1) Se beviset av "Theorem 5"

s. 290-291 i kap 4.2 i

"Calculus" av Stewart.

2) Se beviset av punkt 3 i

Teori-PM på kurshemsidan.

3a) $\lim\limits_{x \to +\infty} \sqrt{e^{2x}+1} - e^{x} = \{\text{förläng med konjugat.}\}$

$$= \lim_{x \to +\infty} \frac{\cancel{e^{2x}} + 1 - \cancel{e^{2x}}}{\sqrt{e^{2x}+1} + e^{x}} = \lim_{x \to +\infty} \frac{1}{\underbrace{\sqrt{e^{2x}+1}}_{\substack{\to \infty \\ x \to \infty}} + \underbrace{e^{x}}_{\substack{\to \infty \\ x \to \infty}}} = 0$$

Svar: $\lim\limits_{x \to +\infty} \sqrt{e^{2x}+1} - e^{x} = 0$

b) $\lim\limits_{x \to 0^{+}} \frac{\ln(\sin x)}{\ln x}$ $\left[\frac{\infty}{\infty}\right]$

$$= \{\text{l'Hôpital}\} = \lim_{x \to 0^{+}} \frac{\frac{1}{\sin x} \cdot \cos x}{\frac{1}{x}}$$

$$= \underbrace{\lim_{x \to 0^{+}} \cos x}_{= 1} \cdot \lim_{x \to 0^{+}} \frac{1}{\frac{\sin x}{x}} = \left\{ \lim_{x \to 0^{+}} \frac{\sin x}{x} = 1 \right\}$$

$$= 1 \cdot \frac{1}{1} = 1$$

Svar: $\lim\limits_{x \to 0^{+}} \frac{\ln(\sin x)}{\ln x} = 1$

4a) $\int \frac{x+7}{x^2-5x+6}$ Integranden rationell $\frac{P(x)}{Q(x)}$

med $\deg(P) < \deg(Q)$

=> Partialbråksuppdelning:

$$\frac{x+7}{x^2-5x+6} = \frac{x+7}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-3)}{(x-3)(x-2)}$$

$$\Rightarrow \begin{cases} A+B = 1 \\ -2A-3B = 7 \end{cases} \Rightarrow \begin{cases} A = 1-B \\ -B = 9 \end{cases} \Rightarrow \begin{cases} A = 10 \\ B = -9 \end{cases}$$

$$\Rightarrow \int \frac{x+7}{x^2-5x+6}\,dx = \int \left(\frac{10}{x-3} - \frac{9}{x-2} \right) dx$$

$$= 10 \ln|x-3| - 9 \ln|x-2| + C$$

Svar: $\int \frac{x+7}{x^2-5x+6}\,dx = 10 \ln|x-3| - 9 \ln|x-2| + C$

b) $\int \sin x \cos x \, e^{\cos x} \, dx$

$= \{ u = \cos x, \ du = -\sin x \, dx \}$

$= -\int u e^u \, du \overset{PI}{=} -u e^u + \int 1 \cdot e^u \, du$

$= -u e^u + e^u + C = e^u (1 - u) + C$

$= e^{\cos x} (1 - \cos x) + C$

<u>Svar:</u> $\int \sin x \cos x \, e^{\cos x} \, dx = e^{\cos x} (1 - \cos x) + C$

c) $\int_{-1}^{1} \frac{1}{x^2}\,dx$

Integranden $\frac{1}{x^2}$ är diskontinuerlig i $x=0$

$$\Rightarrow \int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

om båda integralerna i högerledet konvergerar.

$$\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx = \lim_{t \to 0^-} \left[-\frac{1}{x}\right]_{-1}^{t}$$

$$= \lim_{t \to 0^-} \left(-\frac{1}{t} - 1\right) = \infty$$

$$\Rightarrow \int_{-1}^{0} \frac{1}{x^2}\,dx \text{ divergent}$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{x^2}\,dx \text{ divergent, dvs existerar ej.}$$

Svar: $\int_{-1}^{1} \frac{1}{x^2}\,dx$ är divergent.

5a) $y' = x e^{x^2 - y}$, $y(0) = 0$

$$\frac{dy}{dx} = x e^{x^2} e^{-y}$$ <u>separabel</u>

$$\Rightarrow \quad e^y \, dy = x e^{x^2} \, dx$$

$$\Rightarrow \quad \int e^y \, dy = \int x e^{x^2} dx$$

$$\Rightarrow \quad e^y = \frac{1}{2} e^{x^2} + C \quad , \quad C \geq -\frac{1}{2}$$

$$\Rightarrow \quad y = \ln\left(\frac{1}{2} e^{x^2} + C\right)$$

$$y(0) = \ln\left(\frac{1}{2} + C\right) = 0 \quad \Rightarrow \quad C = \frac{1}{2}$$

$$\Rightarrow \quad y(x) = \ln\left(\frac{1}{2}\left(e^{x^2} + 1\right)\right)$$

<u>Svar:</u> $y(x) = \ln\left(\frac{1}{2}\left(e^{x^2} + 1\right)\right)$

b) $y'' - y = x^2 + 1 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0$

Homogenlösning: $r^2 - 1 = 0$ (karaktäristiska ekv.)

$\Rightarrow r = \pm 1$

$\Rightarrow y_h(x) = c_1 e^{-x} + c_2 e^{x}$

Partikulärlösning: $G(x) = x^2 + 1 \Rightarrow$ Ansätt $y_p(x) = Ax^2 + Bx + C$

$\Rightarrow y_p'(x) = 2Ax + B \quad , \quad y_p''(x) = 2A$

$\Rightarrow 2A - (Ax^2 + Bx + C) = x^2 + 1$

$$\Rightarrow \begin{cases} -A = 1 \\ -B = 0 \\ 2A - C = 1 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 0 \\ C = -3 \end{cases}$$

$\Rightarrow y_p(x) = -x^2 - 3$

$\Rightarrow y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{x} - (x^2 + 3)$

$y'(x) = -c_1 e^{-x} + c_2 e^{x} - 2x$

$$\Rightarrow \begin{cases} y(0) = c_1 + c_2 - 3 = 1 \\ y'(0) = -c_1 + c_2 = 0 \end{cases}$$

$\Rightarrow \quad c_2 = c_1 \quad , \quad 2c_1 - 3 = 1$

$\Rightarrow \quad c_1 = c_2 = 2$

$\Rightarrow \quad y(x) = 2(e^{-x} + e^{x}) - (x^2 + 3)$

<u>Svar:</u> $y(x) = 2(e^{-x} + e^{x}) - (x^2 + 3)$

c) $y' + 2xy = x \quad , \quad y(0) = 0$

Integrerande faktor: $e^{\int 2x\,dx} = e^{x^2}$

$$\Rightarrow e^{x^2} y' + 2x e^{x^2} y = x e^{x^2}$$

$$\Rightarrow \left(e^{x^2} y \right)' = x e^{x^2}$$

$$\Rightarrow e^{x^2} y = \int x e^{x^2}\,dx = \frac{1}{2} e^{x^2} + C$$

$$\Rightarrow y(x) = e^{-x^2}\left(\frac{1}{2} e^{x^2} + C \right) = \frac{1}{2} + C e^{-x^2}$$

$$y(0) = \frac{1}{2} + C = 0 \quad \Rightarrow \quad C = -\frac{1}{2}$$

$$\Rightarrow y(x) = \frac{1}{2}\left(1 - e^{-x^2}\right)$$

Svar: $y(x) = \frac{1}{2}\left(1 - e^{-x^2}\right)$

6a) $y(x) = \sin\left(\frac{\pi}{\ln x}\right)$

$$y'(x) = \cos\left(\frac{\pi}{\ln x}\right)\cdot\left(-\frac{\pi}{(\ln x)^2}\cdot\frac{1}{x}\right)$$

$$y'(e) = \cos\left(\frac{\pi}{\ln e}\right)\cdot\left(-\frac{\pi}{(\ln e)^2}\cdot\frac{1}{e}\right) = \{\ln e = 1\}$$

$$= \underbrace{\cos(\pi)}_{=-1}\cdot\left(-\pi\cdot\frac{1}{e}\right) = \pi\cdot\frac{1}{e} = \pi e^{-1}$$

Svar: Lutningen i $x=e$ är $y' = \pi e^{-1}$

b) $$x^2e^y + \sin\left(\frac{\pi x}{2}\right) + \cos\left(\frac{\pi x}{2}\right) = xy$$

Derivera implicit map x:

$$\overset{\frac{d}{dx}}{\Rightarrow} 2xe^y + x^2y'e^y + \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right) = y + xy'$$

$$\Rightarrow y'(x^2e^y - x) = \left(y - \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right) + \frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right) - 2xe^y\right)$$

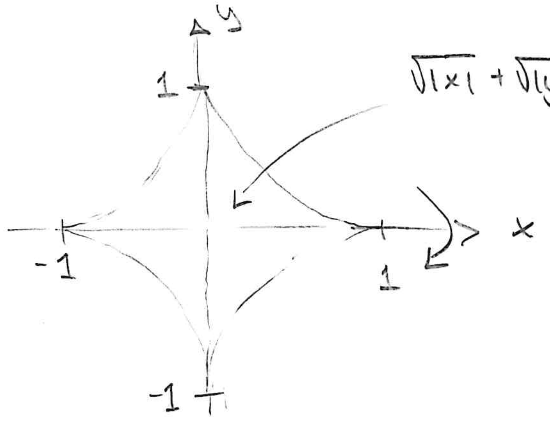
$$\Rightarrow y' = \frac{\left(y - \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right) + \frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right) - 2xe^y\right)}{(x^2e^y - x)}$$

I $(x,y) = (-1,0)$ har vi då

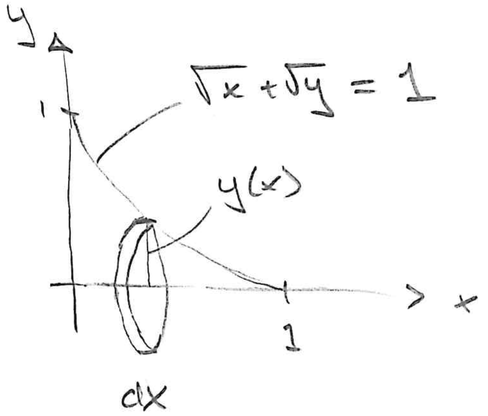
$$y' = \frac{\left(0 - \frac{\pi}{2}\cdot 0 + \frac{\pi}{2}\cdot(-1) + 2\cdot 1\right)}{(1\cdot 1 + 1)} = 1 - \frac{\pi}{4}$$

Svar: Lutningen i $(x,y) = (-1,0)$ är $y' = 1 - \frac{\pi}{4}$

7)



Pga symmetri räcker det att betrakta $x, y \geq 0$



$$V_1 = \int_0^1 \pi (y(x))^2 \, dx \quad \text{med}$$

$$y(x) = (1 - \sqrt{x})^2$$

$$\Rightarrow V_1 = \pi \int_0^1 (1-\sqrt{x})^4 \, dx = \pi \int_0^1 (1 - 2x^{1/2} + x)^2 \, dx$$

$$= \pi \int_0^1 (1 - 4x^{1/2} + 6x - 4x^{3/2} + x^2) \, dx$$

$$= \pi \left[x - \frac{8}{3} x^{3/2} + 3x^2 - \frac{8}{5} x^{5/2} + \frac{1}{3} x^3 \right]_0^1$$

$$= \pi \left[1 - \frac{8}{3} + 3 - \frac{8}{5} + \frac{1}{3} \right] =$$

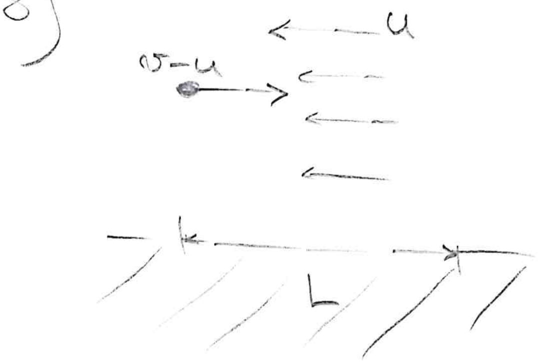
$$= \frac{\pi}{15} \left[15 - 40 + 45 - 24 + 5 \right] = \frac{\pi}{15} (65 - 64) = \frac{\pi}{15}$$

Den sökta volymen ges av

$$V = 2V_1 = \frac{2\pi}{15}$$

<u>Svar:</u> Kroppens volym är $V = \frac{2\pi}{15}$ v.e.

8)



Relativ hastighet: v

Absolut hastighet: $v-u$

Energi per tidsenhet: $P = av^3$

Tidsåtgång: $t = \dfrac{L}{v-u}$

$\Rightarrow$ Energiförbrukning $E(v) = P \cdot t = aL \dfrac{v^3}{v-u}$

Vi söker minimera $E(v)$ för $v > u$

$$E'(v) = aL\left[\frac{3v^2}{v-u} - v^3\frac{1}{(v-u)^2}\right]$$

$$= aL\left[\frac{3v^2(v-u) - v^3}{(v-u)^2}\right] = aL\left[\frac{2v^3 - 3v^2u}{(v-u)^2}\right]$$

$$E'(v) = 0 \Rightarrow 2v^3 - 3v^2u = 0$$

$$\Rightarrow v = \frac{3u}{2}$$

Vi ser att $E'(v) = aL\dfrac{v^2(2v-3u)}{(v-u)^2}$

$$\Rightarrow \begin{cases} E'(v) < 0 & \forall v < \frac{3u}{2} \\ E'(v) > 0 & \forall v > \frac{3u}{2} \end{cases}$$

Enligt test med förstaderivatan är

då $v = \dfrac{3u}{2}$ ett globalt minimum på (u, ∞)

$$E\left(\frac{3u}{2}\right) = aL\,\frac{\left(\frac{3}{2}\right)^3 u^3}{\left(\frac{3u}{2}-u\right)} = aL\,\frac{\frac{27}{8}u^3}{\frac{1}{2}u}$$

$$= \frac{27}{4}aLu^2$$

<u>Svar:</u> Den relativa hastigheten $v = \frac{3u}{2}$ ger en minimal energiåtgång $E = \frac{27}{4}aLu^2$.