frac{1}{3}+4-4\ln\left|\frac{3}{-1}\right|+\underbrace{4\ln\left|\frac{2}{-2}\right|}_{=\ln 1 = 0}$$

$$= \frac{13}{3}-4\ln 3$$

Svar: $\int_0^1 \frac{x^4}{x^2-4}\,dx = \frac{13}{3}-4\ln 3$

c) $\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x^2}\,dx$

$$\int_1^{t} \frac{\ln x}{x^2}\,dx = \{PI\} = \left[-\frac{1}{x}\ln x\right]_1^t - \int_1^t -\frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= -\frac{\ln t}{t} + \underbrace{\ln 1}_{=0} - \left[\frac{1}{x}\right]_1^t$$

$$= -\frac{\ln t}{t} - \frac{1}{t} + 1$$

$$\Rightarrow \int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left(-\underbrace{\frac{\ln t}{t}}_{\to 0} - \underbrace{\frac{1}{t}}_{\to 0} + 1\right) = 1$$

ty $\lim_{t\to\infty} \frac{\ln t}{t}$ $\left[\frac{\infty}{\infty}\right]$ $= \{\text{l'Hôpital}\}$

$$= \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = \lim_{t\to\infty} \frac{1}{t} = 0$$

Svar: $\int_1^{\infty} \frac{\ln x}{x^2}\,dx = 1$

5a) $y' = 2e^x\sqrt{y}$, $y > 0$ separabel

$$\frac{dy}{dx} = 2e^x\sqrt{y} \Rightarrow \frac{1}{2}\frac{1}{\sqrt{y}}\,dy = e^x\,dx$$

$$\Rightarrow \int \frac{1}{2}\frac{1}{\sqrt{y}}\,dy = \int e^x\,dx$$

$$\Rightarrow \sqrt{y} = e^x + C \quad , \quad C \geq 0$$

$$\Rightarrow y = (e^x + C)^2 = e^{2x} + 2Ce^x + C^2$$

$$y(0) = (1 + C)^2 = 4 \Rightarrow 1 + C = \pm 2$$

$$\Rightarrow C = 1 \quad \text{eftersom} \quad C \geq 0$$

$$\Rightarrow y(x) = (e^x + 1)^2 = e^{2x} + 2e^x + 1 > 0 \quad \forall x \in \mathbb{R}$$

Svar: $y(x) = (e^x + 1)^2 = e^{2x} + 2e^x + 1$

b) $y' + y = \cos x$

Integrerande faktor: $e^{\int 1 \, dx} = e^x$

$\Rightarrow \quad e^x y' + e^x y = e^x \cos x$

$\Rightarrow \quad (e^x y)' = e^x \cos x$

$\Rightarrow \quad e^x y = \int e^x \cos x \, dx$

$$\int e^x \cos x \, dx \overset{PI}{=} e^x \cos x + \int e^x \sin x \, dx$$

$$\overset{PI}{=} e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\Rightarrow \int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

Lösningen till diff.ekv. ges då av

$$y = e^{-x} \int e^x \cos x \, dx = \frac{1}{2} (\sin x + \cos x) + Ce^{-x}$$

$$y(0) = \frac{1}{2}(0+1) + C = 0 \quad \Rightarrow \quad C = -\frac{1}{2}$$

$$\Rightarrow \quad y(x) = \frac{1}{2}(\sin x + \cos x) - \frac{1}{2} e^{-x}$$

Svar: $y(x) = \frac{1}{2}(\sin x + \cos x) - \frac{1}{2} e^{-x}$

c) $y'' + 4y' + 4y = 0$

Karaktäristisk ekv. : $r^2 + 4r + 4 = (r+2)^2 = 0$

$\Rightarrow \quad r_1 = r_2 = -2$

$\Rightarrow \quad y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

Randvärdena $y(0) = 0$, $y(1) = 1$ ger

$$\begin{cases} y(0) = c_1 = 0 \\ y(1) = (c_1 + c_2) e^{-2} = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = e^2 \end{cases}$$

$\Rightarrow \quad y(x) = e^2 \cdot x \cdot e^{-2x} = x e^{2(1-x)}$

<u>Svar:</u> $y(x) = x e^{2(1-x)}$

6 a) $y(x) = \cos(\arctan x)$

$$y'(x) = -\sin(\arctan x) \cdot \frac{1}{1+x^2}$$

$$y'(1) = -\sin(\underbrace{\arctan(1)}_{=\frac{\pi}{4}}) \cdot \frac{1}{1+1^2} = -\frac{1}{2}\sin\left(\frac{\pi}{4}\right)$$

$$= -\frac{1}{2\sqrt{2}}$$

<u>Svar:</u> Lutningen i $x=1$ är $y' = -\frac{1}{2\sqrt{2}}$

b) $x^2 e^y + x^5 = xy$

Derivera implicit map x:

$$\frac{d}{dx} \Rightarrow 2xe^y + x^2 y' e^y + 5x^4 = y + xy'$$

$$\Rightarrow y'(x^2e^y - x) = (y - 5x^4 - 2xe^y)$$

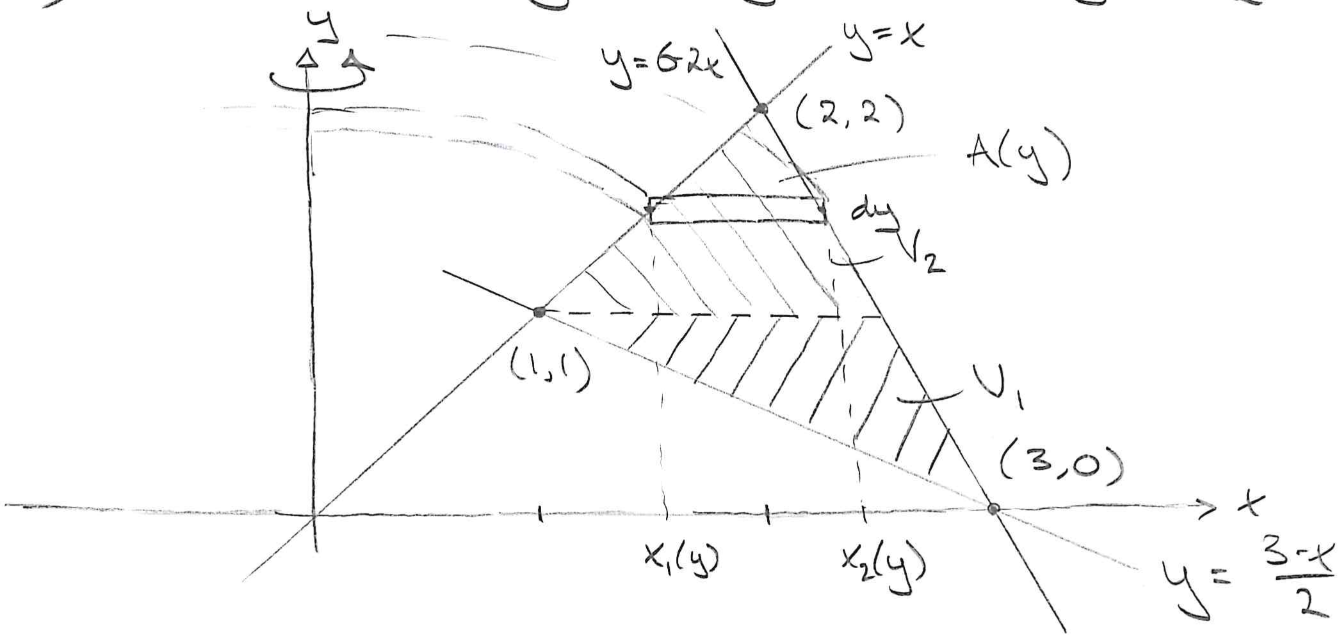
$$\Rightarrow y' = \frac{(y - 5x^4 - 2xe^y)}{(x^2e^y - x)}$$

I $(x,y) = (-1,0)$ har vi då

$$y' = \frac{0 - 5\cdot 1 + 2\cdot 1}{1\cdot 1 + 1} = -\frac{3}{2}$$

<u>Svar:</u> Lutningen i $(x,y) = (-1,0)$ är $y' = -\frac{3}{2}$

7) Räta linjer $y = x$, $y = 6 - 2x$, $y = \frac{3-x}{2}$



Skärningspunkter:

i) $x = 6 - 2x \Rightarrow x = 2 \Rightarrow y = 2$

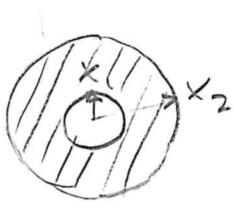
ii) $x = \frac{3-x}{2} \Rightarrow x = 1 \Rightarrow y = 1$

iii) $6 - 2x = \frac{3-x}{2} \Rightarrow 12 - 4x = 3 - x \Rightarrow x = 3 \Rightarrow y = 0$

$\Rightarrow \quad (2,2) \; ; \; (1,1) \; ; \; (3,0)$

Dela in i skivor med tvärsnitt $y =$ konstant

Tvärsnittets area:



$$A(y) = \pi \left(x_2^2(y) - x_1^2(y) \right)$$

$$\Rightarrow V = \int_0^2 A(y)\,dy$$

Vi delar upp integralen i två delar beroende på vilken linje som ger yttre resp inre radie

$$V_1 = \int_0^1 \pi \Big(\big(\underbrace{\tfrac{1}{2}(6-y)}_{\substack{y = 6-2x \\ \text{yttre}}} \big)^2 - \big(\underbrace{3-2y}_{\substack{y = \frac{3-x}{2} \\ \text{inre}}} \big)^2 \Big)\, dy$$

$$= \pi \int_0^1 \left(\cancel{9} - 3y + \tfrac{1}{4}y^2 - \cancel{9} + 12y - 4y^2 \right) dy$$

$$= \pi \int_0^1 \left(9y - \tfrac{15}{4}y^2 \right) = \pi \left[\tfrac{9}{2}y^2 - \tfrac{5}{4}y^3 \right]_0^1$$

$$= \pi \left(\tfrac{9}{2} - \tfrac{5}{4} \right) = \frac{13\pi}{4}$$

$$V_2 = \int_1^2 \pi \Big(\big(\underbrace{\tfrac{1}{2}(6-y)}_{\substack{y = 6-2x \\ \text{yttre}}} \big)^2 - \underbrace{y}_{\substack{y = x \\ \text{inre}}}^2 \Big)\, dy$$

$$= \pi \int_1^2 \left(9 - 3y + \tfrac{1}{4}y^2 - y^2 \right) dy = \pi \int_1^2 \left(9 - 3y - \tfrac{3}{4}y^2 \right) dy$$

$$= \pi \left[9y - \tfrac{3}{2}y^2 - \tfrac{1}{4}y^3 \right]_1^2 = \pi \left(18 - 6 - 2 - 9 + \tfrac{3}{2} + \tfrac{1}{4} \right)$$

$$= \pi \left(1 + \tfrac{3}{2} + \tfrac{1}{4} \right) = \frac{11\pi}{4}$$

$$\Rightarrow \quad V = V_1 + V_2 = \frac{13\pi}{4} + \frac{11\pi}{4} = \frac{24\pi}{4} = 6\pi$$

Svar: Rotationsvolymen är $V = 6\pi$

Alt: Cylindriska skal

$$V = \int_1^3 \underbrace{2\pi x}_{\text{omkrets}} \overbrace{(y_2 - y_1)}^{\text{höjd}} \underbrace{dx}_{\text{tjocklek}}$$

övre undre

Återigen delar vi upp integralen i två delar beroende på vilken linje som ger övre resp. undre begränsning av höjden

$$V_1 = \int_1^2 2\pi x \left(x - \frac{3-x}{2}\right) dx = 2\pi \int_1^2 \frac{3}{2}(x^2 - x)\, dx$$

$$= 3\pi \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_1^2 = 3\pi\left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right)$$

$$= \pi\left(8 - 6 - 1 + \frac{3}{2}\right) = \frac{5\pi}{2}$$

$$V_2 = \int_2^3 2\pi x \left(6 - 2x - \frac{3-x}{2}\right) dx = \pi \int_2^3 (9x - 3x^2)\, dx$$

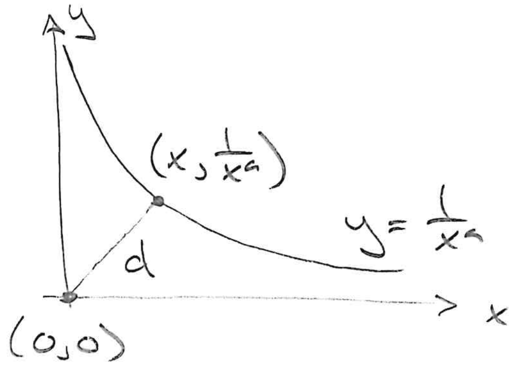
$$= \pi\left[\frac{9}{2}x^2 - x^3\right]_2^3 = \pi\left(\frac{81}{2} - 27 - 18 + 8\right)$$

$$= \frac{\pi}{2}(81 - 74) = \frac{7\pi}{2}$$

$$\Rightarrow V = V_1 + V_2 = \frac{5\pi}{2} + \frac{7\pi}{2} = \frac{12\pi}{2} = 6\pi$$

Svar: Rotationsvolymen är $V = 6\pi$

8)



$$y = \frac{1}{x^a} \qquad a > 0$$
$$x > 0$$

Vi söker minimera avståndet $d = \sqrt{x^2 + \frac{1}{x^{2a}}}$ från origo till punkt på kurvan $y = \frac{1}{x^a}$

Betrakta $D = d^2 = x^2 + \dfrac{1}{x^{2a}}$

Eftersom $d = \sqrt{D}$ strängt växande för $D > 0$ gäller att x som minimerar D också minimerar d

$$D(x) = x^2 + \frac{1}{x^{2a}} = x^2 + x^{-2a}$$

$$D'(x) = 2x - 2a\,x^{-2a-1} = 0$$

$$\Rightarrow \quad 2x = 2a\,x^{-2a-1} \quad \Rightarrow \quad a = x^{2(a+1)}$$

$$\Rightarrow \quad x = a^{\frac{1}{2(a+1)}}$$ <u>kritisk punkt</u>

$$D''(x) = 2 + 2a(2a+1)\,x^{-2(a+1)} > 0 \quad \forall\, x > 0$$

$$\Rightarrow \quad x = a^{\frac{1}{2(a+1)}}$$ <u>lokalt min</u>

Vi har också att

$$D' = 2x^{-2a-1}\left(x^{2(a+1)} - a\right) \begin{cases} < 0 & \text{för } x < a^{\frac{1}{2(a+1)}} \\ > 0 & \text{för } x > a^{\frac{1}{2(a+1)}} \end{cases}$$

$\Rightarrow \quad x = a^{\frac{1}{2(a+1)}}$ <u>globalt min</u> på $(0, \infty)$

Slutligen beräknar vi motsvarande y-koordinat.

$$y\left(a^{\frac{1}{2(a+1)}}\right) = \left(a^{\frac{1}{2(a+1)}}\right)^{-a} = a^{\frac{-a}{2(a+1)}}$$

<u>Svar:</u> Punkten $\left(a^{\frac{1}{2(a+1)}}, a^{\frac{-a}{2(a+1)}}\right)$ ligger närmast origo.