Spectral Isometries

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(joint work with J. Extremera and A. R. Villena)

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Theorem

Let H be a separable Hilbert space. Then for each $k, K, \varepsilon > 0$ *there is* $\delta > 0$ *such that if* $\Phi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ *is a surjective linear map with*

$$\sup_{\|T\|=1} \operatorname{dist}_{\mathrm{H}}\Big(\operatorname{sp}(\Phi(T)), \operatorname{sp}(T)\Big) < \delta,$$

$\kappa(\Phi) > k$*, and* $\|\Phi\| < K$*, then*

$$\operatorname{dist}\Big(\Phi, \operatorname{Aut}(\mathcal{B}(H)) \cup \operatorname{AntiAut}(\mathcal{B}(H))\Big) < \varepsilon.$$

A, B complex Banach algebras

$\Phi \colon A \to B$ linear map

Questions

- Identify the *multiplicative* linear maps in terms of *spectra* or related notions.
- Stability

$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \mathbf{1} \text{ is not invertible}\}$

$r(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}$

Φ isomorphism

$$\Phi(ab) = \Phi(a)\Phi(b) \; a, b \in A$$

Φ anti-isomorphism

$$\Phi(ab) = \Phi(b)\Phi(a) \; a, b \in A$$

$$\Downarrow$$

Φ preserves the spectrum

$$\sigma\big(\Phi(a)\big) = \sigma(a) \; a \in A$$

Φ isomorphism
$\Phi(ab) = \Phi(a)\Phi(b)\ a, b \in A$

Φ anti-isomorphism
$\Phi(ab) = \Phi(b)\Phi(a)\ a, b \in A$

$\Downarrow$

Φ Jordan isomorphism
$\Phi(a^2) = \Phi(a)^2\ a \in A$

$\Downarrow$

Φ preserves the spectrum
$\sigma\big(\Phi(a)\big) = \sigma(a)\ a \in A$

Introducing the problem

Section 1

Kaplansky's problem

I. Kaplansky, 1970

Let A and B be complex unital Banach algebras and let $\Phi\colon A \to B$ be a linear map with the property that

$$\sigma\big(\Phi(a)\big) \subset \sigma(a) \quad (a \in A).$$

Is it true that Φ is a Jordan homomorphism, i.e. $\Phi(a^2) = \Phi(a)^2$, $(a \in A)$?

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Is it true that Φ is a Jordan homomorphism, i.e. $\Phi(a^2) = \Phi(a)^2$, $(a \in A)$?

$$\Phi\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{13} & a_{12} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = 0, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 \neq 0$$

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B. Aupetit, 2000

Let A and B be semisimple complex unital Banach algebras and let $\Phi\colon A \to B$ be a bijective linear map with the property that

$$\sigma\big(\Phi(a)\big) = \sigma(a) \quad (a \in A).$$

Is it true that Φ is a Jordan homomorphism?

Previous results: Kaplansky's problem

1897 Frobenius
1913 Polya
40s Morita
40s Hua
1949 Dieudonné
1959 Marcus, Purves

Theorem

Let $n \in \mathbb{N}$. A linear map $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$\det\big(\Phi(M)\big) = \det(M) \quad (M \in \mathbb{M}_n)$$

if, and only, if $\Phi = W\Psi$ for some automorphism or anti-automorphism Ψ of the Banach algebra $\mathbb{M}_n$ and some invertible matrix $W \in \mathbb{M}_n$ with $\det W = 1$.

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Theorem

Let $n \in \mathbb{N}$. A bijective linear map $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$M \in \mathbb{M}_n,\ \det(M) = 0 \ \Rightarrow\ \det\big(\Phi(M)\big) = 0$$

if, and only if, $\Phi = W\Psi$ for some automorphism or anti-automorphism Ψ of the Banach algebra $\mathbb{M}_n$ and some invertible matrix $W \in \mathbb{M}_n$.

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if, and only if, Φ is either an automorphism or an anti-automorphism of the Banach algebra $\mathbb{M}_n$.

Previous results: Kaplansky's problem

1967–69	Gleason, Kahane, Żelazko
1994	Aupetit, Mouton
2000	Aupetit
2003	Brešar, Fošner, Šmerl

- B commutative.

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- B commutative.
- Φ surjective, $\sigma(\Phi(a)) = \sigma(a)$ and $\mathrm{soc}(B)$ essential in B.

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- A and B von Neumann algebras.

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- A and B von Neumann algebras.
- Φ bijective, $\sigma(\Phi(a)) \subset \sigma(a)$ and $\mathrm{soc}(B)$ essential in B.

Kaplansky's problem for operator algebras

Theorem (A. A. Jafarian and A. R. Sourour, 1986)

Let X and Y be complex Banach spaces and let $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ be a bijective linear map with the property that

$$\sigma\big(\Phi(T)\big) = \sigma(T) \quad (T \in \mathcal{L}(X)).$$

Then one of the following assertions holds:

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1. *the map Φ is an* isomorphism,
2. *the map Φ is an anti-isomorphism,*

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1. *the map Φ is an* isomorphism, *in which case there exists a linear homeomorphism $U\colon X \to Y$ such that $\Phi(T) = UTU^{-1}$ for each $T \in \mathcal{L}(X)$;*
2. *the map Φ is an anti-isomorphism, in which case the spaces X and Y are reflexive and there exists a linear homeomorphism $V\colon X^* \to Y$ such that $\Phi(T) = VT^*V^{-1}$ for each $T \in \mathcal{L}(X)$.*

After Kaplanky's problem: spectral isometries

Φ preserves the spectrum
$\sigma\big(\Phi(a)\big) = \sigma(a) \quad (a \in A)$

$\Longrightarrow$

Φ spectral isometry
$r\big(\Phi(a)\big) = r(a) \quad (a \in A)$

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M. Mathieu - G. J. Schick (2002)

Let A and B be unital C^*-algebras and let $\Phi\colon A \to B$ be a unital surjective spectral isometry. Is it true that Φ is a Jordan homomorphism ?

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Let A and B be semisimple complex unital Banach algebras and let $\Phi\colon A \to B$ be a unital surjective spectral isometry. Is it true that Φ is a Jordan homomorphism ?

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Theorem (M. Brešar and P. Šemrl (1996))

Let X be a complex Banach space and let $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(X)$ be a bijective linear mapping with the property that

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Then there exist $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and either

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1. *an isomorphism $\Psi\colon \mathcal{L}(X) \to \mathcal{L}(X)$ such that $\Phi = \lambda\Psi$,*
2. *an anti-isomorphism $\Psi\colon \mathcal{L}(X) \to \mathcal{L}(X)$ such that $\Phi = \lambda\Psi$,*

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After Kaplanky's problem: spectral isometries

- 1959 Nagasawa
- 2002 Mathieu, Schick
- 2003 Mathieu, Schick
- 2004 Mathieu, Sourour
- 2007 Mathieu, Ruddy
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- 2011 Mathieu, Sourour
- 2010 Costara, Repovš

- A and B commutative.

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- A and B commutative.
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- A and B commutative.
- some classes of C^*-algebras.
- B having a separating family of finite-dimensional irreducible representations.

The problem (1)

Let $\Phi\colon A \to B$ be a linear map between complex unital Banach algebras A and B

$\sigma\big(\Phi(a)\big)$ is near $\sigma(a)$ for each $a \in A$ $\overset{?}{\Longrightarrow}$ Φ is near a (Jordan) multiplicative map

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Let $\Phi\colon A \to B$ be a linear map between complex unital Banach algebras A and B

$\sigma\big(\Phi(a)\big)$ is near $\sigma(a)$ for each $a \in A$	$\overset{?}{\Longrightarrow}$	Φ is near a (Jordan) multiplicative map
$\sup_{\|a\|=1} \operatorname{dist}_H\big(\sigma(\Phi(a)), \sigma(a)\big)$ is small	$\overset{?}{\Longrightarrow}$	$\operatorname{dist}\big(\Phi, \operatorname{Isom}(A,B) \cup \operatorname{AntiIsom}(A,B)\big)$ is small

The problem (2)

Let $\Phi\colon A \to B$ be a linear map between complex unital Banach algebras A and B,

if $r\big(\Phi(a)\big)$ is near $r(a)$ for each $a \in A$, what can we say about Φ?

The problem (2)

Let $\Phi\colon A \to B$ be a linear map between complex unital Banach algebras A and B,

if $r\big(\Phi(a)\big)$ is near $r(a)$ for each $a \in A$, what can we say about Φ?

$\sup_{\|a\|=1} \big|r(\Phi(a)) - r(a)\big|$ is small $\overset{?}{\Longrightarrow}$ $\operatorname{dist}\big(\Phi, \mathbb{T}\,\mathrm{Isom}(A,B) \cup \mathbb{T}\,\mathrm{AntiIsom}(A,B)\big)$ is small

Some answers

Section 2

Finite dimensional case

Theorem (M. Marcus and R. Purves)

Let $n \in \mathbb{N}$. A linear map $\Phi \colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$\sigma\big(\Phi(M)\big) = \sigma(M) \quad (M \in \mathbb{M}_n)$$

if and only if $\Phi \in \mathrm{Aut}(\mathbb{M}_n) \cup \mathrm{AntiAut}(\mathbb{M}_n)$.

Finite dimensional case

Theorem (M. Marcus and R. Purves)

Let $n \in \mathbb{N}$. *A linear map* $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ *satisfies the property*

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if and only if $\Phi \in \mathrm{Aut}(\mathbb{M}_n) \cup \mathrm{AntiAut}(\mathbb{M}_n)$.

Theorem

Let $n \in \mathbb{N}$. *Then for each* $K, \varepsilon > 0$ *there is* $\delta > 0$ *such that if* $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ *is a linear map with*

$$\sup_{\|M\|=1} \mathrm{dist}_{\mathrm{H}}\Big(\sigma\big(\Phi(M)\big), \sigma(M)\Big) < \delta,$$

and $\|\Phi\| < K$, *then* $\mathrm{dist}\Big(\Phi, \mathrm{Aut}(\mathbb{M}_n) \cup \mathrm{AntiAut}(\mathbb{M}_n)\Big) < \varepsilon$.

Finite dimensional case

Theorem (M. Brešar and P. Šemrl "finite-dimensional version")

Let $n \in \mathbb{N}$ *and let* $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ *be a bijective linear mapping with the property*

$$r\big(\Phi(M)\big) = r(M) \quad (M \in \mathbb{M}_n).$$

Then $\Phi \in \mathbb{T}\mathrm{Aut}\big(\mathbb{M}_n\big) \cup \mathbb{T}\mathrm{AntiAut}\big(\mathbb{M}_n\big)$.

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Theorem

Let $n \in \mathbb{N}$. Then for each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ is a bijective linear mapping with

$$\sup_{\|M\|=1} \Big| r\big(\Phi(M)\big) - r(M) \Big| < \delta,$$

and $\|\Phi\| < K$, then $\Phi \in \mathbb{T}\mathrm{Aut}(\mathbb{M}_n) \cup \mathbb{T}\mathrm{AntiAut}(\mathbb{M}_n)$.

Proof

Given $K, \varepsilon, \delta > 0$ put

- $C = \left\{ \Phi \in \mathcal{L}(\mathbb{M}_n) : \|\Phi\| \leq K, \ \operatorname{dist}\big(\Phi, \mathbb{T}\mathrm{Aut}(\mathbb{M}_n) \cup \mathbb{T}\mathrm{AntiAut}(\mathbb{M}_n)\big) \geq \varepsilon \right\}$ is compact.
- $G_\delta = \bigcup_{\|M\|=1} \left\{ \Phi \in \mathcal{L}(\mathbb{M}_n) : \big| r(\Phi(M)) - r(M) \big| > \delta \right\}$ is open.
- $C \subset \bigcup_{\delta > 0} G_\delta \implies C \subset G_\delta$ for some $\delta > 0$ and this proves the theorem.

The role of the boundedness

For each $\varepsilon > 0$, we define $\Phi_\varepsilon : \mathbb{M}_2 \to \mathbb{M}_2$ by

$$\Phi_\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \varepsilon^{-2} b + \varepsilon^{-1} a \\ \varepsilon^2 c & d \end{pmatrix}.$$

Then

$$\sigma\left(\Phi_\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \{\lambda \in \mathbb{C} : (a-\lambda)(d-\lambda) = bc + \varepsilon ac\},$$

while

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{\lambda \in \mathbb{C} : (a-\lambda)(d-\lambda) = bc\}.$$

This implies that

- $\operatorname{dist}_H\big(\sigma(\Phi_\varepsilon(M)), \sigma(M)\big) \le \sqrt{\varepsilon} \quad (\|M\| = 1)$.
- $\operatorname{dist}\big(\Phi_\varepsilon, \operatorname{Aut}(\mathbb{M}_n) \cup \operatorname{AntiAut}(\mathbb{M}_n)\big) \ge \varepsilon^{-1} - 1$.

Infinite dimensional case

An approximate version of Jafarian-Sourour Theorem

Theorem

For each $\varepsilon > 0$ *there is* $\delta > 0$ *such that if* X *and* Y *are Banach spaces and* $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ *is a bijective linear map with*

$$\sup_{\|T\|=1} \mathrm{dist}_{\mathrm{H}}\Big(\sigma\big(\Phi(T)\big), \sigma(T)\Big) < \delta$$

then

$$\mathrm{dist}\Big(\Phi, \mathrm{Iso}\big(\mathcal{L}(X), \mathcal{L}(Y)\big) \cup \mathrm{AntiIso}\big(\mathcal{L}(X), \mathcal{L}(Y)\big)\Big) < \varepsilon.$$

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An approximate version of Jafarian-Sourour Theorem

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For each $K, \varepsilon > 0$ *there is* $\delta > 0$ *such that if* X *and* Y *are Banach spaces and* $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ *is a bijective linear map with*

$$\sup_{\|T\|=1} \mathrm{dist}_{\mathrm{H}}\Big(\sigma\big(\Phi(T)\big), \sigma(T)\Big) < \delta$$

and $\|\Phi^{-1}\|, \|\Phi\| < K$, *then*

$$\mathrm{dist}\Big(\Phi, \mathrm{Iso}\big(\mathcal{L}(X), \mathcal{L}(Y)\big) \cup \mathrm{AntiIso}\big(\mathcal{L}(X), \mathcal{L}(Y)\big)\Big) < \varepsilon.$$

Infinite dimensional case

Spectral isometries in operator algebras

Theorem (M. Brešar and P. Šemrl, 1996)

Let X be a complex Banach space and let $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(X)$ be a bijective linear map with the property that

$$r\big(\Phi(T)\big) = r(T) \quad (T \in \mathcal{L}(X)).$$

Then there exist $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and either

1. *a linear homeomorphism $U\colon X \to X$ such that $\Phi(T) = \lambda U T U^{-1}$ for each $T \in \mathcal{L}(X)$; or*
2. *a linear homeomorphism $V\colon X^* \to X$ such that $\Phi(T) = \lambda V T^* V^{-1}$ for each $T \in \mathcal{L}(X)$, in this case the space X is reflexive.*

Main ingredient

Preservation of nilpotency

Operators preserving nilpontents

Based on a characterization of rank one nilpotents in terms of spectral properties:

P. Šemrl, 1995

Let $N(X)$ be the set of all nilpotent operators in $\mathcal{L}(X)$, and let $N \in N(X)$, $N \neq 0$. Then N has rank one if and only if for every $A \in N(X)$ satisfying $A + N \notin N(X)$ we have $A + \lambda N \notin N(X)$ for every nonzero $\lambda \in \mathbb{C}$.

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Theorem (P. Šemrl, 1995)

Let X be a complex Banach space and let $\Phi\colon \mathcal{L}_0(X) \to \mathcal{L}_0(X)$ be a surjective linear map that preserve nilpotent operators in both directions.
Then there exist $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and either

1. *a linear homeomorphism $U\colon X \to X$ such that $\Phi(T) = \lambda UTU^{-1}$ for each $T \in \mathcal{L}_0(X)$; or*
2. *a linear homeomorphism $V\colon X^* \to X$ such that $\Phi(T) = \lambda VT^*V^{-1}$ for each $T \in \mathcal{L}_0(X)$, in this case the space X is reflexive.*

Operators preserving ...

How to construct the linear homeomorphism?

Study the behaviour on rank-1 elements:

$$L_x = \{x \otimes h : h \in X^*\}$$

$$R_f = \{u \otimes f : u \in X\}$$

$$\Phi(L_x) = L_y \text{ o } \Phi(L_x) = R_f$$

Infinite dimensional case

An approximate version of Brešar-Šemrl Theorem

Theorem

For each $K, \varepsilon > 0$ *there is* $\delta > 0$ *such that if* X *and* Y *are Banach spaces and* $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ *is a bijective linear map with*

$$\sup_{\|T\|=1} \Big| r\big(\Phi(T)\big) - r(T) \Big| < \delta$$

and $\|\Phi^{-1}\|, \|\Phi\| < K$, *then*

$$\operatorname{dist}\Big(\Phi, \mathbb{T}\mathrm{Iso}\big(\mathcal{L}(X), \mathcal{L}(Y)\big) \cup \mathbb{T}\mathrm{AntiIso}\big(\mathcal{L}(X), \mathcal{L}(Y)\big)\Big) < \varepsilon.$$

Proof's sketch

The main tool: *ultraproducts*

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

1. The ultraproduct of a sequence (X_n) of Banach spaces is the Banach space $(X_n)^{\mathcal{U}} = \ell^\infty(\mathbb{N}, X_n)/\left\{x \in \ell^\infty(\mathbb{N}, X_n) : \lim_{\mathcal{U}}\|x_n\| = 0\right\}$.
2. The ultraproduct of a bounded sequence (T_n) of operators $T_n \in \mathcal{L}(X_n, Y_n)$ is the operator $(T_n)^{\mathcal{U}} \in \mathcal{L}\left((X_n)^{\mathcal{U}}, (Y_n)^{\mathcal{U}}\right)$ defined through $(x_n) \mapsto (T_n(x_n))$.

Proof's sketch

Assume towards a contradiction that the theorem is false.

Then there exist $\Phi_n \colon \mathcal{L}(X_n) \to \mathcal{L}(Y_n)$ $(n \in \mathbb{N})$ such that

1. $\sup_{\|T\|=1} \Big| r\big(\Phi(T)\big) - r(T) \Big| \to 0,$
2. $\|\Phi_n^{-1}\|,\ \|\Phi_n\| < K,$
3. $\inf_{n\in\mathbb{N}} \operatorname{dist}\Big(\Phi_n, \mathbb{T}\mathrm{Iso}\big(\mathcal{L}(X_n), \mathcal{L}(Y_n)\big) \cup \mathbb{T}\mathrm{AntiIso}\big(\mathcal{L}(X_n), \mathcal{L}(Y_n)\big)\Big) > 0.$

Proof's sketch

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

Let us consider the mapping

$$\Phi = (\Phi_n)^{\mathcal{U}} \colon \underbrace{\big(\mathcal{L}(X_n)\big)^{\mathcal{U}}}_{\subset \mathcal{L}\left((X_n)^{\mathcal{U}}\right)} \longrightarrow \underbrace{\big(\mathcal{L}(Y_n)\big)^{\mathcal{U}}}_{\subset \mathcal{L}\left((Y_n)^{\mathcal{U}}\right)}, \qquad \Phi(\mathbf{T}) = \big(\Phi_n(T_n)\big)^{\mathcal{U}}.$$

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Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

Let us consider the mapping

$$\Phi = (\Phi_n)^{\mathcal{U}} \colon \underbrace{\big(\mathcal{L}(X_n)\big)^{\mathcal{U}}}_{\subset \mathcal{L}\left((X_n)^{\mathcal{U}}\right)} \longrightarrow \underbrace{\big(\mathcal{L}(Y_n)\big)^{\mathcal{U}}}_{\subset \mathcal{L}\left((Y_n)^{\mathcal{U}}\right)}, \qquad \Phi(\mathbf{T}) = \big(\Phi_n(T_n)\big)^{\mathcal{U}}.$$

1. Φ is bijective.

Proof's sketch

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1. Φ is bijective.
2. $r\big(\Phi(\mathbf{T})\big) = r(\mathbf{T}) \ \left(\mathbf{T} \in \big(\mathcal{L}(X_n)\big)^{\mathcal{U}}\right)$.

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1. Φ is bijective.
2. $r\big(\Phi(\mathbf{T})\big) = r(\mathbf{T}) \ \left(\mathbf{T} \in \big(\mathcal{L}(X_n)\big)^{\mathcal{U}}\right)$.
3. Both $\big(\mathcal{L}(X_n)\big)^{\mathcal{U}}$ and $\big(\mathcal{L}(Y_n)\big)^{\mathcal{U}}$ are irreducible unital closed subalgebras of the Banach algebras $\mathcal{L}\left((X_n)^{\mathcal{U}}\right)$ and $\mathcal{L}\left((Y_n)^{\mathcal{U}}\right)$, respectively, which have finite-rank operators.

Proof's sketch

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

Let us consider the mapping

Theorem

Let A and B be unital primitive Banach algebras and let A and B have minimal idempotents. Assume that $\Phi \colon A \to B$ is a surjective spectral isometry. Then there exist $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and eihter an isomorphism or an anti-isomorphism $\Psi \colon A \to B$ such that $\Phi = \lambda \Psi$.

3. Both $(\mathcal{L}(X_n))^{\mathcal{U}}$ and $(\mathcal{L}(Y_n))^{\mathcal{U}}$ are irreducible unital closed subalgebras of the Banach algebras $\mathcal{L}\left((X_n)^{\mathcal{U}}\right)$ and $\mathcal{L}\left((Y_n)^{\mathcal{U}}\right)$, respectively, which have finite-rank operators.

Ψ is an isomorphism

Then there exists a linear homeomorphism $\mathbf{U}\colon (X_n)^{\mathcal{U}} \to (Y_n)^{\mathcal{U}}$ such that $\Phi(\mathbf{T}) = \lambda \mathbf{U}\mathbf{T}\mathbf{U}^{-1}$. Then

- $\mathbf{U} = (U_n)^{\mathcal{U}}$ for some $U_n \in \mathcal{L}(X_n, Y_n)$ for each $n \in \mathbb{N}$, and
- $\lim_{\mathcal{U}} \sup_{\|T\|=1} \left\| \Phi_n(T) - \lambda U_n T U_n^{-1} \right\| = 0.$

Hence $\lim_{\mathcal{U}} \operatorname{dist}\Big(\Phi_n, \mathbb{T}\mathrm{Iso}\big(\mathcal{L}(X_n), \mathcal{L}(Y_n)\big)\Big) = 0$, a contradiction !

Ψ is an anti-isomorphism

Then there exists a linear homeomorphism $\mathbf{V}\colon (X_n^*)^{\mathcal{U}} \to (Y_n)^{\mathcal{U}}$ such that $\Phi(\mathbf{T}) = \lambda \mathbf{V}\mathbf{T}^*\mathbf{V}^{-1}$. Then

- $\mathbf{V} = (V_n)^{\mathcal{U}}$ for some $V_n \in \mathcal{L}(X_n^*, Y_n)$ for each $n \in \mathbb{N}$, and
- $\lim_{\mathcal{U}} \sup_{\|T\|=1} \left\| \Phi_n(T) - \lambda V_n T^* V_n^{-1} \right\| = 0.$

Hence $\lim_{\mathcal{U}} \operatorname{dist}\Big(\Phi_n, \mathbb{T}\mathrm{AntiIso}\big(\mathcal{L}(X_n), \mathcal{L}(Y_n)\big)\Big) = 0$, a contradiction !

One step further: the pseudospectrum

Section 3

Recap

$$\sup_{\|a\|=1} \operatorname{dist}_H\big(\sigma(\Phi(a)), \sigma(a)\big) \quad \overset{\checkmark}{\longrightarrow} \quad \operatorname{dist}\big(\Phi, \operatorname{JIsom}(A,B)\big)$$

$$\sup_{\|a\|=1} \big|r(\Phi(a)) - r(a)\big| \quad \overset{\checkmark}{\longrightarrow} \quad \operatorname{dist}\big(\Phi, \mathbb{T}\operatorname{JIsom}(A,B)\big)$$

Recap

$$\sup_{\|a\|=1} \operatorname{dist}_{\mathrm{H}}\big(\sigma(\Phi(a)), \sigma(a)\big) \quad \overset{\checkmark}{\underset{?}{\rightleftarrows}} \quad \operatorname{dist}\big(\Phi, \mathrm{JIsom}(A,B)\big)$$

$$\sup_{\|a\|=1} \big|r(\Phi(a)) - r(a)\big| \quad \overset{\checkmark}{\underset{?}{\rightleftarrows}} \quad \operatorname{dist}\big(\Phi, \mathbb{T}\mathrm{JIsom}(A,B)\big)$$

Recap

Kakutani

There exists a sequence $(T_n) \in \mathcal{L}(\ell^2)$ such that

- $\|T_n - T_0\| \to 0$ for some $T_0 \in \mathcal{L}(\ell^2)$, with $\|T_0\| = 1$,
- $r(T_0) = r_0 \neq 0$, while
- $r(T_n) = 0$ for each $n \in \mathbb{N}$.

Recap

Kakutani

There exists a sequence $(T_n) \in \mathcal{L}(\ell^2)$ such that

- $\|T_n - T_0\| \to 0$ for some $T_0 \in \mathcal{L}(\ell^2)$, with $\|T_0\| = 1$,
- $r(T_0) = r_0 \neq 0$, while
- $r(T_n) = 0$ for each $n \in \mathbb{N}$.

Pick $\varphi \in \left(\mathcal{L}(\ell^2)\right)^*$, $\varphi(T_0) = \|\varphi\| = 1$ and define $\Phi_n \colon \mathcal{L}(\ell^2) \to \mathcal{L}(\ell^2)$ by

$$\Phi_n(T) = T + \varphi(T)(T_n - T_0) \quad (T \in \mathcal{L}(\ell^2),\ n \in \mathbb{N}).$$

Then $\|\Phi_n - \mathbf{1}\| = \|T_n - T_0\| \to 0$, and

- $\lim_{n\to\infty} \operatorname{dist}(\Phi_n, \mathrm{JIsom}) = 0$,
- $r_0 \leq \sup\left\{|r(\Phi_n(T)) - r(T)| \colon T \in B_{\mathcal{L}(\ell^2)}\right\}$ for each $n \in \mathbb{N}$.

Recap

$$\sup_{\|a\|=1} \operatorname{dist}_H\big(\sigma(\Phi(a)), \sigma(a)\big)$$

→ ✓ ← ✖

$$\operatorname{dist}\big(\Phi, \operatorname{JIsom}(A, B)\big)$$

$$\sup_{\|a\|=1} \big|r(\Phi(a)) - r(a)\big|$$

→ ✓ ← ✖

$$\operatorname{dist}\big(\Phi, \mathbb{T}\operatorname{JIsom}(A, B)\big)$$

The pseudospectrum
and the pseudospectral radius

Definition

Let X be a Banach space and let $T \in \mathcal{L}(X)$ and $\varepsilon > 0$

- The *ε-pseudospectrum* of T is defined to be the set

$$\sigma_\varepsilon(T) = \left\{\lambda \in \mathbb{C} : \left\|(T - \lambda I_X)^{-1}\right\| > \varepsilon^{-1}\right\}$$

(we adopt the convention that $\|(T - \lambda I_X)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$).

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- The *ε-pseudospectral radius* of T is defined as

$$r_\varepsilon(T) = \sup\{|\lambda| : \lambda \in \sigma_\varepsilon(T)\}.$$

The pseudospectrum

$$A = \begin{pmatrix} 1+i & 0 & i \\ -i & 0.2 & 0 \\ 0.7i & 0.2 & 0.5 \end{pmatrix}$$

Albrecht Böttcher and Marko Lindner (2008) Pseudospectrum. Scholarpedia, 3(3):2680.

Figure : Pseudospectra of A

Preservation of the pseudospectrum

How to measure the pseudospectral behaviour?

Let X and Y be Banach spaces and let $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ be a linear map.

- Preserving the pseudospectrum

$$\mathcal{P}'_\sigma(\Phi) = \inf\{\varepsilon > 0 : \sigma(\Phi(T)) \subset \sigma_\varepsilon(T) \text{ and } \sigma(T) \subset \sigma_\varepsilon(\Phi(T)) \quad (T \in B_{\mathcal{L}(X)})\}$$

- Preserving the pseudospectral radius

$$\mathcal{P}'_r(\Phi) = \inf\{\varepsilon > 0 : r(\Phi(T)) \leq r_\varepsilon(T) \text{ and } r(T) \leq r_\varepsilon(\Phi(T)) \quad (T \in B_{\mathcal{L}(X)})\}.$$

Remark

- Φ preserves the spectrum $\iff$ $P'_\sigma(\Phi) = 0$.
- $\mathcal{P}'_r < \infty$ and Φ surjective $\implies$ Φ is continuous.

State of the art

$\sup_{\|a\|=1} \mathrm{dist}_H(\sigma(\Phi(a)), \sigma(a))$	$\geq$	$\mathcal{P}'_\sigma(\Phi)$		$\mathrm{dist}(\Phi, \mathrm{JIsom}(A,B))$
$\mathrel{\rotatebox{90}{$\geq$}}$		$\mathrel{\rotatebox{90}{$\geq$}}$		
$\sup_{\|a\|=1} \|r(\Phi(a)) - r(a)\|$	$\geq$	$\mathcal{P}'_r(\Phi)$		$\mathrm{dist}(\Phi, \mathbb{T}\mathrm{JIsom}(A,B))$

State of the art

$$\sup_{\|a\|=1} \operatorname{dist}_{\mathrm{H}}\big(\sigma(\Phi(a)), \sigma(a)\big)$$

$$\mathcal{P}'_{\sigma}(\Phi)$$

$$\operatorname{dist}\big(\Phi, \mathrm{JIsom}(A, B)\big)$$

$$\sup_{\|a\|=1} \big|r(\Phi(a)) - r(a)\big|$$

$$\mathcal{P}'_{r}(\Phi)$$

$$\operatorname{dist}\big(\Phi, \mathbb{T}\mathrm{JIsom}(A, B)\big)$$

State of the art

$$\sup_{\|a\|=1} \operatorname{dist}_{\mathrm{H}}\big(\sigma(\Phi(a)), \sigma(a)\big) \;\;\Longrightarrow\;\; \mathcal{P}'_{\sigma}(\Phi) \;\;\Longleftrightarrow\;\; \operatorname{dist}\big(\Phi, \operatorname{JIsom}(A,B)\big)$$

$$\sup_{\|a\|=1} \operatorname{dist}_{\mathrm{H}}\big(\sigma(\Phi(a)), \sigma(a)\big) \;\;\not\Longleftarrow\;\; \mathcal{P}'_{\sigma}(\Phi)$$

$$\sup_{\|a\|=1} \big|r(\Phi(a)) - r(a)\big| \;\;\Longrightarrow\;\; \mathcal{P}'_{r}(\Phi) \;\;\Longleftrightarrow\;\; \operatorname{dist}\big(\Phi, \mathbb{T}\operatorname{JIsom}(A,B)\big)$$

$$\sup_{\|a\|=1} \big|r(\Phi(a)) - r(a)\big| \;\;\not\Longleftarrow\;\; \mathcal{P}'_{r}(\Phi)$$

State of the art

$$\sup_{\|a\|=1} \mathrm{dist}_{\mathrm{H}}\big(\sigma(\Phi(a)), \sigma(a)\big) \;\underset{\not\Leftarrow}{\Longrightarrow}\; \mathcal{P}'_{\sigma}(\Phi) \;\Longleftrightarrow\; \mathrm{dist}\big(\Phi, \mathrm{JIsom}(A, B)\big)$$

$$\sup_{\|a\|=1} \big|r(\Phi(a)) - r(a)\big| \;\underset{\not\Leftarrow}{\Longrightarrow}\; \mathcal{P}'_{r}(\Phi) \;\Longleftrightarrow\; \mathrm{dist}\big(\Phi, \mathbb{T}\mathrm{JIsom}(A, B)\big)$$

Thank you!