Spectral Properties of Toeplitz Operators on the Unit Sphere and on the Unit Ball

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BANACH ALGEBRAS AND APPLICATIONS
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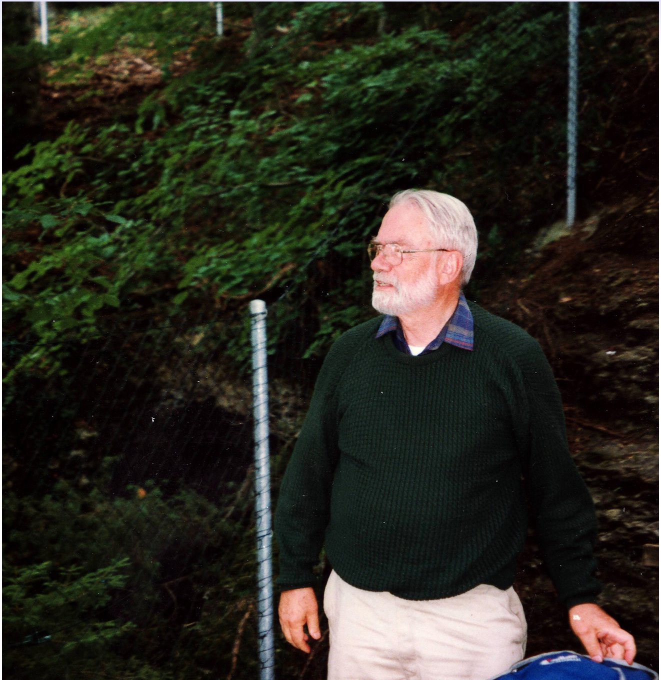

Weighted Bergman spaces

$\mathbb{B}_N := \{z \in \mathbb{C}^N; |z| < 1\}$

For $\lambda > -1$ consider the closed subspace

$A^2_\lambda(\mathbb{B}_N) := L^2(\mathbb{B}_N, dv^N_\lambda) \cap \mathcal{O}(\mathbb{B}_N)$ of $L^2(\mathbb{B}_N, dv^N_\lambda)$ with respect to

$$dv^N_\lambda(z) = \frac{\Gamma(N+\lambda+1)}{\pi^N \Gamma(\lambda+1)}(1-|z|^2)^\lambda dV_N(z),$$

where V_N is the Lebesgue measure in $\mathbb{C}^N$.

An orthonormal basis $(e^{(\lambda,N)}_\alpha)_{\alpha \in \mathbb{Z}^N_+}$ for $A^2_\lambda(\mathbb{B}_N)$ is given by

$$e^{(\lambda,N)}_\alpha(z) := \left(\frac{\Gamma(N+\lambda+|\alpha|+1)}{\alpha!\Gamma(N+\lambda+1)}\right)^{1/2} z^\alpha. \qquad (1)$$

The Toeplitz operator with symbol $\varphi \in L^\infty(\mathbb{B}_N)$ is defined by

$$T_\varphi f := P_\lambda(\varphi f), \qquad (f \in A^2_\lambda(\mathbb{B}_N)),$$

P_λ the orthogonal projection from $L^2(\mathbb{B}_N, v^N_\lambda)$ onto $A^2_\lambda(\mathbb{B}_N)$.

The Hardy space

$d\sigma_N$ normalised surface measure on $\mathbb{S}_N = \{z \in \mathbb{C}^N; |z| = 1\}$.
$H^2(\mathbb{S}_N)$ is the closure in $L^2(\mathbb{S}_N, d\sigma_N)$ of $\mathbb{C}[\zeta_1, \dots, \zeta_N]$.
The functions

$$e_\alpha^{(-1,N)} : \zeta \mapsto \left(\frac{\Gamma(N + |\alpha|)}{\alpha!\Gamma(N)}\right)^{1/2} \zeta^\alpha = \left(\frac{(N-1+|\alpha|)!}{\alpha!(N-1)!}\right)^{1/2} \zeta^\alpha \quad (2)$$

form a orthonormal basis $(e_\alpha^{(-1,N)})_{\alpha \in \mathbb{Z}_+^N}$ in $H^2(\mathbb{S}_N)$.

For $\varphi \in L^\infty(\mathbb{S}_N)$, the Toeplitz operator $T_\varphi \in \mathcal{L}(H^2(\mathbb{S}_N))$ is:

$$T_\varphi f := P_{-1}(\varphi f), \qquad (f \in H^2(\mathbb{B}_N)),$$

P_{-1} is the orthogonal projection from $L^2(\mathbb{S}_N, d\sigma_N)$ onto $H^2(\mathbb{S}_N)$.

Spectral inclusion estimates

If $\mathcal{A} \subseteq L^\infty(\mathbb{B}_N)$ (resp. $\mathcal{A} \subseteq L^\infty(\mathbb{S}_N)$), then $\mathcal{T}(\mathcal{A})$ denotes the closed C^*-algebra generated by $\{T_\varphi\,;\,\varphi \in \mathcal{A}\}$.
Let $\mathfrak{C}$ be the closed two-sided ideal in $\mathcal{T}(L^\infty(\mathbb{S}_N))$ generated by the set of all semi-commutators $T_{\varphi\psi} - T_\varphi T_\psi$, $\varphi, \psi \in L^\infty(\mathbb{S}_N)$.
Davie and Jewell (1977):

1. The continuous, unital, $*$-linear mapping $\eta : \varphi \mapsto T_\varphi$ induces an isometric $*$-isomorphism $\eta_{\mathfrak{C}} : L^\infty(\mathbb{S}_N) \to \mathcal{T}(L^\infty(\mathbb{S}_N))/\mathfrak{C}$.
2. For all $\varphi \in L^\infty(\mathbb{S}_N))$ we have the inclusions

$$\mathfrak{R}(\varphi) \subseteq \mathrm{sp}_e(T_\varphi) \subseteq \mathrm{sp}(T_\varphi) \subseteq \mathrm{co}(\mathfrak{R}(\varphi)). \tag{3}$$

Joint spectra

Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}^n$. We define

$$\mathrm{sp}_L(a, \mathcal{A}) := \{z \in \mathbb{C}^n;\ \sum_{j=1}^n \mathcal{A}(z_j - a_j) \neq \mathcal{A}\},$$

$$\mathrm{sp}_R(a, \mathcal{A}) := \{z \in \mathbb{C}^n;\ \sum_{j=1}^n (z_j - a_j)\mathcal{A} \neq \mathcal{A}\},$$

$$\mathrm{sp}_{2-\mathrm{sid}}(a, \mathcal{A}) := \{z \in \mathbb{C}^n;\ \sum_{j=1}^n \mathcal{A}(z_j - a_j)\mathcal{A} \neq \mathcal{A}\}, \text{ and}$$

$$\mathrm{sp}_H(a, \mathcal{A}) := \mathrm{sp}_L(a, \mathcal{A}) \cup \mathrm{sp}_R(a, \mathcal{A}).$$

These sets are compact and satisfy

$$\mathrm{sp}_{2-\mathrm{sid}}(a, \mathcal{A}) \subseteq \mathrm{sp}_L(a, \mathcal{A}) \cap \mathrm{sp}_R(a, \mathcal{A}) \subseteq \mathrm{sp}_H(a, \mathcal{A}) \subseteq \prod_{j=1}^n \mathrm{sp}(a_j, \mathcal{A}).$$

Coburn (1973):

$$\mathcal{K}(H^2(\mathbb{S}_N)) \subset \mathcal{T}(C(\mathbb{S}_N)), \quad \mathcal{K}(A^2_\lambda(\mathbb{B}_N)) \subset \mathcal{T}(C(\mathbb{B}_N)).$$

The quotient algebras $\mathcal{T}_Q(C(\overline{\mathbb{B}_N})) := \mathcal{T}(C(\overline{\mathbb{B}_N}))/\mathcal{K}(A^2_\lambda(\mathbb{B}_N))$ and $\mathcal{T}_Q(C(\mathbb{S}_N)) := \mathcal{T}(C(\mathbb{S}_N))/\mathcal{K}(H^2(\mathbb{B}_N))$ are isometrically $*$-isomorphic to $C(\mathbb{S}_N)$ and contained in the center of

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Proposition

For all $\varphi = (\varphi_1, \ldots, \varphi_n) \in L^\infty(\mathbb{S}_N)^n$ we have

$$\mathfrak{R}(\varphi) \subseteq \mathrm{sp}_{2-\mathrm{sid}}([T_\varphi], \mathcal{T}_\mathcal{Q}(L^\infty(\mathbb{S}_N))) \subseteq \mathrm{sp}_H([T_\varphi], \mathcal{T}_\mathcal{Q}(L^\infty(\mathbb{S}_N))) \subseteq$$
$$\subseteq \mathrm{sp}_H(T_\varphi, \mathcal{L}(H^2(\mathbb{S}_N))) \subseteq \mathrm{co}(\mathfrak{R}(\varphi)). \tag{4}$$

The Local Principle

For $\lambda > -1$: $H := A^2_\lambda(\mathbb{B}_N)$, $L^\infty := L^\infty(\mathbb{B}_N)$, $C := C(\overline{\mathbb{B}_N})$.
For $\lambda = -1$: $H := H^2(\mathbb{S}_N)$, $L^\infty := L^\infty(\mathbb{S}_N)$, $C := C(\mathbb{S}_N)$.

For $\omega \in \mathbb{S}_N$ let $\mathcal{I}_\mathcal{Q}(\omega)$ be the closed 2-sided ideal in $\mathcal{T}_\mathcal{Q}(L^\infty)$ generated by

$$\{T_\varphi + \mathcal{K}(H); \varphi \in C, \varphi(\omega) = 0\}$$

The *local algebra over* ω is defined as

$$\mathcal{T}(\omega) := \mathcal{T}_\mathcal{Q}(L^\infty)/\mathcal{I}_\mathcal{Q}(\omega).$$

If $\mathcal{I}(\omega)$ denotes the closed 2-sided ideal in $\mathcal{T}(L^\infty)$ generated by

$$\{T_\varphi; \varphi \in C, \varphi(\omega) = 0\},$$

then we have a natural isomorphisms

$$\mathcal{T}(\omega) \cong \mathcal{T}(L^\infty)/\mathcal{I}(\omega).$$

$$\Gamma_\omega : \mathcal{T}_\mathcal{Q}(L^\infty) \to \mathcal{T}(\omega)$$

denotes the canonical unital $*$-epimorphisms onto the local algebra. For $T \in \mathcal{T}(L^\infty)$ we write

$$\mathrm{sp}_\omega(T) := \mathrm{sp}(\Gamma_\omega([T]), \mathcal{T}(\omega))$$

for the spectrum in the local algebra. The unit element in the local algebras is $1_\omega := \Gamma_\omega([I])$.
$\omega \in \mathbb{S}_N$ will be called a *point of continuity* for a symbol $\varphi \in L^\infty)$ if there exists some $y_\omega \in \mathbb{C}$ such that

$$\inf_{\delta>0} \|\varphi|_{U_\delta(\omega)} - y_\omega\|_\infty = 0.$$

We then put $\varphi(\omega) := y_\omega$.

Theorem

$$\Gamma : \mathcal{T}_{\mathcal{Q}}(L^\infty) \to \bigoplus_{\omega \in \mathbb{S}_N} \mathcal{T}(\omega), \quad [T] \mapsto (\Gamma_\omega([T]))_{\omega \in \mathbb{S}_N},$$

is an isometric $*$-monomorphism. For all $T \in \mathcal{T}(L^\infty)$ we have:

1. *The function $\omega \mapsto \|\Gamma_\omega([T])\|$ is upper semi-continuous.*
2. *$\{(\omega, \mu) \in \mathbb{S}_N \times \mathbb{C}\,;\, (\mu 1_\omega - \Gamma_\omega([T]))^{-1}$ exists in $\mathcal{T}(\omega)\}$ is open in $\mathbb{S}_N \times \mathbb{C}$.*
3. *T is a Fredholm operator if and only if $\Gamma_\omega([T]))$ is invertible in $\mathcal{T}(\omega))$ for all $\omega \in \mathbb{S}_N$. In particular,*

$$\mathrm{sp}_e(T) = \bigcup_{\omega \in \mathbb{S}_N} \mathrm{sp}_\omega(T)\Big).$$

4. *If $\omega \in \mathbb{S}_N$ is a point of continuity for $\varphi \in L^\infty$ then $\Gamma_\omega([T_\varphi]) = \varphi(\omega)1_\omega$. Hence, $\mathrm{sp}_\omega(T_\varphi) = \{\varphi(\omega)\}$.*

Local spectral inclusion estimates

For $\varphi \in L^\infty(\mathbb{S}_N)^n$ and $\omega \in \mathbb{S}_N$ we define the *local essential range* of φ at ω by

$$\mathfrak{R}_\omega(\varphi) := \bigcap_{\delta>0} \mathfrak{R}(\varphi|_{U_\delta(\omega)}).$$

$\mathfrak{R}_\omega(\varphi)$ is a non-empty compact set satisfying

$$\max_{w\in\mathfrak{R}_\omega(\varphi)} |w| = \inf_{\varepsilon>0} \max_{w\in\mathfrak{R}(\varphi_{|U_\varepsilon(\omega)})} |w| = \lim_{\varepsilon\to 0} \max_{w\in\mathfrak{R}(\varphi_{|U_\varepsilon(\omega)})} |w|.$$

$\mathcal{J}_\omega := \{\psi \in L^\infty(\mathbb{S}_N)\,;\, \mathfrak{R}_\omega(\psi) = \{0\}\}$ is a closed ideal in $L^\infty(\mathbb{S}_N)$ and $L^\infty_\omega(\mathbb{S}_N) := L^\infty(\mathbb{S}_N)/\mathcal{J}_\omega$ is a commutative C^*-algebra with

$$\|[\psi]_\omega\|_{\infty,\omega} := \max_{w\in\mathfrak{R}_\omega(\psi)} |w| \quad ([\psi]_\omega = \psi + \mathcal{J}_\omega \in L^\infty_\omega(\mathbb{S}_N)).$$

For all $\varphi \in L^\infty(\mathbb{S}_N)^n$ we have $\mathrm{sp}([\varphi]_\omega, L^\infty_\omega(\mathbb{S}_N)) = \mathfrak{R}_\omega(\varphi)$.

Let now $\mathfrak{C}_\omega$ be the closed two-sided ideal in $\mathcal{T}(L^\infty(\mathbb{S}_N))$ generated by $\mathfrak{C} + \mathcal{I}(\omega, \mathbb{S}_N)$.

Theorem

The continuous, unital, $*$-linear mapping $\eta : \varphi \mapsto T_\varphi$ induces an isometric $*$-isomorphism $\eta_{\mathfrak{C},\omega} : L^\infty_\omega(\mathbb{S}_N) \to \mathcal{T}(L^\infty(\mathbb{S}_N))/\mathfrak{C}_\omega$.

Theorem

For all $\varphi = (\varphi_1, \ldots, \varphi_n) \in L^\infty(\mathbb{S}_N)^n$, $\omega \in \mathbb{S}_N$, we have with $\Gamma_\omega([T_\varphi]) := (\Gamma_\omega([T_{\varphi_1}]) \ldots, \Gamma_\omega([T_{\varphi_n}]))$,

$$\begin{aligned}\mathfrak{R}_\omega(\varphi) \subseteq \mathrm{sp}_{2-\mathrm{sid}}(\Gamma_\omega([T_\varphi]), \mathcal{T}(\omega)) &\subseteq \mathrm{sp}_H(\Gamma_\omega([T_\varphi]), \mathcal{T}(\omega)) \\ &\subseteq \mathrm{co}(\mathfrak{R}_\omega(\varphi)).\end{aligned} \tag{5}$$

Symbols which are independent with respect to some variables

Let $\lambda \geq -1$. As before we put:
For $\lambda > -1$: $H := A^2_\lambda(\mathbb{B}_N)$, $L^\infty := L^\infty(\mathbb{B}_N)$, $C := C(\overline{\mathbb{B}_N})$.
For $\lambda = -1$: $H := H^2(\mathbb{S}_N)$, $L^\infty := L^\infty(\mathbb{S}_N)$, $C := C(\mathbb{S}_N)$.

Let N, k be integers with $1 \leq k < N$. We consider symbols $\varphi \in L^\infty$ that are independent of the first k variables, i.e. with some $a_\varphi \in L^\infty(\mathbb{B}_{N-k})$, we have

$$\varphi(z) = a_\varphi(z'') \tag{6}$$

for all $z = (z', z'') \in \mathbb{B}_N$ resp. $z = (z', z'') \in \mathbb{S}_N$ with $z' \in \mathbb{C}^k$, $z'' \in \mathbb{C}^{N-k}$.

For $\alpha' \in \mathbb{Z}^k_+$ we define $H(\alpha') := \overline{\text{LH}}\{e^{(\lambda,N)}_{(\alpha',\alpha'')} \,;\, \alpha'' \in \mathbb{Z}^{N-k}_+\}$. Thus,

$$H = \bigoplus_{\alpha' \in \mathbb{Z}^k_+} H(\alpha').$$

Theorem (Quiroga-Barrasco and Vasilevski(2007))

If G is a maximal commutative subgroup of $Aut(\mathbb{B}_N)$, then the C^*-subalgebra $\mathfrak{A}_G$ in $\mathcal{L}(A^2_\lambda(\mathbb{B}_N))$ generated by all Toeplitz operators with G-invariant symbols is commutative.

Using

$$A^2_\alpha(\mathbb{B}_N) = \mathcal{O}(\mathbb{B}_N) \cap L^2(\mathbb{B}_N, d\nu_\alpha) = L^2(\mathbb{B}_N, d\nu_\alpha) \cap \ker \overline{\partial}$$

and appropriate Fourier- and Mellin-transforms they obtained for each such maximal commutative subgroup G an explicit spectral representation of $\mathfrak{A}_G$.

A corresponding result is true in the Hardy space situation (Akkar 2012).

Theorem

If $\varphi \in L^\infty$ is independent of the first k variables such that there is a function $a_\varphi \in L^\infty(\mathbb{B}_{N-k})$ satisfying (6). *Then:*

1. *Each $H(\alpha')$, $\alpha' \in \mathbb{Z}_+^k$, is a reducing subspace for T_φ.*
2. *For all $\alpha' \in \mathbb{Z}_+^k$ the restriction $T_\varphi|_{H(\alpha')}$ is unitarily equivalent to the Toeplitz operator T_{a_φ} on the weighted Bergman space $A^2_{k+|\alpha'|+\lambda}(\mathbb{B}_{N-k})$.*
3. *If G is any maximal commutative subgroup of $\mathrm{Aut}(\mathbb{B}_{N-k})$ then the C^*-subalgebra generated in $\mathcal{L}(H)$ by all such symbols φ for which a_φ is G-invariant is commutative.*

Application to certain piecewise continuous symbols

McDonald (1979) considered piecewise continuous symbols with discontinuities along hyperplanes of real codimension 1.

We say that a Jordan arc $\gamma : [0,1] \to \mathbb{C}$ with $\gamma((0,1)) \subset \mathbb{D}$ has the *graph property* at $z \in \gamma((0,1))$, if there exist some $\delta > 0$ with $U_\delta(t) \subseteq (0,1)$ such that $\gamma(U_\delta(t))$ is the graph of a continuous function over some interval on a line through 0.

$$G(\gamma) := \{z \in \gamma((0,1))\,;\, \gamma \text{ has the graph property at } z\}.$$

$z \in G(\gamma)$ if and only if there exist some open interval $I_0 \subseteq (-1,1)$, some $\alpha \in [0,2\pi)$, a bijective, monotone increasing continuous function $u : U_\delta(t) \to I_0$ and a function $v \in C(I_0,\mathbb{R})$ such that

$$\begin{aligned} \forall s \in U_\delta(t): \quad & \gamma(s) = e^{i\alpha}(u(s) + iv(u(s))) \\ \forall \xi \in I_0: \quad & \gamma(u^{-1}(\xi)) = e^{i\alpha}(\xi + iv(\xi)). \end{aligned} \tag{7}$$

Let now $\gamma : [0, 1] \to \mathbb{C}$ be a Jordan arc with $|\gamma(0)| = |\gamma(1)| = 1$ and $\gamma((0, 1)) \subset \mathbb{D}$. Let also E be a linear subspace in $\mathbb{C}^N$ of (complex) codimension 1, let $u_0 \in E^\perp$ with $|u''| = 1$ and consider

$$E_\gamma := \{u' + \gamma(t)u_0 \,;\, u' \in E,\ t \in [0, 1]\}.$$

As in the hyperplane situation considered by McDonald, the set $\mathbb{B}_N \setminus E_\gamma$ (respectively $\mathbb{S}_N \setminus E_\gamma$) consists of two connected components which will be denoted by B_γ^+ and B_γ^- (respectively S_γ^+ and S_γ^-).
Note, that $\gamma([0, 1])$ may have positive area measure.

If $\varphi_\gamma^+ \in C(\overline{B_\gamma^+})$ and $\varphi_- \in C(\overline{B_\gamma^-})$ are given and if $\gamma([0,1])$ has area measure 0, then there exists precisely one element $\varphi \in L^\infty(\mathbb{B}_N)$ with $\varphi|_{B_\gamma^+} \equiv \varphi_\gamma^+$ and $\varphi|_{B_\gamma^-} \equiv \varphi_-$.
If $\gamma([0,1])$ has positive area measure this is no longer true. In this case $\mathbb{B}_N \cap E_\gamma$ and $\mathbb{S}_N \cap E_\gamma$ have positive volume respectively surface measure. We therefore consider the following symbol classes

$$HC_\gamma^+(\mathbb{B}_N) := \{\varphi_+ \chi_{\overline{B_\gamma^+}} + \varphi_- \,;\, \varphi_+, \varphi_- \in C(\overline{\mathbb{B}_N})\}$$

and

$$PC_\gamma^+(\mathbb{S}_N) := \{\varphi_+ \chi_{\overline{S_\gamma^+}} + \varphi_- \,;\, \varphi_+, \varphi_- \in C(\mathbb{S}_N)\},$$

where χ_M denotes the characteristic function of the corresponding set M.

Theorem

Let γ, E_γ be as above and suppose that $G(\gamma)$ is dense in $\gamma([0,1])$. If $\varphi = \varphi_+ \chi_{\overline{B_\gamma^+}} + \varphi_- \in HC_\gamma^+(\mathbb{B}_N)$ with $\varphi_+, \varphi_- \in C(\overline{\mathbb{B}_N})$ or $\varphi = \varphi_+ \chi_{\overline{S_\gamma^+}} + \varphi_- \in PC_\gamma^+(\mathbb{S}_N)$ with $\varphi_+, \varphi_- \in C(\mathbb{S}_N)$ then, in both situations, we obtain

$$\mathrm{sp}_e(T_\varphi) = \varphi(S_\gamma^+ \cup S_\gamma^-) \cup \{t\varphi_+(\omega) + \varphi_-(\omega)\,;\, 0 \leq t \leq 1, \omega \in E_\gamma \cap \mathbb{S}_N\}.$$

Remark (Lance and Thomas, 1991)

There exist Jordan arcs with strictly positive area measure satisfying the assumptions of this theorem.

Symbols that are uniformly continuous on sectors

Let $\theta := (\theta_0, \theta_1, \ldots, \theta_n) \in [0, 2\pi]^{n+1}$ with $\theta_0 = 0, \theta_n = 2\pi$ and $\theta_{j-1} < \theta_j$ for $j = 1, \ldots, n$. We define the sectors

$$S_j := \{(z', re^{is})\,;\, z' \in \mathbb{C}^{N-1}, 0 < r \leq 1, \theta_{j-1} < s < \theta_j\} \qquad (j = 1, \ldots, n)$$

$S_0 := S_n$, and consider the symbol classes $PC_\theta(\mathbb{B}_N)$ of all those L^∞-functions φ, that are uniformly continuous on $S_j \cap \mathbb{B}_N$ for all $j = 1, \ldots, n$. We write $\tilde{\varphi}_j$ for the continuous extension of $\varphi|_{S_j \cap \mathbb{B}_N}$ to $\overline{S_j \cap \mathbb{B}_N}$.
For $\omega' \in \mathbb{S}_{N-1}$ let $\gamma_{\varphi,\omega'} : [0, 2\pi] \to \mathbb{C}$ be a parametrisation of the closed polygon with the corners $\gamma_{\varphi,\omega'}((\theta_j + \theta_{j-1})/2) = \tilde{\varphi}_j(\omega', 0)$, $j = 1, \ldots, n$.

Proposition

Consider $\varphi \in PC_\theta(\mathbb{B}_n)$ *and* $\omega = (\omega', \omega_N) \in \mathbb{S}_N$.

1. *If* $\omega \in S_j$ *for some* $j \in \{1, \ldots, n\}$, *then* $\mathrm{sp}_\omega(T_\varphi) = \{\varphi(\omega)\}$.
2. *If* $\omega \in \overline{S_j} \cap \overline{S_{j-1}}$ *for some* $j \in \{1, \ldots, n\}$ *and* $\omega_N \neq 0$, *then* $\mathrm{sp}_\omega(T_\varphi) = [\tilde{\varphi}_{j-1}(\omega), \tilde{\varphi}_j(\omega)]$.
3. *If* $\omega_N = 0$ *then*

$$\begin{aligned}\Sigma_\omega(\varphi) :=& \gamma_{\varphi,\omega'}([0,2\pi]) \cup \{w \in \mathbb{C} \setminus \gamma_{\varphi,\omega'}([0,2\pi])\,;\, n(\gamma_{\varphi,\omega'}, w) \neq 0\} \\ \subseteq& \mathrm{sp}^\lambda_\omega(T_\varphi).\end{aligned}$$

Proposition

Consider $\varphi \in PC_\theta(\mathbb{B}_n)$ *and* $\omega = (\omega', \omega_N) \in \mathbb{S}_N$*.*

1. *If* $\omega \in S_j$ *for some* $j \in \{1, \ldots, n\}$*, then* $\mathrm{sp}_\omega(T_\varphi) = \{\varphi(\omega)\}$*.*
2. *If* $\omega \in \overline{S_j} \cap \overline{S_{j-1}}$ *for some* $j \in \{1, \ldots, n\}$ *and* $\omega_N \neq 0$*, then* $\mathrm{sp}_\omega(T_\varphi) = [\tilde{\varphi}_{j-1}(\omega), \tilde{\varphi}_j(\omega)]$*.*
3. *If* $\omega_N = 0$ *then*

$$\Sigma_\omega(\varphi) := \gamma_{\varphi,\omega'}([0,2\pi]) \cup \{w \in \mathbb{C} \setminus \gamma_{\varphi,\omega'}([0,2\pi])\,;\ n(\gamma_{\varphi,\omega'}, w) \neq 0\} \subseteq \mathrm{sp}^\lambda_\omega(T_\varphi).$$

Note, that $\Sigma_\omega(\varphi)$ contains $\mathfrak{R}_\omega(\varphi)$. Hence, if $\Sigma_\omega(\varphi)$ is convex, then equality holds in statement 3 of the proposition.

In the cases $n = 2$ and $n = 3$, this is always the case.

For $n \geq 4$, the set $\Sigma_\omega(\varphi)$ need no longer to be convex and it is not clear wether we have equality in 3.

Symbols with weaker invariance properties

Following a suggestion of Davie and Jewell, we consider symbols $\phi \in L^\infty(\mathbb{S}_N)$ with the weaker symmetry property

$$\forall \theta \in \mathbb{R}, z \in \mathbb{S}_N : \quad \phi(e^{i\theta} z) = \phi(z). \tag{8}$$

Proposition

If ϕ satisfies (8), then for all $m \in \mathbb{Z}_+$ the space $\mathcal{H}_m$ of all m-homogeneous holomorphic polynomials is a finite dimensional reducing subspace for T_ϕ. In particular, we have

1. $\mathrm{sp}_p(T_\phi)$ *is countable.*
2. T_ϕ *and* T_ϕ^* *have the SVEP.*
3. $T_\phi = \bigoplus_{m=0}^{\infty} T_\phi \big|_{\mathcal{H}(m)}$.

Example

Consider the case $N = 2$ and the symbol ϕ with

$$\phi(z_1, z_2) := \frac{z_1 \overline{z_2}}{|z_1 z_2|}.$$

Then T_ϕ has the additional properties:

1. $\mathrm{sp}_p(T_\phi) = \{0\}$.
2. $X_{T_\phi}(\{0\})$ is dense in $H^2(\mathbf{S})$.
3. $\mathrm{sp}_e(T_\phi) = \mathrm{sp}(T) = \overline{\mathbb{D}}$.
4. T_ϕ is a $C_{0,0}$ contraction.

In particular, T_ϕ is a universal dilation in the sense of Bercovici, Foiaş and Pearcy.