C*-algebras of 2-groupoids

Massoud Amini

Tarbiat Modares University
Institute for Fundamental Researches (IPM)

Banach Algebras 2013
Gothenburg, Sweden
July 30, 2013
Table of contents

1 Abstract
   - motivation

2 2-groupoids
   - 2-categories
   - algebraic 2-groupoids
   - topological 2-groupoids and 2-Haar systems

3 $C^*$-algebras of 2-groupoids
   - quasi-invariant measures
   - full $C^*$-algebras
   - induced representations and reduced $C^*$-algebras
   - $r$-discrete principal 2-groupoids
Abstract

We define topological 2-groupoids and study locally compact 2-groupoids with 2-Haar systems. We consider quasi-invariant measures on the sets of 1-arrows and unit space and build the corresponding vertical and horizontal modular functions. For a given 2-Haar system we construct the vertical and horizontal full $C^*$-algebras of a 2-groupoid and show that its is unique up to strong Morita equivalence, and make a correspondence between their bounded representations on Hilbert spaces and those of the 2-groupoid on Hilbert bundles.
Abstract

We define topological **2-groupoids** and study locally compact 2-groupoids with 2-Haar systems. We consider quasi-invariant measures on the sets of 1-arrows and unit space and build the corresponding vertical and horizontal modular functions. For a given 2-Haar system we construct the vertical and horizontal **full** $C^*$-**algebras** of a 2-groupoid and show that its is **unique** up to strong Morita equivalence, and make a correspondence between their bounded representations on Hilbert spaces and those of the 2-groupoid on Hilbert bundles.

We show that representations of certain closed 2-subgroupoids are **induced to representations** of the 2-groupoid and use regular representation to build the vertical and horizontal **reduced** $C^*$-**algebras** of the 2-groupoid. We establish strong Morita equivalence between $C^*$-**algebras** of the 2-groupoid of composable pairs and those of the 1-arrows and unit space. We describe the reduced $C^*$-**algebras** of **r-discrete** principal 2-groupoids and find their ideals and masas.
Motivation

In noncommutative geometry, certain quotient spaces are described by non-commutative $C^*$-algebras, when the symmetry groups of such quotient spaces are non Hausdorff, it is more appropriate to describe such symmetry groups and groupoids using crossed modules of groupoids (Buss-Meyer-Zhu, 2012).
Motivation

In noncommutative geometry, certain quotient spaces are described by non-commutative $C^*$-algebras, when the symmetry groups of such quotient spaces are non Hausdorff, it is more appropriate to describe such symmetry groups and groupoids using crossed modules of groupoids (Buss-Meyer-Zhu, 2012).

One motivating example is the gauge action on the irrational rotation algebra $A_\vartheta$, which is the universal $C^*$-algebra generated by two unitaries $U$ and $V$ satisfying the commutation relation $UV = \lambda VU$ with $\lambda := \exp(2\pi i \vartheta)$. Since $A_\vartheta$ is the crossed product $C(\mathbb{T}) \rtimes_\lambda \mathbb{Z}$, for the canonical action of $\mathbb{Z}$ on $\mathbb{T}$ by $n \cdot z := \lambda^n \cdot z$, it could be viewed as the noncommutative analog of the non Hausdorff quotient space $\mathbb{T}/\lambda^\mathbb{Z}$. This latter group acts on itself by translations, thus $\mathbb{T}/\lambda^\mathbb{Z}$ is a symmetry group of $A_\vartheta$. 
Motivation

More generally, one may define actions of crossed modules on $C^*$-algebras similar to the twisted actions in the sense of Philip Green (Green, 1978) and build crossed products for such actions. The resulting crossed product is functorial: If two actions are equivariantly Morita equivalent in a suitable sense, their crossed products are Morita–Rieffel equivalent $C^*$-algebras.
More generally, one may define actions of crossed modules on $C^*$-algebras similar to the twisted actions in the sense of Philip Green (Green, 1978) and build crossed products for such actions. The resulting crossed product is functorial: If two actions are equivariantly Morita equivalent in a suitable sense, their crossed products are Morita–Rieffel equivalent $C^*$-algebras.

Crossed modules of discrete groups are used in homotopy theory to classify 2-connected spaces up to homotopy equivalence. They are equivalent to strict 2-groups (Baez, 1997, Noohi, 2007).
Motivation

One could write every locally Hausdorff groupoid as the truncation of a Hausdorff topological weak 2-groupoid. Also the crossed modules of topological groupoids are equivalent to strict topological 2-groupoids.
Motivation

One could write every locally Hausdorff groupoid as the truncation of a Hausdorff topological weak 2-groupoid. Also the crossed modules of topological groupoids are equivalent to strict topological 2-groupoids.

For a Hausdorff étale groupoid $G$ and the interior $H \subseteq G$ of the set of loops (arrows with same source and target) in $G$, the quotient $G/H$ is a locally Hausdorff, étale groupoid, and the pair $(G, H)$ together with the embedding $H \to G$ and the conjugation action of $G$ on $H$ is a crossed module of topological groupoids. The corresponding C*-algebra $C^*(G, H)$ is the C*-algebra of foliations in the sense of Alan Connes (Connes, 1982). The C*-algebra of general (non Hausdorff) groupoids are studied in details by Jean Renault (Renault, 1980).
We define a strict 2-category as a category enriched over categories. We adapt the notations and terminology of (Buss-Meyer-Zhu, 2013); see also (Baez, 1997). For two objects \( x \) and \( y \) of the first order category, we have a category of morphisms from \( x \) to \( y \), and the composition of morphisms lifts to a bifunctor between these morphism categories.
We define a **strict 2-category** as a category enriched over categories. We adapt the notations and terminology of (Buss-Meyer-Zhu, 2013); see also (Baez, 1997). For two objects $x$ and $y$ of the first order category, we have a category of morphisms from $x$ to $y$, and the composition of morphisms lifts to a bifunctor between these morphism categories.

The arrows between objects $u : x \rightarrow y$ are called **1-morphisms**. We write $x = d(u)$ and $y = r(u)$. The arrows between arrows

\[ \begin{array}{ccc}
  y & \downarrow^a & x \\
  \downarrow & & \\
  v & \leftarrow & u \\
\end{array} \]

are called **2-morphisms** (or **bigons**). We write $u = d(a)$, $v = r(a)$ and $x = d^2(a)$, $y = r^2(a)$. 
Composition

The category structure on the space of arrows $x \rightarrow y$ gives a vertical composition of 2-morphisms

$$
\begin{array}{c}
\xymatrix{y & v & x \\
\downarrow^b & \downarrow^a & \\
w & \downarrow^w & } \\
\xymatrix{y & \ar[rr]^a \ar[rr]^v & x. \\
\downarrow^u & \downarrow^w & }
\end{array}
$$
Composition

The vertical product $a \cdot_v b$ is defined if $r(b) = d(a)$. The composition functor between the arrow categories gives a composition of 1-morphisms

$$z \leftarrow^u y \leftarrow^v x \quad \mapsto \quad z \leftarrow^{uv} x,$$

which is defined if $r(v) = d(u)$, and a horizontal composition of 2-morphisms

The horizontal product $a \cdot_h b$ is defined if $r^2(b) = d^2(a)$. 
Composition

These three compositions are assumed to be associative and unital, with the same units for the vertical and horizontal products. The horizontal and vertical products commute: given a diagram

![Diagram](attachment:image.png)

composing first vertically and then horizontally or vice versa produces the same 2-morphism $u_1u_2 \Rightarrow v_1v_2$. 
Composition

These three compositions are assumed to be associative and unital, with the same units for the vertical and horizontal products. The horizontal and vertical products commute: given a diagram

composing first vertically and then horizontally or vice versa produces the same 2-morphism $u_1 u_2 \Rightarrow v_1 v_2$.

We denote the inverse of a 1-morphism $u$ by $u^{-1}$ and vertical and horizontal inverses of a 2-morphism $a$ by $a^{-v}$ and $a^{-h}$. 
Examples

Categories form a strict 2-category with small categories as objects, functors between categories as arrows, and natural transformations between functors as 2-morphisms. The composition of 1-morphisms is the composition of functors and the vertical composition of 2-morphisms is the composition of natural transformations. The horizontal composition of 2-morphisms yields a canonical natural transformation. Another example of a strict 2-category has $C^*$-algebras as objects, non-degenerate $\ast$-homomorphisms as 1-morphisms, and unitary intertwiners between such $\ast$-homomorphisms as 2-morphisms.
Definition

A (strict) 2-groupoid is a strict 2-category in which all 1-morphisms and 2-morphisms are invertible (both for the vertical and horizontal product).
Definition

A (strict) 2-groupoid is a strict 2-category in which all 1-morphisms and 2-morphisms are invertible (both for the vertical and horizontal product).

2-group

All 2-groupoids are assumed to be small 2-categories, namely the classes of objects and morphisms are sets. A (strict) 2-group is a strict 2-groupoid with a single object. Given a 2-groupoid $G$, its objects $G^0$ and 1-morphisms $G^1$ form a groupoid, and so does the 1-morphisms and 2-morphisms $G^2$ with vertical composition.
Notation

We usually write \( \mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0) \) and denote the subset of composable elements in \( \mathcal{G}^1 \times \mathcal{G}^1 \) by \( \mathcal{G}^{(1)} \) and the subsets of elements in \( \mathcal{G}^2 \times \mathcal{G}^2 \) which are vertically or horizontally composable by \( \mathcal{G}^{(2v)} \) or \( \mathcal{G}^{(2h)} \). We may use horizontal products with unit 2-morphisms to produce any 2-morphism from a 2-morphisms that starts at a unit 1-morphism:

\[
\begin{align*}
  & y & 1_y & y \downarrow & u & 1_u & x \quad \mapsto \quad y & u \downarrow & a \cdot h 1_u x.

  & \\ & r(a) & u & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & r(a)u
\end{align*}
\]
Crossed module

The 2-morphisms starting at the identity 1-morphisms at the object \( x \) form a group \( \mathcal{G}_x \) with respect to horizontal composition, and the range map is a homomorphism from the set of such 2-morphisms to the isotropy group bundle \( H = \bigsqcup_{x \in \mathcal{G}^0} \mathcal{G}_x \) of the groupoid \((\mathcal{G}^0, \mathcal{G}^1)\). This map is onto when \( \mathcal{G} \) is 2-transitive (i.e. for each \( u, v \in \mathcal{G}^1 \) there is \( a \in \mathcal{G}^2 \) with \( d(a) = u \) and \( r(a) = v \)). Furthermore, the groupoid \( \mathcal{G} \) acts on the group bundle \( H \) by horizontal conjugation:

\[
\begin{align*}
&x \quad \downarrow 1_g \quad y \\
&\quad \swarrow u \quad \searrow 1_y \\
&u \quad \downarrow a \quad \uparrow 1_y \\
&\quad \swarrow 1_y \quad \searrow r(a) \\
&\quad \downarrow 1_{u^{-1}} \quad \uparrow \quad \downarrow x \\
&\quad \swarrow u^{-1} \quad \searrow 1_{u^{-1}} \\
&\quad \downarrow x \quad \uparrow \quad \downarrow y \\
&\quad \swarrow v \quad \searrow \downarrow b \\
&\quad \downarrow u r(a) u^{-1} \quad \uparrow \quad \downarrow x, \\
\end{align*}
\]

where \( b = 1_u \cdot_h a \cdot_h 1_{u^{-1}} \).
Crossed module

We may consider the map

$$r : \bigsqcup_{x \in G^0} G_x \to \bigsqcup_{x \in G^0} G_x^x$$

and regard $\left( H, G^1, r \right)$ as a crossed module of groupoids. Conversely, for each crossed module $\left( H, G^1, r \right)$ where $H$ is a bundle of groups, $G^1$ is a groupoid and $r : H \to G^1$ is a groupoid homomorphism, there is a unique 2-groupoid $G$ whose isotropic group bundle is isomorphic to $H$, whose set of 1-morphisms is isomorphic to $G^1$, and its range map realizes (after identification) as $r$. 
Example

As a concrete example, consider the map $r_\theta : \mathbb{Z} \to \mathbb{T}; \ n \mapsto e^{2\pi in\theta}$ where $\theta \in \mathbb{R}$, then $\mathbb{T}$ on $\mathbb{Z}$ by conjugation and the corresponding crossed module is the symmetry of the rotation algebra $A_\theta$. This gives a 2-groupoid with a single object, 1-morphisms $\mathbb{T}$ and 2-morphisms $\mathbb{Z} \times \mathbb{T}$. 
algebraic 2-groupoid

Let \( \mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0) \) be a 2-groupoid, then \( \mathcal{G} \) is called 1-principal if the map \((r, d) : \mathcal{G}^1 \to \mathcal{G}^0 \times \mathcal{G}^0\) is one-to-one, 2-principal if the map \((r, d) : \mathcal{G}^2 \to \mathcal{G}^1 \times \mathcal{G}^1\) is one-to-one, and 1+2-principal if both 1-principal and 2-principal. If we replace one-to-one with onto, we get the notions of 1-transitive, 2-transitive, and 1+2-transitive. Note that 2-transitivity here is different from the property of each two nodes being connected by paths of length 2.
algebraic 2-groupoid

Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a 2-groupoid, then $\mathcal{G}$ is called 1-principal if the map $(r, d) : \mathcal{G}^1 \to \mathcal{G}^0 \times \mathcal{G}^0$ is one-to-one, 2-principal if the map $(r, d) : \mathcal{G}^2 \to \mathcal{G}^1 \times \mathcal{G}^1$ is one-to-one, and 1+2-principal if both 1-principal and 2-principal. If we replace one-to-one with onto, we get the notions of 1-transitive, 2-transitive, and 1+2-transitive. Note that 2-transitivity here is different from the property of each two nodes being connected by paths of length 2.

For each $x \in \mathcal{G}^0$ and $u \in \mathcal{G}^1$, $\mathcal{G}^x_u = \{ u \in \mathcal{G}^1 : d(u) = r(u) = x \}$, $\mathcal{G}^u = \{ a \in \mathcal{G}^2 : d(a) = r(a) = u \}$, and $\mathcal{G}^u;_x = \{ a \in \mathcal{G}^2 : d(a) = r(a) = u, d^2(a) = r^2(a) = x \}$. We also have the isotropy groupoid $\mathcal{G}(x) = (\mathcal{G}_x^2(x), \mathcal{G}_x^1(x))$ where $\mathcal{G}_x^2(x) = \{ a \in \mathcal{G}^2 : d^2(a) = r^2(a) = x \}$ and $\mathcal{G}_x^1(x) = \{ r(a) : a \in \mathcal{G}_x^2(x) \}$ with vertical multiplication.
Example

We give three basic examples of 2-groupoids.

(i) (Transformation 2-group) Let $S$ be an additive group with identity $0$ acting from right on a set $U$ and put $G^1 = U \times S$ and $G^0 = U \times \{0\}$. Let $T$ be a multiplicative group with identity $1$ acting from left on $S$ and acting trivially from right on $U$ and put $G^2 = T \times U \times S$ and identify $U \times S \{1\} \times U \times S$. Assume that the left action of $T$ on $S$ is distributive

$$t \cdot (s + s') = t \cdot s + t \cdot s',$$

for $s, s' \in S$ and $t \in T$. Define $r(u, s) = (u, 0)$ and $d(u, s) = (u \cdot s, 0)$ and partial multiplication by $(u, s) \cdot (u \cdot s, s') = (u, s + s')$ with $(u, s)^{-1} = (u \cdot s, -s)$. Also define $r(t, u, s) = (1, u, s)$ and $d(t, u, s) = (1, t \cdot s)$ and vertical multiplication by

$$(t, u, t' \cdot s') \cdot_v (t', u, s') = (tt', u, s')$$

with $(t, u, s)^{-v} = (t^{-1}, u, t \cdot s)$ and horizontal multiplication by

$$(t, u, s) \cdot_h (t, u \cdot s, s') = (t, u, s + s')$$

with $(t, u, s)^{-h} = (t, u \cdot s, -s)$. 

Example

(ii) (Principal 2-groupoid) Let $X$ be a set and put $\mathcal{G}^2 = X^{(5)}$, $\mathcal{G}^1 = X^{(3)}$, $\mathcal{G}^0 = X$. Define $r(x, y, z) = z$ and $d(y, z) = x$ and $(x, y, z) \cdot (z, u, v) = (x, y, v)$ with $(x, y, z)^{-1} = (z, y, x)$. Define $r(x, y, z, u, v) = (x, u, v)$ and $d(x, y, z, u, v) = (x, y, v)$ and vertical multiplication by $(x, y, z, u, v) \cdot_v (x, u, s, t, v) = (x, y, z, t, v)$ with $(x, y, z, u, v)^{-v} = (x, u, z, y, v)$ and horizontal multiplication by $(x, y, z, u, v) \cdot_h (v, w, s, t, r) = (x, y, s, u, r)$ with $(x, y, z, u, v)^{-h} = (v, u, z, y, x)$.

(iii) (Groupoid bundle) If $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ satisfies $d(u) = r(u)$ for each $u \in \mathcal{G}^1$ then $\mathcal{G} = \bigcup_{x \in \mathcal{G}^0} \mathcal{G}(x)$ is a groupoid bundle.
Similarity

For 2-groupoids $\mathcal{G}$ and $\mathcal{H}$, a vertical morphism $\varphi : \mathcal{G} \to \mathcal{H}$ of 2-groupoids is a pair $\varphi = (\varphi^2, \varphi^1)$ such that $\varphi^2(a \cdot_v b) = \varphi^2(a) \cdot_v \varphi^2(b)$ and $\varphi^1(uv) = \varphi^1(u)\varphi^2(v)$, for $a, b \in \mathcal{G}^2$ and $u, v \in \mathcal{G}^1$, whenever both sides are defined. Two vertical morphisms $\varphi, \psi$ from $\mathcal{G}$ to $\mathcal{H}$ are called similar if there are maps $\vartheta^2 : \mathcal{G}^1 \to \mathcal{H}^2$ and $\vartheta^1 : \mathcal{G}^0 \to \mathcal{H}^1$ such that

$$d(\vartheta^2(u)) = \vartheta^1(d(u)), \quad r(\vartheta^2(u)) = \vartheta^1(r(u))$$

and

$$\vartheta^2 \circ r(a) \cdot_v \varphi^2(a) = \psi^2(a) \cdot_v \vartheta^2 \circ d(a), \quad \vartheta^1 \circ r(u)\varphi^1(u) = \psi^1(u)\vartheta^1 \circ r(u)$$

for $u \in \mathcal{G}^1$ and $a \in \mathcal{G}^2$. We write $\varphi \sim_v \psi$. We say that $\mathcal{G}$ and $\mathcal{H}$ are v-similar if there are vertical morphisms $\varphi : \mathcal{G} \to \mathcal{H}$ and $\psi : \mathcal{H} \to \mathcal{G}$ such that $\varphi \circ \psi \sim_v id_{\mathcal{H}}$ and $\psi \circ \varphi \sim_v id_{\mathcal{G}}$. The notions of horizontal morphisms and h-similarity are defined similarly and the latter is denoted by $\sim_h$. 
Definition

Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a 2-groupoid and $\mathcal{E} = (\mathcal{E}^1, \mathcal{E}^0)$ with $\mathcal{E}^0 \subseteq \mathcal{G}^0$ and $\mathcal{E}^1 \subseteq \{ u \in \mathcal{G}^1 : d(u), r(u) \in \mathcal{E}^0 \}$, the 2-groupoid $\mathcal{G}_\mathcal{E} = (\mathcal{E}^2, \mathcal{E}^1, \mathcal{E}^0)$, where $\mathcal{E}^2 = \{ a \in \mathcal{G}^2 : d(a), r(a) \in \mathcal{E}^1 \}$, is called the restriction of $\mathcal{G}$ to $\mathcal{E}$. We say that $\mathcal{E}$ is full if $\mathcal{E}^0$ meets each equivalence class in $\mathcal{G}^0$ and $\mathcal{E}^1$ meets each equivalence class in $\mathcal{G}^1$. 
The next lemma is proved by Ramsay for groupoids (Ramsay, 1971).

**Lemma**

If $\mathcal{E}$ is full then $\mathcal{G}_\mathcal{E} \sim_\nu \mathcal{G}$. 
The next lemma is proved by Ramsay for groupoids (Ramsay, 1971).

**Lemma**

If $E$ is full then $\mathcal{G}_E \sim_v \mathcal{G}$.

**Corollary**

Every 2-groupoid is $v$-similar to a groupoid bundle. A 2-groupoid is $v$-similar to a groupoid if and only if its objects consists of only one equivalence class.
Identification

We identify $G^0$ with a subset of $G^1$ and $G^1$ with a subset of $G^2$ by identifying $x \in G^0$ with $1_x$ and $u \in G^1$ with $1_u$.

Definition

A topological 2-groupoid is a 2-groupoid $G = (G^2, G^1, G^0)$ and a topology on $G^2$ such that

(i) The maps $u \mapsto u^{-1}$ and $a \mapsto a^{-v}$, $a \mapsto a^{-h}$ are continuous on $G^1$ and $G^2$.

(ii) The maps $(u, v) \mapsto uv$ and $(a, b) \mapsto a \cdot_v b$, $(a, b) \mapsto a \cdot_h b$ are continuous on their domains.
Lemma

For any topological 2-groupoid \( G = (G^2, G^1, G^0) \),

(i) The maps \( u \mapsto u^{-1} \) and \( a \mapsto a^{-v}, a \mapsto a^{-h} \) are homeomorphisms on \( G^1 \) and \( G^2 \).

(ii) The source and range maps \( d, r \) are continuous on \( G^1 \) and \( G^2 \).

(iii) If \( G^1 \) is Hausdorff, \( G^0 \subseteq G^1 \) is closed, and if \( G^2 \) is Hausdorff, \( G^0 \subseteq G^1, G^1 \subseteq G^2 \) and \( G^0 \subseteq G^2 \) are closed.

(iv) If \( G^0 \) is Hausdorff, \( G^{(1)} \subseteq G^1 \times G^1 \) is closed, and if \( G^1 \) is Hausdorff, \( G^{(2v)} \subseteq G^2 \times G^2 \) and \( G^{(2h)} \subseteq G^2 \times G^2 \) are closed.

(v) For the range equivalence \( a \sim_r b \) defined by \( r(a) = r(b) \), the orbit space \( G^2/\sim_r \) is homeomorphic to \( G^1 \). Similarly \( G^1/\sim_r \) is homeomorphic to \( G^0 \).
Definition

A **locally compact 2-groupoid** is a topological 2-groupoid \( \mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0) \) such that \( \mathcal{G}^0, \mathcal{G}^1 \) are Hausdorff Borel subsets of \( \mathcal{G}^2 \) and every point of \( \mathcal{G}^2 \) has an open, relatively compact, Hausdorff neighborhood.
Definition

A **locally compact 2-groupoid** is a topological 2-groupoid $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ such that $\mathcal{G}^0, \mathcal{G}^1$ are Hausdorff Borel subsets of $\mathcal{G}^2$ and every point of $\mathcal{G}^2$ has an open, relatively compact, Hausdorff neighborhood.

For the rest of this talk, $\mathcal{G}$ is a locally compact 2-groupoid. We put

$$C_c(\mathcal{G}) = \{f : \mathcal{G}^2 \to \mathbb{C} : f \text{ is continuous and } \text{supp}(f) \subseteq \mathcal{G}^2 \text{ is compact}\},$$

where $\text{supp}(f)$ is the complement of the union of open Hausdorff subsets of $\mathcal{G}^2$ on which $f$ vanishes. By assumption $\mathcal{G}^2$ is a union of compact Hausdorff sets $K$ and on the algebraic direct limit $C_c(\mathcal{G})$ is endowed with an **inductive limit** topology.
Definition

Let $\mathcal{G}$ be a locally compact 2-groupoid. A continuous left 2-Haar system on $\mathcal{G}$ consists of two families of positive Borel measures $\{\lambda^u_v\}$ and $\{\lambda^x_h\}$ on $\mathcal{G}^2$, where $u$ ranges over $\mathcal{G}^1$ and $x$ ranges over $\mathcal{G}^0$, such that

(i) $\text{supp}(\lambda^u_v) = \mathcal{G}^u$ and $\text{supp}(\lambda^x_h) = \mathcal{G}^x$, for each $u \in \mathcal{G}^1$ and $x \in \mathcal{G}^0$.

(ii) For any $f \in C_c(\mathcal{G})$, the map $u \mapsto \int f \, d\lambda^u_v$ is continuous on $\mathcal{G}^1$ and the map $x \mapsto \int f \, d\lambda^x_h$ is continuous on $\mathcal{G}^0$.

(iii) For any $f \in C_c(\mathcal{G})$,

\[
\int f(a \cdot_v b) \, d\lambda^d(a)(b) = \int f(b) \, d\lambda^r(a)(b)
\]

and

\[
\int f(a \cdot_h b) \, d\lambda^d(a)(b) = \int f(b) \, d\lambda^r(a)(b).
\]
Definition

Let $G$ be a locally compact 2-groupoid. A **continuous left 2-Haar system** on $G$ consists of two families of positive Borel measures $\{\lambda^u_v\}$ and $\{\lambda^x_h\}$ on $G^2$, where $u$ ranges over $G^1$ and $x$ ranges over $G^0$, such that

(i) $\text{supp}(\lambda^u_v) = G^u$ and $\text{supp}(\lambda^x_h) = G^x$, for each $u \in G^1$ and $x \in G^0$.

(ii) For any $f \in C_c(G)$, the map $u \mapsto \int f d\lambda^u_v$ is continuous on $G^1$ and the map $x \mapsto \int f d\lambda^x_h$ is continuous on $G^0$.

(iii) For any $f \in C_c(G)$,

$$\int f(a \cdot_v b) d\lambda^{d(a)}_v(b) = \int f(b) d\lambda^{r(a)}_v(b)$$

and

$$\int f(a \cdot_h b) d\lambda^{d^2(a)}_h(b) = \int f(b) d\lambda^{r^2(a)}_h(b).$$

$$\int f(uv) d\lambda^{d(u)}_v(v) = \int f(v) d\lambda^{r(u)}_v(v).$$
Proposition

If $\mathcal{G}$ has a continuous 2-Haar system, we have the continuous surjections:

$$\lambda_v : C_c(\mathcal{G}^2) \to C_c(\mathcal{G}^1); \quad f \mapsto \lambda_v(f), \quad \lambda_v(f)(u) = \int fd\lambda_v^u,$$

and

$$\lambda_h : C_c(\mathcal{G}^2) \to C_c(\mathcal{G}^0); \quad f \mapsto \lambda_h(f), \quad \lambda_h(f)(x) = \int fd\lambda_h^x.$$

Moreover the maps $r : \mathcal{G}^2 \to \mathcal{G}^1$, $r : \mathcal{G}^1 \to \mathcal{G}^0$ and $r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ are open and the associated equivalence relations on $\mathcal{G}^1$ and $\mathcal{G}^0$ are open.
Example

The 2-Haar systems of the above examples are as follows:

\( \text{Transformation 2-group} \) Let \( S, T \) be locally compact groups with Haar measures \( \lambda_S \) and \( \lambda_T \) acting continuously on a locally compact Hausdorff space \( U \) as in Example 3.1(i) and \( G_2 = T \times U \times S \), then the vertical and horizontal left Haar systems on \( G \) are given by

\[
\lambda_1(1, u, s)v = \lambda_T \times \delta_u \times \lambda_1(1, u, 0) \\
\lambda_2(1, u, s)h = \lambda_2 \times \delta_u \times \lambda_S(u, s)
\]

where \( \lambda_1, \lambda_2 \) are arbitrary Borel measures with full support on \( S, T \), respectively.
Example

The **2-Haar systems** of the above examples are as follows:

(i) (Transformation 2-group) Let $S$, $T$ be locally compact groups with **Haar measures** $\lambda_S$ and $\lambda_T$ acting continuously on a locally compact Hausdorff space $U$ as in Example 3.1(i) and $G^2 = T \times U \times S$, then the vertical and horizontal left Haar systems on $G$ are given by

$$
\lambda_v^{(1,u,s)} = \lambda_T \times \delta_u \times \lambda_1, \quad \lambda_h^{(1,u,0)} = \lambda_2 \times \delta_u \times \lambda_S \quad (u \in U, s \in S),
$$

where $\lambda_1$, $\lambda_2$ are **arbitrary** Borel measures with full support on $S$, $T$, respectively.
Example

(ii) (Principal 2-groupoid) Let $X$ be a locally compact Hausdorff space and $G^2 = X^{(5)}$. Consider the homeomorphism

$$d : G^{(x,u,v)} \to X^{(2)}; \ (x, y, z, u, v) \mapsto (y, z),$$

let $\alpha$ be any Borel measure on $X^{(2)}$ with full support such that for each $f \in C_c(G)$, the map

$$(x, u, v) \mapsto \int f(x, y, z, u, v) d\alpha(y, z)$$

is continuous on $X^{(3)}$, then $\int f d\lambda_v^{(x,u,v)} = \int f(x, y, z, u, v) d\alpha(y, z)$ defines a vertical left Haar system. The horizontal case is treated similarly.
Example

(iii) (Groupoid bundle) Let $\mathcal{G} = \bigsqcup_{x \in \mathcal{G}^0} \mathcal{G}(x)$ be a locally compact groupoid bundle. The 2-Haar system is essentially unique (if it exists), that is any two systems $\{\lambda^u, \lambda^x\}$ and $\{\sigma^u, \sigma^x\}$ are related via $\lambda^u = h(u)\sigma^u$ and $\lambda^x = k(x)\sigma^x$, where $h \in C(\mathcal{G}^1)_+$ and $k \in C(\mathcal{G}^0)_+$. 
Definition

A locally compact 2-groupoid $\mathcal{G}$ is called \textit{r-discrete} if $\mathcal{G}^0 \subseteq \mathcal{G}^1$ and $\mathcal{G}^1 \subseteq \mathcal{G}^2$ are \textit{open}. 
Definition

A locally compact 2-groupoid $\mathcal{G}$ is called $r$-discrete if $\mathcal{G}^0 \subseteq \mathcal{G}^1$ and $\mathcal{G}^1 \subseteq \mathcal{G}^2$ are open.

Lemma

If $\mathcal{G}$ is $r$-discrete, then
(i) for each $u \in \mathcal{G}^1$ and $x \in \mathcal{G}^0$, $\mathcal{G}^u$ and $\mathcal{G}^x$ are open in $\mathcal{G}^2$,
(ii) if a continuous 2-Haar system exists, it is essentially the system of counting measures. In this case, $d, r : \mathcal{G}^2 \to \mathcal{G}^1$, $d, r : \mathcal{G}^1 \to \mathcal{G}^0$, and $d^2, r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ are local homeomorphisms.
Definition

Let $\mathcal{G}$ be a locally compact 2-groupoid. A subset $s$ of $\mathcal{G}^2$ is called a $\mathcal{G}^1$-set if the restrictions of $d$ and $r$ to $s$ are one-to-one. This is equivalent to $s \cdot_v s^{-1}$ and $s^{-1} \cdot_v s$ being contained in $\mathcal{G}^1$. A subset $s$ of $\mathcal{G}^2$ is called a $\mathcal{G}^0$-set if the restrictions of $d^2$ and $r^2$ to $s$ are one-to-one, or equivalently $s \cdot_h s^{-1}$ and $s^{-1} \cdot_h s$ are contained in $\mathcal{G}^0$. 
Definition

Let $\mathcal{G}$ be a locally compact 2-groupoid. A subset $s$ of $\mathcal{G}^2$ is called a $\mathcal{G}^1$-set if the restrictions of $d$ and $r$ to $s$ are one-to-one. This is equivalent to $s \cdot_v s^{-1}$ and $s^{-1} \cdot_v s$ being contained in $\mathcal{G}^1$. A subset $s$ of $\mathcal{G}^2$ is called a $\mathcal{G}^0$-set if the restrictions of $d^2$ and $r^2$ to $s$ are one-to-one, or equivalently $s \cdot_h s^{-1}$ and $s^{-1} \cdot_h s$ are contained in $\mathcal{G}^0$.

In the above definition the products are considered as products of sets. Note that both $\mathcal{G}^1$-sets and $\mathcal{G}^0$-sets form an inverse semigroup, and for each $a \in \mathcal{G}^2$ and $\mathcal{G}^1$-set $s$, if $d(a) \in r(s)$ (resp. $r(a) \in d(s)$) then the set $a \cdot_v s$ (resp. $s \cdot_v a$) is a singleton, and so defines an element of $\mathcal{G}^2$ denoted again by $a \cdot_v s$ (resp. $s \cdot_v a$). Also there is a map $r(s) \rightarrow d(s); u \mapsto u \cdot s := d(u \cdot_v s)$, satisfying $u \cdot (s \cdot_v t) = (u \cdot s) \cdot_v t$, for $\mathcal{G}^1$-sets $s, t$. Similarly, for $a \in \mathcal{G}^2$ and $\mathcal{G}^0$-set $s$ with $d^2(a) \in r^2(s)$ (resp. $r^2(a) \in d^2(s)$) the element $a \cdot_h s$ (resp. $s \cdot_v a$) of $\mathcal{G}^2$ is defined, and the map $r^2(s) \rightarrow d^2(s); x \mapsto x \cdot s := d^2(x \cdot_h s)$, satisfies $x \cdot (s \cdot_h t) = (x \cdot s) \cdot_h t$, for $\mathcal{G}^0$-sets $s, t$. 
Proposition

For a locally compact 2-groupoid $\mathcal{G}$, the following are equivalent:

(i) $\mathcal{G}$ is r-discrete and has a continuous left 2-Haar system,

(ii) The maps $r : \mathcal{G}^2 \to \mathcal{G}^1$ and $r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ are local homeomorphisms,

(iii) The product maps $\mathcal{G}^{(1)} \to \mathcal{G}^1$, $\mathcal{G}^{(2v)} \to \mathcal{G}^1$ and $\mathcal{G}^{(2h)} \to \mathcal{G}^0$ are local homeomorphisms,

(iv) $\mathcal{G}^2$ has an open basis consisting of open $\mathcal{G}^1$-sets and one consisting of open $\mathcal{G}^0$-sets.
Associated measures

Let $\mathcal{G}$ be a locally compact 2-groupoid with continuous left 2-Haar system $\{\lambda^u_v\}$ and $\{\lambda^x_h\}$, let $\{\lambda^u_{vu}\}$ and $\{\lambda^x_{hx}\}$ be the images of this system under the inverse maps $a \mapsto a^{-v}$ and $a \mapsto a^{-h}$. Then the latter is a continuous right 2-Haar system. Borel measures $\mu^1$ and $\mu^0$ on $\mathcal{G}^1$ and $\mathcal{G}^0$ induce measures

$$\nu_v = \int \lambda^u_v d\mu^1(u), \quad \nu_h = \int \lambda^x_h d\mu^0(x)$$

with images

$$\nu^{-1}_v = \int \lambda^u_{vu} d\mu^1(u), \quad \nu^{-1}_h = \int \lambda^x_{hx} d\mu^0(x)$$

and induced measures

$$\nu^2_v = \int \lambda^u_v \times \lambda^u_{vu} d\mu^1(u), \quad \nu^2_h = \int \lambda^x_h \times \lambda^x_{hx} d\mu^0(x).$$
Definition

The Borel measure $\mu^1$ on $\mathcal{G}^1$ is called \textit{quasi-invariant} if $\nu_v \sim \nu_v^{-1}$. The Borel measure $\mu^0$ on $\mathcal{G}^0$ is called \textit{quasi-invariant} if $\nu_h \sim \nu_h^{-1}$. 
Definition

The Borel measure $\mu^1$ on $\mathcal{G}^1$ is called quasi-invariant if $\nu_v \sim \nu_v^{-1}$. The Borel measure $\mu^0$ on $\mathcal{G}^0$ is called quasi-invariant if $\nu_h \sim \nu_h^{-1}$.

From the uniqueness of the Radon-Nikodym derivative we have the following result which defines vertical and horizontal modular functions. We put $\nu_{v0} = D_v^{\frac{1}{2}} \nu_v$ and $\nu_{h0} = D_h^{\frac{1}{2}} \nu_h$. 
Proposition

For quasi-invariant measure $\mu^1$ on $G^1$, there is a locally $\nu_v$-integrable positive function $D_v$ such that $\nu_v = D_v \nu_v^{-1}$ and

(i) $D_v(a \cdot_v b) = D_v(a) D_v(b) (\nu_v^2 - a.e)$, $D_v(a^{-v}) = D_v(a)^{-1} (\nu_v - a.e)$,

(ii) if $\mu'^1 = g^1 \mu^1$ where $g^1$ is positive and locally $\mu^1$-integrable then $D'_v = (g^1 \circ r) D_v (g^1 \circ d)^{-1}$ satisfies $\nu'_v = D'_v \nu'_v^{-1}$.

Similarly, for quasi-invariant measure $\mu^0$ on $G^0$, there is a locally $\nu_v$-integrable positive function $D_h$ such that $\nu_h = D_h \nu_h^{-1}$ and relations similar to (i) and (ii) above hold.
Non singular units

For locally compact topological spaces $X$ and $Y$ and surjective map $p : X \to Y$, a measure class $\mathcal{C}$ on $X$ and (probability) measure $\mu \in \mathcal{C}$, $p_\ast \mathcal{C}$ is the measure class of $p_\ast \mu := \mu \circ p^{-1}$. A pseudo-image of $\mu \in \mathcal{C}$ is a measure in $p_\ast \mathcal{C}$. If $(X, \mu)$ and $(Y, \nu)$ are measure spaces and $s : X \to Y; x \mapsto x \cdot s$ is a bi-measurable bijection, then $\mu$ lifts to a measure $\mu \cdot s$ on $Y$ defined by

$$\int f(y) d(\mu \cdot s)(y) = \int f(x \cdot s) d\mu(x) \quad (f \in C_c(Y))$$

and when $\mu \cdot s \ll \nu$ we denote the corresponding Radon-Nikodym derivative by $d(\mu \cdot s)/d\nu$ and say that $s$ is non singular if it induces an isomorphism of the corresponding measure algebras.
Ergodic measures

For quasi-invariant measures $\mu^1$ and $\mu^0$, subsets $A^1 \subseteq G^1$ and $A^0 \subseteq G^0$ are called almost invariant if $r(a) \in A^1$ is equivalent to $d(a) \in A^1$ ($\nu_v$-a.e.) and $r^2(a) \in A^0$ is equivalent to $d^2(a) \in A^0$ ($\nu_h$-a.e.). The measures $\mu^1$ and $\mu^0$ are called ergodic if every almost invariant set is null or co-null.

For arbitrary Borel measures $\mu^1$ and $\mu^0$, the pseudo-images $[\mu^1]$ and $[\mu^0]$ of $\nu_v$ and $\nu_h$ under $d$ and $d^2$ are quasi-invariant and in the same measure class as $\mu^1$ and $\mu^0$ if and only if $\mu^1$ and $\mu^0$ are quasi-invariant.

If $\alpha^u_v$ and $\alpha^x_h$ are a pseudo-images of $\lambda^u_v$ and $\lambda^x_h$ then the measure class of $\alpha^u_v$ and $\alpha^x_h$ depend only on the orbits of $u$ and $x$ in $G^1$ and $G^0$ and $\alpha^u_v$ and $\alpha^x_h$ are ergodic, and every quasi-invariant pair carried by the orbits of $u$ and $x$ are equivalent to $\alpha^u_v$ and $\alpha^x_h$. 
Modular functions

Let $\mu^1$ be a Borel measure on $G^1$ with induced measure $\nu_v$ and $s$ be a $\nu_v$-measurable $G^1$-set. The measure $\nu_v$ is called quasi-invariant under $s$ if the map $a \mapsto a \cdot_v s^{-v}$ is non singular from $(d^{-1}(d(s)), \nu_v)$ to $(d^{-1}(r(s)), \nu_v)$. Let $\delta_v(\cdot, s) = d(\nu_v \cdot s^{-v})/d\nu_v$ be the corresponding Radon-Nikodym derivative. The measure $\mu^1$ is called quasi-invariant under $s$ if the map $u \mapsto u \cdot s^{-v}$ is non singular from $d(s), \mu^1)$ to $r(s), \mu^1$ and $\Delta_v(\cdot, s) = d(\mu^1 \cdot s^{-v})/d\mu^1$ is the corresponding Radon-Nikodym derivative. For a Borel measure $\mu^0$ on $G^0$, The horizontal functions $\delta_h$ and $\Delta_h$ are defined similarly.
Lemma

Under the above quasi-invariance properties,

(i) $\delta_v(s(a), s) = \delta_v(a, s)$ (\(\nu_v\)-a.e. \(a \in d^{-1}(r(s))\)),
(ii) $\delta_v(u, s) = D_v(u \cdot s)\Delta_v(u, s)$ (\(\mu^1\)-a.e. \(u \in r(s)\)),

and the same for \(\delta_h\) and \(\Delta_h\).
Invariant sets

A $\mathcal{G}^1$-set $s$ is said to be Borel (continuous) if the restrictions of $d$ and $r$ to $s$ are Borel isomorphisms (homeomorphisms) onto a Borel (open) subset of $G^1$. It is called non singular if there is a Borel (continuous) positive function $\delta_v(\cdot, s)$ on $r(s)$, bounded above and below on compact subsets of $\mathcal{G}^1$, such that $\delta_v(d(a), s) = d\left(\lambda_v^u \cdot s^{-v}\right)/d\lambda_v^u(a)$ for every $u \in \mathcal{G}^1$ and $\lambda_v^u$-a.e. $a \in d^{-1}(r(s))$. A non singular Borel $\mathcal{G}^1$-set $s$ is also non singular with respect to the induced measure $\nu_v$ of any Borel measure $\mu^1$ on $\mathcal{G}^1$ and $\delta_v(d(a), s) = d\left(\nu_v \cdot s^{-v}\right)/d\nu_v(a)$ for $\nu_v$-a.e. $a \in d^{-1}(r(s))$. The set of non singular Borel $\mathcal{G}^1$-sets also form an inverse semigroup and

$$\delta_v(u, s \cdot v t) = \delta_v(u, s)\delta_v(u \cdot s, t) \quad (u \in r(s \cdot v t)), $$

$$ddv(u, s^{-v}) = \delta_v(u \cdot s^{-v}, s)^{-1} \quad (u \in d(s)).$$
Notation

Let $\mathcal{G}$ be a locally compact 2-groupoid with a fixed continuous left 2-Haar system $\{\lambda^u_v\}$ and $\{\lambda^x_h\}$, for $f, g \in C_c(\mathcal{G})$ put

$$f \ast_v g(a) = \int f(a \cdot_v b)g(b^{-v})d\lambda^d_v(a)(b), \quad f \ast_v (a) = \overline{f}(a^{-v}),$$

and

$$f \ast_h g(a) = \int f(a \cdot_h b)g(b^{-h})d\lambda^d_h(a)(b), \quad f \ast_h (a) = \overline{f}(a^{-h}),$$

for each $a \in \mathcal{G}^2$. 
Lemma

$C_c(\mathcal{G})$ is a topological $*$-algebra with respect to both of the vertical and horizontal convolutions and involutions, denoted by $C_{cv}(\mathcal{G})$ and $C_{ch}(\mathcal{G})$, respectively.
Representation

A representation of $C_{cv}(G)$ on a Hilbert space $H$ is a $*$-homomorphism $L : C_{cv}(G) \to B(H)$ which is continuous in the inductive limit topology on the domain and weak operator topology on the range. We have the same definition for representations of $C_{ch}(G)$. We only work with non-degenerate representations.
Representation

A representation of $C_{cv}(G)$ on a Hilbert space $H$ is a *-homomorphism $L : C_{cv}(G) \to B(H)$ which is continuous in the inductive limit topology on the domain and weak operator topology on the range. We have the same definition for representations of $C_{ch}(G)$. We only work with non-degenerate representations.

Boundedness

For $f \in C_{cv}(G)$ put

$$\|f\|_{v,r} = \sup_{u \in G^1} \int |f| d\lambda_v^u, \quad \|f\|_{v,d} = \sup_{u \in G^1} \int |f| d\lambda_{vu}$$

and $\|f\|_v = max\{\|f\|_{v,r}, \|f\|_{v,d}\}$. 
Boundedness

For $f \in C_{cv}(\mathcal{G})$ put

$$
\|f\|_{v,r} = \sup_{u \in \mathcal{G}^1} \int |f| d\lambda^u_v, \quad \|f\|_{v,d} = \sup_{u \in \mathcal{G}^1} \int |f| d\lambda_{vu}_v
$$

and $\|f\|_v = \max\{\|f\|_{v,r}, \|f\|_{v,d}\}$. This is a norm on $C_{cv}(\mathcal{G})$ defining a topology coarser than the inductive limit topology. We say that a representation $L$ is $v$-bounded if there is a constant $M > 0$ such that $\|L(f)\| \leq M \|f\|_v$, for each $f \in C_{cv}(\mathcal{G})$. We put $\|f\|^v = \sup_L \|L(f)\|$, where the supremum is taken over all $v$-bounded non-degenerate representations. This is a $C^*$-seminorm on $C_{cv}(\mathcal{G})$ and $\|f\|^v \leq \|f\|_v$, for each $f \in C_{cv}(\mathcal{G})$. The norms $\|f\|^h$ and $\|f\|^h$ are defined similarly on $C_{ch}(\mathcal{G})$ using $h$-bounded representations.
Definition

A vertical representation of $\mathcal{G}$ (abbreviated as v-representation) consists of a quasi-invariant Borel measure $\mu^1$ on $\mathcal{G}^1$, a $\mathcal{G}^1$-Hilbert bundle $\mathcal{H}$ over $(\mathcal{G}^1, \mu^1)$, and a map $\pi : \mathcal{G}^2 \to \text{Iso}(\mathcal{H})$ such that

(i) $\pi(a)$ is a map from $\mathcal{H}_{d(a)}$ to $\mathcal{H}_{d(a)}$ and $\pi(u) = \text{id}_{\mathcal{H}_u}$, for all $a \in \mathcal{G}^2$ and $u \in \mathcal{G}^1$,

(ii) $\pi(a \cdot_v b) = \pi(a)\pi(b)$ for $\nu_v^2$-a.e. $(a, b)$,

(iii) $\pi(a^{-v}) = \pi(a)^{-1}$ for $\nu_v$-a.e. $a$,

(iv) $a \mapsto \langle \pi(a)\xi \circ d(a), \eta \circ r(a) \rangle$ is measurable on $\mathcal{G}^2$ for all measurable sections $\xi, \eta$.

h-representations are defined similarly using Hilbert bundles over $(\mathcal{G}^0, \mu^0)$. 
Equivalence

Two $\nu$-representations $(\pi_1, \mathcal{H}_1, \mu_1^1)$ and $(\pi_2, \mathcal{H}_2, \mu_2^1)$ are equivalent if $\mu_1^1 \sim \mu_2^1$ and there is an isomorphism $\phi$ of Hilbert bundles from $\mathcal{H}_1$ onto $\mathcal{H}_2$ which intertwines $\pi_1$ and $\pi_2$, that is

$$\pi_2(a)\phi \circ d(a) = \phi \circ r(a)\pi_1(a) \text{ for } \nu_\nu\text{-a.e. } a \in G^2.$$
Equivalence

Two v-representations \((\pi_1, \mathcal{H}_1, \mu^1_1)\) and \((\pi_2, \mathcal{H}_2, \mu^1_2)\) are equivalent if \(\mu^1_1 \sim \mu^1_2\) and there is an isomorphism \(\phi\) of Hilbert bundles from \(\mathcal{H}_1\) onto \(\mathcal{H}_2\) which intertwines \(\pi_1\) and \(\pi_2\), that is

\[
\pi_2(a)\phi \circ d(a) = \phi \circ r(a)\pi_1(a) \quad \text{for } \nu\text{-a.e. } a \in G^2.
\]

Let \((\pi, \mathcal{H}, \mu^1)\) be a v-representation and \(\Gamma_v(\mathcal{H})\) be the Hilbert space of square integrable sections with respect to \(\mu^1\).
Lemma

Let \((\pi, \mathcal{H}, \mu^1)\) be a \(v\)-representation of \(\mathcal{G}\), \(f \in C_{cv}(\mathcal{G})\) and \(\xi, \eta \in \Gamma_v(\mathcal{H})\), then

\[
\langle \tilde{\pi}(f)\xi, \eta \rangle = \int f(a)\langle \pi(a)\xi \circ d(a), \eta \circ r(a) \rangle d\nu_{v0}(a)
\]

defines a \(v\)-bounded representation of \(C_{cv}(\mathcal{G})\) on \(\Gamma_v(\mathcal{H})\), and two equivalent \(v\)-representations of \(\mathcal{G}\) induce equivalent \(v\)-bounded representations of \(C_{cv}(\mathcal{G})\) as above.

When \(\text{dim}(\mathcal{H}_u)\) is constant, namely there is a Hilbert space \(H\) with \(\mathcal{H}_u \simeq H\), for all \(u \in \mathcal{G}^1\),

\[
\tilde{\pi}(f)\xi(u) = \int f(a)\pi(a)\xi \circ d(a)D_v^{\frac{1}{2}}(a)d\lambda_v^u(a),
\]

\(\mu^1\)-a.e., for \(f \in C_{cv}(\mathcal{G})\) and \(\xi \in L^2(\mathcal{G}^1, \mu^1, H)\). In general \(\tilde{\pi}\) is a direct sum of representations on constant fields over all possible dimensions. Similar statements hold for \(h\)-representations \((\pi, \mathcal{H}, \mu^0)\) and Hilbert space \(\Gamma_h(\mathcal{H})\) of square integrable sections with respect to \(\mu^0\).
Regular representation

Consider the measurable field of Hilbert spaces $L^2(\mathcal{G}^2, \lambda^u_v)$ with square integrable sections $L^2(\mathcal{G}^2, \nu_v) = \int \oplus L^2(\mathcal{G}^2, \lambda^u_v) d\mu^1(u)$ where $\mu^1$ is a quasi-invariant Borel measure on $\mathcal{G}^1$. Then

$\pi(a) : L^2(\mathcal{G}^2, \lambda^d_v(a)) \rightarrow L^2(\mathcal{G}^2, \lambda^r_v(a))$; $\pi(a)\xi(b) = \xi(a^{-v} \cdot_v b)$ is a $v$-representation of $\mathcal{G}$ and

$$a \mapsto \langle \pi(a)\xi \circ d(a), \eta \circ r(a) \rangle = \int \xi(a^{-v} \cdot_v b)\bar{\eta}(b) d\lambda^r_v(a)(b)$$

is continuous for $\xi, \eta \in \mathcal{C}_c(\mathcal{G})$ and measurable for $\xi, \eta \in L^2(\mathcal{G}^2, \nu_v)$. This is called the left regular representation of $\mathcal{G}$ with respect to $\mu^1$. Similarly we could define the left regular representation of $\mathcal{G}$ with respect to a quasi-invariant measure $\mu^0$ on $\mathcal{G}^0$. 
Lemma

The topological algebra $C_{cv}(\mathcal{G})$ has a left approximate identity in the inductive limit topology. Same holds for $C_{ch}(\mathcal{G})$. 
Modular function

The above lemma implies that \( v \)-left regular representations with respect to all quasi-invariant measures on \( G^1 \) induce a faithful family of \( v \)-bounded representations of \( C_{cv}(G) \). Also, for each quasi-invariant measure \( \mu^1 \) on \( G^1 \), \( C_{cv}(G) \) is a generalized Hilbert algebra under the inner product of \( L^2(G^2, \nu_v^{-1}) \) whose left regular representation is equivalent to the \( v \)-left regular representation with respect to \( \mu^1 \) \([7, 2.1.10]\) and by Tomita-Takesaki theory we have a modular function \( J_v : L^2(G^2, \nu_v^{-1}) \to L^2(G^2, \nu_v^{-1}) \); \( J_v \xi(a) = D_v^{\frac{1}{2}}(a) \bar{\xi}(a^{-v}) \) and the modular operator

\[
\Delta_v : L^2(G^2, \nu_v) \cap L^2(G^2, \nu_v^{-1}) \to L^2(G^2, \nu_v) \cap L^2(G^2, \nu_v^{-1});
\]

\[
\Delta_v \xi(a) = D_v(a) \xi(a).
\]

The same observations hold for \( C_{ch}(G) \).
Definition

The full vertical (resp. horizontal) $C^*$-algebra of $\mathcal{G}$ is the completion of $C_{cv}(\mathcal{G})$ (resp. $C_{ch}(\mathcal{G})$) in $\|\cdot\|_v$ (resp. $\|\cdot\|_h$).
Lemma

Let \( \{L, H\} \) be a representation of \( C_{cv}(\mathcal{G}) \), there is a unique representation \( \{L^1, H^1\} \) of \( C_c(\mathcal{G}^1) \) such that

\[
L(hf) = L^1(h)L(f), \quad L(fh) = L(f)L^1(h) \quad (h \in C_c(\mathcal{G}^1), f \in C_{cv}(\mathcal{G}))
\]

where

\[
hf(a) = h \circ r(a)f(a), \quad fh(a) = f(a)h \circ d(a) \quad (a \in \mathcal{G}^2).
\]

Moreover for \( f, g \in C_{cv}(\mathcal{G}), h \in C_c(\mathcal{G}^1) \),

\[
f \ast_v h g = fh \ast_v g, \quad hf \ast_v g = h(f \ast_v g), \quad (hf)^* = f^* h^*,
\]

where \( h^*(u) = \overline{h(u)} \), for \( u \in \mathcal{G}^1 \). There is a representation \( \{L^0, H^0\} \) of \( C_c(\mathcal{G}^0) \) with similar relations to the horizontal convolution.
Corollary

\( C^*(\mathcal{G}^1) \) and \( C^*(\mathcal{G}^0) \) are subalgebras of the multiplier algebras \( M(C^*_v(\mathcal{G})) \) and \( M(C^*_v(\mathcal{G})) \), respectively.
**Notation**

Every representation of $C_c(\mathcal{G})$ extends to a representation of $B(\mathcal{G})$ of bounded Borel functions on $\mathcal{G}^2$ with vertical or horizontal convolution. For a non singular Borel $\mathcal{G}^1$-set $s$ and $f \in B(\mathcal{G})$ we define

$s \cdot_v f(a) = \delta_{\frac{1}{2}}(r(a), s)$ for $a \in r^{-1}(r(s))$, and zero otherwise, and

$f \cdot_v s(a) = \delta_{\frac{1}{2}}(d(a), s^{-v})$ for $a \in d^{-1}(d(s))$, and zero otherwise, then

$$(s \cdot_v (t \cdot_v f)) = (st) \cdot_v f, \quad (f \cdot_v s)^*v g = f^*v (s \cdot_v g), \quad (s \cdot_v f)^*v g = s \cdot_v (f^*v g)$$

and $(f \cdot_v s)^* = s^{-v} \cdot_v f^*$, for non singular $\mathcal{G}^1$-sets $s, t$ and $f, g \in B(\mathcal{G})$. 
**Notation**

We denote $B(G)$ with vertical convolution by $B_v(G)$. Same relations hold for $B_h(G)$, that is $B(G)$ with horizontal convolution. Also we could find a unique representation $V^1$ of the Borel ample semigroup of non singular $G^1$-sets such that

$$L(s \cdot_v f) = V^1(s)L(f), \quad L(f \cdot_v s) = L(f)V^1(s), \quad V^1(s)L^1(h)V^1(s)^* = L^1(h^s),$$

for non singular $G^1$-set $s$, $f \in B_v(G)$ and $h \in C_c(G^1)$, where $h^s(u) = h(us)$ for $u \in r(s)$, and zero otherwise. Same holds for representations $L$, $L^0$ and a representation $V^0$ of the Borel ample semigroup of non singular $G^0$-sets.
Theorem

If $G$ is a locally compact second countable 2-groupoid with left 2-Haar system $\{\lambda^u_v\}$ and $\{\lambda^x_h\}$ with sufficiently many non singular $G^1$-sets (resp. $G^0$-sets) then every $v$-bounded (resp. $h$-bounded) representation of $C_{cv}(G)$ (resp. $C_{ch}(G)$) on a separable Hilbert space is the integration of a $v$-representation (resp. an $h$-representation) of $G$. 
Corollary

When $\mathcal{G}$ is second countable with sufficiently many non singular $\mathcal{G}^1$-sets (resp. $\mathcal{G}^0$-sets), every representation of $C_{cv}(\mathcal{G})$ (resp. $C_{ch}(\mathcal{G})$) on a separable Hilbert space is $v$-bounded (resp. $h$-bounded) and there is a one-to-one correspondence between $\mathcal{G}^1$-Hilbert bundles (resp. $\mathcal{G}^0$-Hilbert bundles) and separable Hermitian $C^*_v(\mathcal{G})$-modules (resp. $C^*_v(\mathcal{G})$-modules) preserving intertwining operators.
Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a locally compact 2-groupoid with left 2-Haar system $\{\lambda^u_v\}$ and $\{\lambda^x_h\}$ and $\mathcal{H} = (\mathcal{H}^2, \mathcal{H}^1, \mathcal{H}^0)$ be a closed 2-subgroupoid, that is a 2-subgroupoid such that $\mathcal{H}^i \subseteq \mathcal{G}^i$ is closed for $i = 0, 1, 2$, with left 2-Haar system $\{\sigma^u_v\}$ and $\{\sigma^x_h\}$ such that $\mathcal{G}^1 \subseteq \mathcal{H}^2$ and $\mathcal{G}^0 \subseteq \mathcal{H}^1$. For the equivalence relations $a \sim_v b$ iff $d(a) = r(b)$ and $a \cdot_v b \in \mathcal{H}^2$ and $a \sim_h b$ iff $d^2(a) = r^2(b)$ and $a \cdot_h b \in \mathcal{H}^2$, for $a, b \in \mathcal{G}^2$, the quotient space $\mathcal{H}\backslash\mathcal{G}$ is Hausdorff and locally compact and the quotient map: $\mathcal{G} \rightarrow \mathcal{H}\backslash\mathcal{G}$ is open. Also there are continuous open surjections from the quotient spaces to $\mathcal{G}^1$ and $\mathcal{G}^0$ induced by $d$ and $d^2$, respectively.
Lemma

There are **Bruhat** approximate vertical and horizontal **cross-sections** for \( \mathcal{G} \) over \( \mathcal{H} \backslash \mathcal{G} \), that is non-negative continuous functions \( b_v, b_h \) on \( \mathcal{G} \) whose supports have compact intersections respectively with \( \mathcal{H}^2 \cdot_v K \) and \( \mathcal{H}^2 \cdot_h K \) for each compact subset \( K \) of \( \mathcal{G}^2 \) such that

\[
\int b_v(c^{-v} \cdot_v a) d\sigma^r_v(a)(c) = 1, \quad \int b_h(c^{-h} \cdot_h a) d\sigma^r_h(a)(c) = 1,
\]

for each \( a \in \mathcal{G}^2 \).
Quotients

Consider equivalence relations on $G^{(2v)}$ and $G^{(2h)}$, $(a_1, b_1) \sim_v (a_2, b_2)$ iff $b_1 = b_2$ and $a_1 \cdot_v a_2^{-v} \in H^2$ and $(a_1, b_1) \sim_h (a_2, b_2)$ iff $b_1 = b_2$ and $a_1 \cdot_h a_2^{-h} \in H^2$, then the quotient spaces $H \backslash G^{(2v)}$ and $H \backslash G^{(2h)}$ are locally compact 2-groupoids with set of 1-morphisms $H \backslash G^1$ and $H \backslash G^0$ with left 2-Haar systems $\{\delta_{\dot{a}} \times \lambda_v^{d(\dot{a})}\}$ and $\{\delta_{\dot{a}} \times \lambda_h^{d^2(\dot{a})}\}$ with $a$ ranging respectively over $H \backslash G^{(2v)}$ and $H \backslash G^{(2h)}$. 
Imprimitivity modules

For $\varphi \in C_c(\mathcal{H})$ and $f \in C_c(\mathcal{G})$,

$$\varphi \cdot_v f(a) = \int \varphi(c)f(c^{-v} \cdot_v a) d\sigma^r_v(a)(c),$$

$$f \cdot_v \varphi(a) = \int f(a \cdot_v c)\varphi(c^{-v}) d\sigma^d_v(a)(c),$$

and

$$\varphi \cdot_h f(a) = \int \varphi(c)f(c^{-h} \cdot_h a) d\sigma^r_h(a)(c),$$

$$f \cdot_h \varphi(a) = \int f(a \cdot_h c)\varphi(c^{-h}) d\sigma^d_h(a)(c),$$

for $a \in \mathcal{G}^2$. 
Imprimitivity modules

Also for $\phi \in C_c(\mathcal{H}\backslash G^{(2v)})$, $\psi \in C_c(\mathcal{H}\backslash G^{(2v)})$ and $f \in C_c(G)$,

$$
\phi \cdot_v f(a) = \int \phi(\dot{a}^{-v}, a \cdot_v b)f(b^{-v})d\lambda_{v}^{d(a)}(b),
$$

$$
f \cdot_v \phi(a) = \int f(b)\phi(\dot{b}, b^{-v} \cdot_v a)d\lambda_{v}^{r(a)}(b),
$$

and

$$
\psi \cdot_v f(a) = \int \psi(\dot{a}^{-h}, a \cdot_h b)f(b^{-h})d\lambda_{h}^{d^2(a)}(b),
$$

$$
f \cdot_h \psi(a) = \int f(b)\psi(\dot{b}, b^{-h} \cdot_h a)d\lambda_{h}^{r^2(a)}(b),
$$

for $a \in G^2$. 
Imprimitivity modules

Then $X_v := C_{cv}(G)$ is a bimodule over $B_v := C_{c,v}(H)$ and $E_v := C'_{c,v}(H \backslash G^{(2v)})$ with commuting actions on opposite sides and the action of $C_{c,v}(H)$ as double centralizers on $C_{cv}(G)$ extends to an action on $C_v^*(G)$, giving a $*$-homomorphism of $C_{c,v}(H)$ into the multiplier algebra $M(C_v^*(G))$, and the same holds for $C_{ch}(G)$.
Imprimitivity modules

Consider $X_v$ as a left $E_v$-module and right $B_v$-module with the following vector valued inner products

$$\langle f, g \rangle_{B_v}(c) = \int \bar{f}(a^{-v})g(a^{-v} \cdot_v c) d\lambda_r^v(c)(a)$$

and

$$\langle f, g \rangle_{E_v}(a, a^{-v} \cdot_v b) = \int f(a^{-v} \cdot_v c) \bar{g}(b \cdot_v c) d\sigma_r^v(a)(c),$$

for $c \in \mathcal{H}^2$, $a, b \in \mathcal{G}^2$. Then

$$\langle f, gh \rangle_{B_v} = \langle f, g \rangle_{B_v} h, \quad \langle ef, g \rangle_{B_v} = \langle f, e^* g \rangle_{B_v},$$

and

$$\langle ef, g \rangle_{E_v} = e \langle f, g \rangle_{E_v}, \quad \langle f, gh \rangle_{E_v} = \langle fh^*, g \rangle_{E_v},$$

for $f, g \in X_v$, $h \in B_v$ and $e \in E_v$, and $f_1 \langle g, f_2 \rangle_{B_v} = \langle f_1, g \rangle_{E_v} f_2$, for $f_1, f_2, g \in X_v$. The same holds for the horizontal spaces and modules.
Lemma

The linear span of \( \{ \langle f, g \rangle_{E_v} : f, g \in X_v \} \) contains a left approximate identity for \( E_v \) in the inductive limit topology and is dense in \( E_v \) and \( C^*_v(\mathcal{H}\backslash\mathcal{G}^{(2v)}) \). Similarly the linear span of \( \{ \langle f, g \rangle_{B_v} : f, g \in X_v \} \) is dense in \( B_v \) and \( C^*_v(\mathcal{H}) \). Same holds for \( E_h \) \( B_h \).
Corollary

The $C^*$-algebras $C^*_v(\mathcal{G}(2^\mathbb{V}))$ and $C^*_v(\mathcal{G}^1)$ are strongly Morita equivalent. Similarly, $C^*_h(\mathcal{G}(2^\mathbb{H}))$ and $C^*_v(\mathcal{G}^0)$ are strongly Morita equivalent.
Conditional expectation

Now by Rieffel construction, each v-representation of $C_v^*(\mathcal{G}^1)$ induces a v-representation of $C_v^*(\mathcal{G}^{(2v)})$ and then restricts to a v-representation of $C_v^*(\mathcal{G})$ which acts on $C_v^*(\mathcal{G}^{(2v)})$ as double centralizers, in other words, the restriction map $P_v : C_{c,v}(\mathcal{G}) \to C_{c,v}(\mathcal{G}^1)$ is a generalized conditional expectation in the sense of (Rieffel, 1974). Similarly we have a generalized conditional expectation $P_h : C_{c,h}(\mathcal{G}) \to C_{c,h}(\mathcal{G}^0)$.
Conditional expectation

Now by Rieffel construction, each $v$-representation of $C_v^*(\mathcal{G}^1)$ induces a $v$-representation of $C_v^*(\mathcal{G}^{2v})$ and then restricts to a $v$-representation of $C_v^*(\mathcal{G})$ which acts on $C_v^*(\mathcal{G}^{2v})$ as double centralizers, in other words, the restriction map $P_v : C_{c,v}(\mathcal{G}) \to C_{c,v}(\mathcal{G}^1)$ is a generalized conditional expectation in the sense of (Rieffel, 1974). Similarly we have a generalized conditional expectation $P_h : C_{c,h}(\mathcal{G}) \to C_{c,h}(\mathcal{G}^0)$.

More generally, if $\mathcal{G}$ is second countable and $\mathcal{H}$ is a closed 2-subgroupoid such that both $\mathcal{G}$ and $\mathcal{H}$ have sufficiently many non singular Borel sets, the restriction map from $C_{cv}(\mathcal{G})$ to $C_{cv}(\mathcal{H})$ is a generalized conditional expectation, and the same for $C_{ch}(\mathcal{G})$. 
Induced representation

For the representation of $C^*_v(G^1)$ given by multiplication on $L^2(G^1, \mu^1)$ the induced representation $\text{Ind} \mu^1$ acts on $L^2(G^1, \nu^{-1}_v)$ by convolution on the left, namely

$$\langle \text{Ind} \mu^1(f) \xi, \eta \rangle = \int \int \int f(a \cdot_v b) \xi(b^{-v}) \bar{\eta}(a) d\lambda^u_v(b) \lambda_{v,u}(a) d\mu^1(u),$$

for $f \in C_{cv}(G)$ and $\xi, \eta \in L^2(G^1, \nu^{-1}_v)$. When $\mu^1$ is quasi-invariant, $\text{Ind} \mu^1$ is just the left regular representation on $\mu^1$. In this case, $\ker(\text{Ind} \mu^1)$ consists of those $f \in C_{cv}(G)$ that $f = 0$ on $\text{supp}(\nu^{-1}_v)$. Since $G^1$ has a faithful family of quasi-invariant measures, $C_{cv}(G)$ has a faithful family of $v$-bounded representations (consisting of induced representations of such quasi-invariant measures).
In particular, $\|f\|_{\text{red}}^\vee := \sup_{\mu^1} \|\text{Ind}_{\mu^1}(f)\|$ is a $C^*$-norm, where $\mu^1$ ranges over all quasi-invariant Borel measures on $G^1$, and $\|f\|_{\text{red}}^\vee \leq \|f\|^{\vee}$, for each $f \in C^0_c(G)$. Similarly $\|f\|_{\text{red}}^\h := \sup_{\mu^0} \|\text{Ind}_{\mu^0}(f)\| \leq \|f\|^\h$ is a $C^*$-norm, where $\mu^0$ ranges over all quasi-invariant Borel measures on $G^0$. The completions $C^*_{\text{v,red}}(G)$ and $C^*_{\text{h,red}}(G)$ of $C^0_c(G)$ and $C^*_c(G)$ with respect to these $C^*$-norms are called the vertical and horizontal reduced $C^*$-algebras of $G$, which are quotients of the vertical and horizontal full $C^*$-algebras $C^*_v(G)$ and $C^*_h(G)$ of $G$. 
Proposition

If a second countable locally compact groupoid \( G \) has two 2-Haar systems \( \{ \lambda^{u}_{v} \}, \{ \lambda^{x}_{h} \} \) and \( \{ \sigma^{u}_{v} \}, \{ \sigma^{x}_{h} \} \) and it has sufficiently many nonsingular Borel \( G^{1} \)-sets (resp. \( G^{0} \)-sets) with respect to both systems, then the corresponding \( C^{*} \)-algebras \( C^{*}_{v}(G, \lambda) \) and \( C^{*}_{v}(G, \sigma) \) (resp. \( C^{*}_{h}(G, \lambda) \) and \( C^{*}_{h}(G, \sigma) \)) are strongly Morita equivalent.
we describe the reduced $C^*$-algebras of \textit{r}-discrete principal 2-groupoids and find their \textit{ideals} and \textit{masa's}.
we describe the reduced $C^*$-algebras of \textit{r-discrete principal} 2-groupoids and find their ideals and masa’s.

\textbf{Lemma}

Let $\mathcal{G}$ be an \textit{r-discrete} 2-groupoids with 2-Haar system and $a \in \mathcal{G}^2$. Let $L = \text{Ind}\mu^1$ (resp. $L = \text{Ind}\mu^0$) be the representation of $C_{cv}(\mathcal{G})$ (resp. $C_{ch}(\mathcal{G})$) induced by the point mass $\mu^1 = \delta_{d(a)}$ (resp. $\mu^0 = \delta_{d^2(a)}$), then for every $f \in C_{cv}(\mathcal{G})$ (resp. $f \in C_{ch}(\mathcal{G})$),

$$f(a) = \langle L(f)\delta_u, \delta_a \rangle = L(f)\delta_u(a),$$

where $u = d(a)$ (resp. $u = x := d^2(a)$) and $\delta_u, \delta_a$ are regarded as unit vectors in $L^2(\mathcal{G}, \lambda_{vu})$ (resp. in $L^2(\mathcal{G}, \lambda_{hx})$). In particular, $\max\{\|f\|_\infty, \|f\|_2\} \leq \|f\|_{red}^v$ (resp. the same for $\|f\|_{red}^h$) where $\|\cdot\|_2$ is the norm in $L^2(\mathcal{G}, \lambda_{vu})$ (resp. in $L^2(\mathcal{G}, \lambda_{hx})$).
GNS-representation

Now the inclusion map $j_v : C_{cv}(\mathcal{G}) \to C_0(\mathcal{G})$ extends to a norm decreasing linear map $j_v : C_{v,red}^*(\mathcal{G}) \to C_0(\mathcal{G})$. Let us observe that the latter map is still injective: consider the surjection $p : C_{cv}(\mathcal{G}) \to C_c(\mathcal{G}^1)$, for a quasi-invariant probability measure $\mu^1$ on $\mathcal{G}^1$, the induced representation $\text{Ind}_{\mu^1}$ is the GNS-representation of $\mu^1 \circ p$, namely $\int p(f) d\mu^1 = \langle \text{Ind}_{\mu^1}(f) \xi_0, \xi_0 \rangle$ and $\text{Ind}_{\mu^1}(f) \xi_0 = f * v \xi_0 = j_v(f)$ where $\xi_0 \in L^2(\mathcal{G}, \nu_v^{-1})$ is the characteristic function of $\mathcal{G}^1$ and $j_v$ is now considered as the inclusion from $C_{cv}(\mathcal{G})$ into $L^2(\mathcal{G}, \nu_v^{-1})$, now the above lemma shows that $\text{Ind}_{\mu^1}(g) \xi_0 = j_v(g)$ remains valid for $g \in C_{v,red}^*(\mathcal{G})$ and if $j_v(g) = 0$ then $\text{Ind}_{\mu^1}(g) = 0$ as $\xi_0$ is a cyclic vector, and this, being true for all quasi-invariant probability measures $\mu^1$ on $\mathcal{G}^1$, implies that $g = 0$. Also $\|g\|_\infty \leq \|g\|_{v,\text{red}}$, where on the left hand side $g$ is regarded as a continuous function on $\mathcal{G}$. The same observations hold for $C_{h,\text{red}}^*(\mathcal{G})$. 

Massoud Amini
A 2-groupoid $\mathcal{G}$ is called essentially $v$-principal (resp. $h$-principal), if for every invariant closed subset $F$ of $\mathcal{G}^1$ (resp. $\mathcal{G}^0$) the set of $u \in F$ (resp. $x \in F$) whose isotropy group $\mathcal{G}_u^v$ (resp. $\mathcal{G}_x^x$) is a singleton, is dense in $F$. It is called essentially principal, if for every invariant closed subset $F$ of $\mathcal{G}^0$ the set of $x \in F$ whose isotropy groupoid $\mathcal{G}(x)$ is a singleton, is dense in $F$. 
Lemma

Let $\mathcal{G}$ be an $r$-discrete essentially v-principal (resp. h-principal) 2-groupoids with 2-Haar system and $a \in \mathcal{G}^2$. For any quasi-invariant measure $\mu^1$ on $\mathcal{G}^1$ (resp. $\mu^0$ on $\mathcal{G}^0$) with support $F$, any v-representation (resp. h-representation) $\pi$ on $\mu^1$ (resp. $\mu^0$), and any $f \in C_{cv}(\mathcal{G})$ (resp. $f \in C_{ch}(\mathcal{G})$) we have $\sup_F f \leq \|\tilde{\pi}(f)\|$. 
Correspondence

Let $\mathcal{G}$ be a locally compact groupoid with 2-Haar system. For an invariant open subset $U$ of $\mathcal{G}^1$ (resp. $G^0$) let
\[
I_{cv}(U) = \{ f \in C_{cv}(\mathcal{G}) : f(u) = 0 \ (u \notin \mathcal{G}_U) \} \quad \text{(resp.} \quad I_{ch}(U) = \{ f \in C_{ch}(\mathcal{G}) : f(x) = 0 \ (x \notin \mathcal{G}_U) \})
\]
and $I_v(U)$ (resp. $I_h$) be its closure. Let $F$ be the complement of $U$ in $\mathcal{G}^1$ (resp. $G^0$) then it follows from [7, 2.4.5] that $I_v(U)$ (resp. $I_h$) is isomorphic to $C_{v,red}^*(\mathcal{G}_U)$ (resp. $C_{h,red}^*(\mathcal{G}_U)$), and it is a closed ideal of $C_{v,red}^*(\mathcal{G})$ (resp. $C_{h,red}^*(\mathcal{G})$) whose quotient is isomorphic to $C_{v,red}^*(\mathcal{G}_F)$ (resp. $C_{h,red}^*(\mathcal{G}_F)$). If $\mu^1$ (resp. $\mu^0$) is a quasi-invariant measure on $\mathcal{G}^1$ (resp. on $\mathcal{G}^0$) with support $F$, $U$ is the complement of $F$, then $I_v(U) = ker(\text{Ind}\mu^1)$ (resp. $I_h = ker(\text{Ind}\mu^0)$). This provides a one-to-one correspondence between invariant open subsets of $\mathcal{G}^1$ (resp. $G^0$) and a family of closed ideals of $C_{v,red}^*(\mathcal{G})$ (resp. $C_{h,red}^*(\mathcal{G})$). Both sets are a lattice with respect to inclusion.
When $\mathcal{G}$ is $r$-discrete and essentially $v$-principal (resp. $h$-principal), the above correspondence is an order preserving bijection, namely all closed ideals of $C^*_{v, red}(\mathcal{G})$ (resp. $C^*_{h, red}(\mathcal{G})$) are of the form $I_v(U)$ (resp. $I_h$) for some invariant open subset $U$ of $\mathcal{G}^1$ (resp. $\mathcal{G}^0$) and the correspondence $U \mapsto I_v(U)$ (resp. $I_h$) preserves inclusion. Indeed, in this case, the surjection $p$ defined above is a conditional expectation and $Ind\mu^1$ (resp. $Ind\mu^0$) is the GNS-representation of $\mu^1 \circ p$ (resp. $\mu^0 \circ p$) and so $\|Ind\mu^1(f)\| \leq \|\tilde{\pi}(f)\|$ for $f \in C_{cv}(\mathcal{G})$ (resp. $\|Ind\mu^0(f)\| \leq \|\tilde{\pi}(f)\|$ for $f \in C_{ch}(\mathcal{G})$) hence $ker(\tilde{\pi})$ is equal to $I_v(U)$ (resp. $I_h$) where $U$ is the complement of the support of $\mu^1$ (resp. $\mu^0$).
Lemma

Let $\mathcal{G}$ be an $r$-discrete with 2-Haar system. An element $g$ of $C^*_v,\text{red}(\mathcal{G})$ (resp. $C^*_h,\text{red}(\mathcal{G})$) commutes with each element of $C^*_v(\mathcal{G}^1)$ (resp. $C^*_h(\mathcal{G}^0)$) iff it vanishes off the isotropy group bundle $\bigcup_{u \in \mathcal{G}^1} \mathcal{G}^u_u$ (resp. $\bigcup_{x \in \mathcal{G}^0} \mathcal{G}^x_x$).
Corollary

If $\mathcal{G}$ is an $r$-discrete with 2-Haar system, $C_v^*(\mathcal{G}^1)$ (resp. $C_h^*(\mathcal{G}^0)$) is a masa in $C_{v,\text{red}}^*(\mathcal{G})$ (resp. $C_{h,\text{red}}^*(\mathcal{G})$) iff $\mathcal{G}^1$ (resp. $\mathcal{G}^0$) is the interior of the isotropy group bundle $\bigcup_{u \in \mathcal{G}^1} \mathcal{G}^1_u$ (resp. $\bigcup_{x \in \mathcal{G}^0} \mathcal{G}^0_x$).
Ample semigroup

In the above corollary, if moreover $\mathcal{G}$ is essentially v-principal (resp. h-principal), the restriction map $p : C^*_{v,\text{red}}(\mathcal{G}) \to C^*_v(\mathcal{G}^1)$ (resp. $p : C^*_{h,\text{red}}(\mathcal{G}) \to C^*_h(\mathcal{G}^0)$) is a faithful surjective conditional expectation and there is a one-to-one correspondence between the ample semigroup of compact open $\mathcal{G}^1$-sets (resp. $\mathcal{G}^1$-sets) and the inverse semigroup of partial homeomorphisms of $C^*_v(\mathcal{G}^1)$ (resp. $C^*_h(\mathcal{G}^0)$) defined by conjugation with respect to the elements in the normalizer of $C^*_v(\mathcal{G}^1)$ (resp. $C^*_h(\mathcal{G}^0)$) in $C^*_{v,\text{red}}(\mathcal{G})$ (resp. $C^*_{h,\text{red}}(\mathcal{G})$) (c.f. [7, 2.4.8]).


