1) Se beviset "The sum law"

s.111-112 i kap 2.4 i

"Calculus" av Stewart

2) Se härledning s.472 i kap 7.1

i "Calculus" av Stewart.

3a) $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x}$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = 1$

Svar: $\lim_{x \to 0} \frac{\tan x}{x} = 1$

b) $\lim_{x \to 1} \frac{x^2+6x-7}{x^2+2x-3} \left[\frac{0}{0}\right] = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+7)}{\cancel{(x-1)}(x+3)}$

$= \lim_{x \to 1} \frac{x+7}{x+3} = \frac{8}{4} = 2$

Svar: $\lim_{x \to 1} \frac{x^2+6x-7}{x^2+2x-3} = 2$

4a) $$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \left\{ \begin{array}{ll} x = \sin\theta & \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ dx = \cos\theta\, dx & \\ x=-1 \Rightarrow \theta = -\frac{\pi}{2}, & x = 1 \Rightarrow \theta = \frac{\pi}{2} \end{array} \right\}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\sqrt{\cos^2\theta}}_{= \cos\theta \text{ for } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \left\{ \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \right\}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} + \frac{1}{4}\underbrace{\sin\pi}_{=0} - \left(-\frac{\pi}{4} + \frac{1}{4}\underbrace{\sin(-\pi)}_{=0}\right) = \frac{\pi}{2}$$

Svar: $\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$

4 b) $\int \frac{5x}{x^2+2x+1}\,dx$ Integranden rationell $\frac{P(x)}{Q(x)}$

med $\deg(P) < \deg(Q)$

$\rightsquigarrow$ Partialbråksuppdelning:

$$\frac{5x}{x^2+2x+1} = \frac{5}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{Ax+A+B}{(x+1)^2}$$

$$\Rightarrow \begin{cases} A = 5 \\ A+B = 0 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = -5 \end{cases}$$

$$\Rightarrow \int \frac{5x}{x^2+2x+1}\,dx = \int \left( \frac{5}{x+1} - \frac{5}{(x+1)^2} \right) dx$$

$$= 5 \ln|x+1| - \left(-5 \frac{1}{x+1}\right) + C$$

$$= 5\left(\ln|x+1| + \frac{1}{x+1}\right) + C$$

<u>Svar:</u> $\int \frac{5x}{x^2+2x+1}\,dx = 5\left(\ln|x+1| + \frac{1}{x+1}\right) + C$

4c) $$\int_0^1 (x^2e^{2x}+x)\,dx = \int_0^1 x^2 e^{2x}\,dx + \underbrace{\left[\frac{x^2}{2}\right]_0^1}_{=\frac{1}{2}}$$

$$\overset{PI}{=} \frac{1}{2} + \underbrace{\left[x^2\cdot\frac{1}{2}e^{2x}\right]_0^1}_{=\frac{1}{2}e^2} - \int \cancel{2}x\cdot\frac{1}{\cancel{2}}e^{2x}\,dx$$

$$\overset{PI}{=} \frac{1}{2}(1+e^2) - \underbrace{\left[x\cdot\frac{1}{2}e^{2x}\right]_0^1}_{=\frac{1}{2}e^2} + \int\frac{1}{2}e^{2x}\,dx$$

$$= \frac{1}{2}(1+\cancel{e^2}-\cancel{e^2}) + \left[\frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{2}+\frac{1}{4}(e^2-1)$$

$$= \frac{1}{4}(1+e^2)$$

<u>Svar:</u> $$\int_0^1 (x^2e^{2x}+x)\,dx = \frac{1}{4}(1+e^2)$$

5 a) $yy'(1+x^2) = x$ , $y(0) = 2$

$y \frac{dy}{dx}(1+x^2) = x$ <u>separabel</u>

$\Rightarrow \quad y\,dy = \frac{x}{1+x^2}\,dx$

$\Rightarrow \quad \int y\,dy = \int \frac{x}{1+x^2}\,dx$

$\Rightarrow \quad \frac{y^2}{2} = \frac{1}{2}\ln(1+x^2) + C$

$\Rightarrow \quad y = \pm\sqrt{\ln(1+x^2) + C}$

Eftersom vi söker lösning $y(0) = 2 > 0$:

$y(0) = \sqrt{\underbrace{\ln(1)}_{=0} + C} = \sqrt{C} = 2$

$\Rightarrow \quad C = 4 \quad \Rightarrow \quad y(x) = \sqrt{\ln(1+x^2) + 4}$

<u>Svar:</u> $y(x) = \sqrt{\ln(1+x^2) + 4}$

b) $y'' - 2y' + y = xe^{2x}$ , $y(0) = 1$ , $y'(0) = 1$

Homogenlösning: $r^2 - 2r + 1 = 0$ (karaktäristisk ekv.)

$\Rightarrow r_{1,2} = 1$

$\Rightarrow y_h(x) = C_1 e^x + C_2 x e^x$

Partikulärlösning: $G(x) = xe^{2x}$

$\Rightarrow$ Ansätt $y_p(x) = (Ax + B)e^{2x}$

$y_p'(x) = Ae^{2x} + 2(Ax+B)e^{2x} = (2Ax + (A+2B))e^{2x}$

$y_p''(x) = 2Ae^{2x} + 2(2Ax + (A+2B))e^{2x} = (4Ax + (4A+4B))e^{2x}$

$\Rightarrow y_p'' - 2y_p' + y_p = (Ax + (2A+B))e^{2x} = xe^{2x}$

$\Rightarrow \begin{cases} A = 1 \\ 2A + B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -2 \end{cases}$

$\Rightarrow y_p(x) = (x-2)e^{2x}$

Allmänna lösningen ges alltså av

$y(x) = C_1 e^x + C_2 x e^x + (x-2)e^{2x}$

$$y'(x) = C_1 e^x + C_2 e^x + C_2 x e^x + 2(x-2)e^{2x} + e^{2x}$$

$$= (C_1 + C_2)e^x + C_2 x e^x + (2x-3)e^{2x}$$

Begynnelsevillkoren ger då

$$\begin{cases} y(0) = C_1 - 2 = 1 \\ y'(0) = C_1 + C_2 - 3 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = 3 \\ C_2 = 1 \end{cases}$$

$$\Rightarrow \quad y(x) = 3e^x + xe^x + (x-2)e^{2x}$$

<u>Svar:</u> $y(x) = (x+3)e^x + (x-2)e^{2x}$

c) $y' + \sin(x)\, y = \sin(x)\,, \quad y(\frac{\pi}{2}) = 0$

Integrerande faktor: $e^{\int \sin x \, dx} = e^{-\cos x}$

$\Rightarrow \quad e^{-\cos x} y' + \sin x \, e^{-\cos x} y = \sin x \, e^{-\cos x}$

$\Rightarrow \quad (e^{-\cos x} y)' = \sin x \, e^{-\cos x}$

$\Rightarrow \quad e^{-\cos x} y = \int \sin x \, e^{-\cos x} dx = e^{-\cos x} + C$

$\Rightarrow \quad y(x) = e^{\cos x} (e^{-\cos x} + C) = 1 + Ce^{\cos x}$

$y(\frac{\pi}{2}) = 1 + C e^{\cos \frac{\pi}{2}} = 1 + C e^0 = 1 + C = 0$

$\Rightarrow \quad C = -1 \quad \Rightarrow \quad y(x) = 1 - e^{\cos x}$

Svar: $y(x) = 1 - e^{\cos x}$

6a) $y(x) = \sin(\pi x + \arctan(x))$

$$y'(x) = \cos(\pi x + \arctan(x)) \cdot \left(\pi + \frac{1}{1+x^2}\right)$$

$$y'(1) = \cos(\pi + \underbrace{\arctan(1)}_{=\frac{\pi}{4}}) \cdot \left(\pi + \frac{1}{1+1}\right)$$

$$= \underbrace{\cos\left(\frac{5\pi}{4}\right)}_{= -\cos\left(\pi - \frac{5\pi}{4}\right) = -\cos\left(-\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{2}} \left(\pi + \frac{1}{2}\right) = -\frac{1}{\sqrt{2}}\left(\pi + \frac{1}{2}\right)$$

Svar: Lutningen i $x = 1$ är $y' = -\frac{1}{\sqrt{2}}\left(\pi + \frac{1}{2}\right)$

b) $\sqrt{x} - e^y = \sin\left(\frac{\pi x}{4}\right) y$ Derivera implicit map $x$

$$\overset{\frac{d}{dx}}{\Longrightarrow} \frac{1}{2\sqrt{x}} - y' e^y = \sin\left(\frac{\pi x}{4}\right) y' + \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) y$$

$$\Rightarrow y'\left(e^y + \sin\left(\frac{\pi x}{4}\right)\right) = \frac{1}{2\sqrt{x}} - \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) y$$

$$\Rightarrow y' = \frac{\frac{1}{2\sqrt{x}} - \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) y}{e^y + \sin\left(\frac{\pi x}{4}\right)}$$

I punkten $(x,y) = (1,0)$ har vi då

$$y' = \frac{\frac{1}{2} - 0}{1 + \frac{1}{\sqrt{2}}} = \frac{1}{2 + \sqrt{2}}$$

Svar: Lutningen i $(x,y) = (1,0)$ är $y' = \frac{1}{2+\sqrt{2}}$

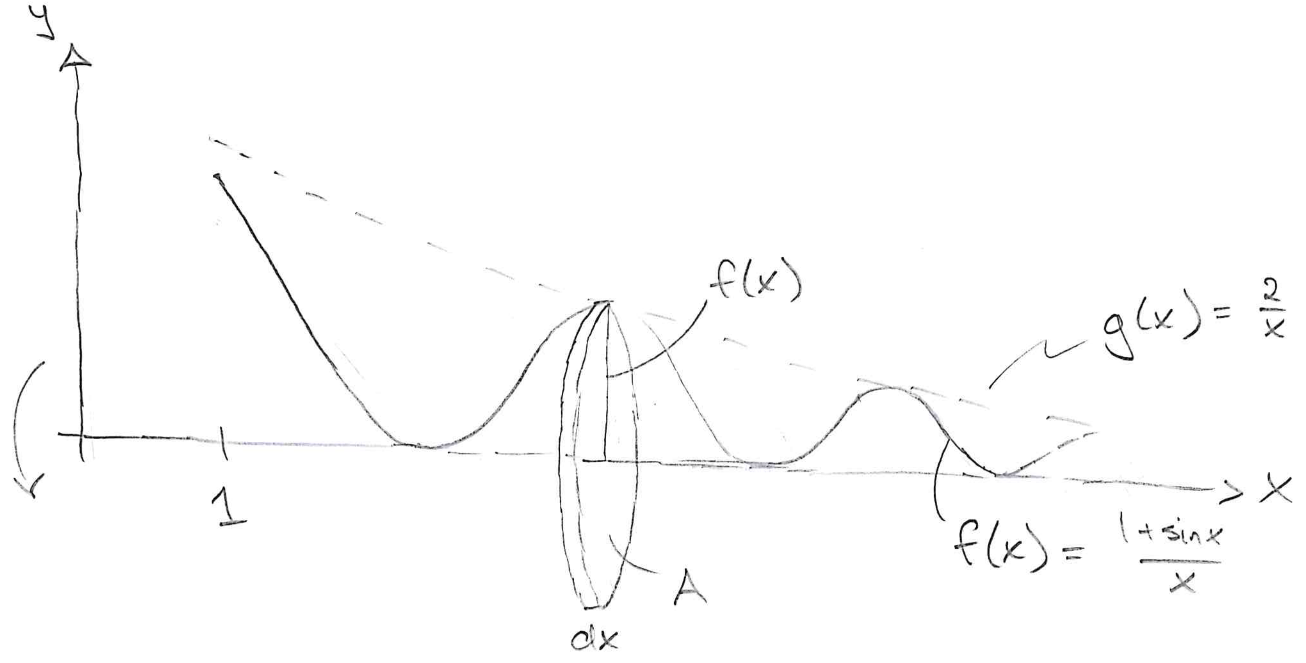
7) Med $f(x) = \frac{1+\sin x}{x}$ och $g(x) = \frac{2}{x}$

har vi $f(x) = \frac{1+\sin x}{x} \leq \frac{1+1}{x} = \frac{2}{x} = g(x)$

och $f(x) = \frac{1+\sin x}{x} \geq \frac{1-1}{x} = 0$

eftersom $|\sin x| \leq 1$

$\Rightarrow \quad 0 \leq f(x) \leq g(x) \quad \forall x \in [1,\infty)$




$\Rightarrow \quad A = \pi(f(x))^2 \Rightarrow dV = \pi(f(x))^2 dx$

Den sökta rotationsvolymen är

$$V = \int_1^\infty dV = \pi \int_1^\infty \left(\frac{1+\sin x}{x}\right)^2 dx$$

Vi har $f(x)^2 = \left(\frac{1+\sin x}{x}\right)^2 \leq \frac{4}{x^2} = g(x)^2 \quad \forall x \in [1,\infty)$

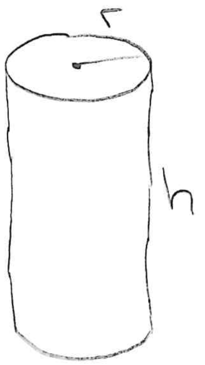
$$\Rightarrow \quad V \leq \pi \int_1^\infty \frac{4}{x^2}\,dx = 4\pi \int_1^\infty \frac{1}{x^2}\,dx$$

$$= 4\pi \left[-\frac{1}{x}\right]_1^\infty = 4\pi \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t$$

$$= 4\pi \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 4\pi \qquad \boxed{B}$$

<u>Svar:</u> $V = \pi \int_1^\infty \left(\frac{1+\sin x}{x}\right)^2 dx \leq 4\pi$

8) Cylindrisk behållare




Area: $A = 2\pi r h + 2\pi r^2$

$\underset{\text{mantel}}{\uparrow}$ $\quad$ $\underset{\text{lock + botten}}{\uparrow}$

$$\Rightarrow h = \frac{1}{2\pi r}\left(A - 2\pi r^2\right)$$

$r, h > 0$

Vi ser att $h > 0 \Rightarrow r < \sqrt{\frac{A}{2\pi}}$

$\Rightarrow$ Vi söker maximum på $r \in \left(0, \sqrt{\frac{A}{2\pi}}\right)$ för volymen

$$V(r) = \pi r^2 h(r) = \frac{\pi r^2}{2\pi r}\left(A - 2\pi r^2\right) = \frac{1}{2}Ar - \pi r^3$$

$$V'(r) = \frac{A}{2} - 3\pi r^2 = 0 \quad \Rightarrow \quad r^2 = \frac{A}{6\pi}$$

$$\Rightarrow \quad r = \underset{(-)}{+}\sqrt{\frac{A}{6\pi}} \quad \text{ty} \quad r > 0$$

$$V''(r) = -6\pi r < 0 \quad \forall r > 0$$

$$\Rightarrow \quad r = \sqrt{\frac{A}{6\pi}} \quad \underline{\text{lokalt max}}$$

Dessutom har vi $\begin{cases} V'(r) > 0 & \forall r < \sqrt{\frac{A}{6\pi}} \\ V'(r) < 0 & \forall r > \sqrt{\frac{A}{6\pi}} \end{cases}$

$\Rightarrow$ Enligt test med förstaderivata är

$r = \sqrt{\frac{A}{6\pi}}$ globalt max på $\left(0, \sqrt{\frac{A}{2\pi}}\right)$

För $r = \sqrt{\frac{A}{6\pi}}$ har vi

$$h\left(\sqrt{\frac{A}{6\pi}}\right) = \frac{A}{2\pi}\sqrt{\frac{6\pi}{A}} - \sqrt{\frac{A}{6\pi}} = \sqrt{\frac{A}{\pi}} \cdot \left(\frac{\sqrt{6}}{2} - \frac{1}{\sqrt{6}}\right)$$

$$= \sqrt{\frac{A}{\pi}}\left(\frac{6-2}{2\sqrt{6}}\right) = \sqrt{\frac{4A}{6\pi}} = \sqrt{\frac{2A}{3\pi}}$$

$$V\left(\sqrt{\frac{A}{6\pi}}\right) = \pi r^2 h(r) = \cancel{\pi} \cdot \frac{A}{6\cancel{\pi}}\sqrt{\frac{2A}{3\pi}} = \frac{\sqrt{2} \cdot A^{3/2}}{2 \cdot 3\sqrt{3} \cdot \sqrt{\pi}} = \frac{A^{3/2}}{3\sqrt{6\pi}}$$

Svar: Den maximala volymen $V = \frac{A^{3/2}}{3\sqrt{6\pi}}$ v.e.

ges för dimensionerna $r = \sqrt{\frac{A}{6\pi}}$ l.e.

och $h = \sqrt{\frac{2A}{3\pi}}$ l.e.