C*-algebras of 2-groupoids

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Abstract

We define topological 2-groupoids and study locally compact 2-groupoids with 2-Haar systems. We consider quasi-invariant measures on the sets of 1-arrows and unit space and build the corresponding vertical and horizontal modular functions. For a given 2-Haar system we construct the vertical and horizontal full C^*-algebras of a 2-groupoid and show that its is unique up to strong Morita equivalence, and make a correspondence between their bounded representations on Hilbert spaces and those of the 2-groupoid on Hilbert bundles.

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We show that representations of certain closed 2-subgroupoids are induced to representations of the 2-groupoid and use regular representation to build the vertical and horizontal reduced C^*-algebras of the 2-groupoid. We establish strong Morita equivalence between C^*-algebras of the 2-groupoid of composable pairs and those of the 1-arrows and unit space. We describe the reduced C^*-algebras of r-discrete principal 2-groupoids and find their ideals and masas.

Motivation

In noncommutative geometry, certain quotient spaces are described by non-commutative C^*-algebras, when the symmetry groups of such quotient spaces are non Hausdorff, it is more appropriate to describe such symmetry groups and groupoids using crossed modules of groupoids (Buss-Meyer-Zhu, 2012).

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In noncommutative geometry, certain quotient spaces are described by non-commutative C^*-algebras, when the symmetry groups of such quotient spaces are non Hausdorff, it is more appropriate to describe such symmetry groups and groupoids using crossed modules of groupoids (Buss-Meyer-Zhu, 2012).

One motivating example is the gauge action on the irrational rotation algebra A_ϑ, which is the universal C*-algebra generated by two unitaries U and V satisfying the commutation relation $UV = \lambda VU$ with $\lambda := \exp(2\pi i\vartheta)$. Since A_ϑ is the crossed product $C(\mathbb{T}) \rtimes_\lambda \mathbb{Z}$, for the canonical action of $\mathbb{Z}$ on $\mathbb{T}$ by $n \cdot z := \lambda^n \cdot z$, it could be viewed as the noncommutatove analog of the non Hausdorff quotient space $\mathbb{T}/\lambda^{\mathbb{Z}}$. This latter group acts on itself by translations, thus $\mathbb{T}/\lambda^{\mathbb{Z}}$ is a symmetry group of A_ϑ.

Motivation

More generally, one may define actions of crossed modules on C*-algebras similar to the twisted actions in the sense of Philip Green (Green, 1978) and build crossed products for such actions. The resulting crossed product is functorial: If two actions are equivariantly Morita equivalent in a suitable sense, their crossed products are Morita–Rieffel equivalent C*-algebras.

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Crossed modules of discrete groups are used in homotopy theory to classify 2-connected spaces up to homotopy equivalence. They are equivalent to strict 2-groups (Baez, 1997, Noohi, 2007).

Motivation

One could write every locally Hausdorff groupoid as the truncation of a Hausdorff topological weak 2-groupoid. Also the crossed modules of topological groupoids are equivalent to strict topological 2-groupoids.

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For a Hausdorff étale groupoid G and the interior $H \subseteq G$ of the set of loops (arrows with same source and target)in G, the quotient G/H is a locally Hausdorff, étale groupoid, and the pair (G, H) together with the embedding $H \to G$ and the conjugation action of G on H is a crossed module of topological groupoids. The corresponding C*-algebra C*(G, H) is the C*-algebra of foliations in the sense of Alan Connes (Connes, 1982). The C*-algebra of general (non Hausdorff) groupoids are studied in details by Jean Renault (Renault, 1980).

strict 2-category

We define a strict 2-category as a category enriched over categories. We adapt the notations and terminology of (Buss-Meyer-Zhu, 2013); see also (Baez, 1997). For two objects x and y of the first order category, we have a category of morphisms from x to y, and the composition of morphisms lifts to a bifunctor between these morphism categories.

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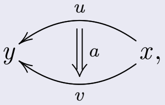
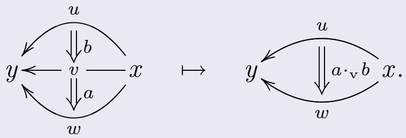
The arrows between objects $u : x \to y$ are called 1-morphisms. We write $x = d(u)$ and $y = r(u)$. The arrows between arrows



are called 2-morphisms (or bigons). We write $u = d(a)$, $v = r(a)$ and $x = d^2(a)$, $y = r^2(a)$.

Composition

The category structure on the space of arrows $x \to y$ gives a vertical composition of 2-morphisms

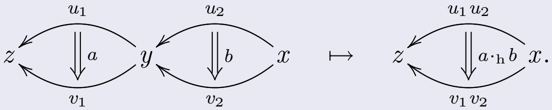


Composition

The vertical product $a \cdot_{\mathrm{v}} b$ is defined if $r(b) = d(a)$. The composition functor between the arrow categories gives a composition of 1-morphisms

$$z \xleftarrow{u} y \xleftarrow{v} x \quad \mapsto \quad z \xleftarrow{uv} x,$$

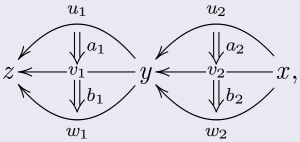
which is defined if $r(v) = d(u)$, and a horizontal composition of 2-morphisms



The horizontal product $a \cdot_{\mathrm{h}} b$ is defined if $r^2(b) = d^2(a)$.

Composition

These three compositions are assumed to be associative and unital, with the same units for the vertical and horizontal products. The horizontal and vertical products commute: given a diagram



composing first vertically and then horizontally or vice versa produces the same 2-morphism $u_1 u_2 \Rightarrow v_1 v_2$.

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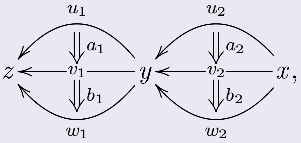

composing first vertically and then horizontally or vice versa produces the same 2-morphism $u_1 u_2 \Rightarrow v_1 v_2$.

We denote the inverse of a 1-morphism u by u^{-1} and vertical and horizontal inverses of a 2-morphism a by $a^{-\mathrm{v}}$ and $a^{-\mathrm{h}}$.

Examples

Categories form a strict 2-category with small categories as objects, functors between categories as arrows, and natural transformations between functors as 2-morphisms. The composition of 1-morphisms is the composition of functors and the vertical composition of 2-morphisms is the composition of natural transformations. The horizontal composition of 2-morphisms yields a canonical natural transformation. Another example of a strict 2-category has C^*-algebras as objects, non-degenerate *-homomorphisms as 1-morphisms, and unitary intertwiners between such *-homomorphisms as 2-morphisms.

Definition

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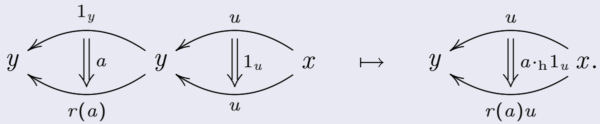
2-group

All 2-groupoids are assumed to be small 2-categories, namely the classes of objects and morphisms are sets. A (strict) 2-group is a strict 2-groupoid with a single object. Given a 2-groupoid $\mathcal{G}$, its objects $\mathcal{G}^0$ and 1-morphisms $\mathcal{G}^1$ form a groupoid, and so does the 1-morphisms and 2-morphisms $\mathcal{G}^2$ with vertical composition.

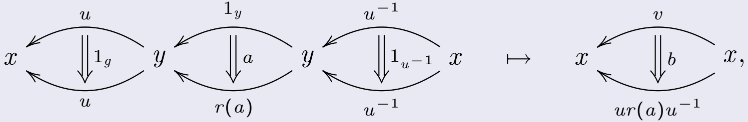
Notation

We usually write $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ and denote the subset of composable elements in $\mathcal{G}^1 \times \mathcal{G}^1$ by $\mathcal{G}^{(1)}$ and the subsets of elements in $\mathcal{G}^2 \times \mathcal{G}^2$ which are vertically or horizontally composable by $\mathcal{G}^{(2\mathrm{v})}$ or $\mathcal{G}^{(2\mathrm{h})}$. We may use horizontal products with unit 2-morphisms to produce any 2-morphism from a 2-morphisms that starts at a unit 1-morphism:



Crossed module

The 2-morphisms starting at the identity 1-morphisms at the object x form a group $\mathcal{G}_x$ with respect to horizontal composition, and the range map is a homomorphism from the set of such 2-morphisms to the isotropy group bundle $H = \bigsqcup_{x\in\mathcal{G}^0} \mathcal{G}_x$ of the groupoid $(\mathcal{G}^0, \mathcal{G}^1)$. This map is onto when $\mathcal{G}$ is 2-transitive (i.e. for each $u, v \in \mathcal{G}^1$ there is $a \in \mathcal{G}^2$ with $d(a) = u$ and $r(a) = v$). Furthermore, the groupoid $\mathcal{G}$ acts on the group bundle H by horizontal conjugation:



where $b = 1_u \cdot_{\mathrm{h}} a \cdot_{\mathrm{h}} 1_{u^{-1}}$.

Crossed module

We may consider the map

$$r : \bigsqcup_{x \in \mathcal{G}^0} \mathcal{G}_x \to \bigsqcup_{x \in \mathcal{G}^0} \mathcal{G}_x^x$$

and regard $(H, \mathcal{G}^1, r)$ as a crossed module of groupoids. Conversely, for each crossed module $(H, \mathcal{G}^1, r)$ where H is a bundle of groups, $\mathcal{G}^1$ is a groupoid and $r : H \to \mathcal{G}^1$ is a groupoid homomorphism, there is a unique 2-groupoid $\mathcal{G}$ whose isotropic group bundle is isomorphic to H, whose set of 1-morphisms is isomorphic to $\mathcal{G}^1$, and its range map realizes (after identification) as r.

Example

As a concrete example, consider the map $r_\theta : \mathbb{Z} \to \mathbb{T}; n \mapsto e^{2\pi i n\theta}$ where $\theta \in \mathbb{R}$, then $\mathbb{T}$ on $\mathbb{Z}$ by conjugation and the corresponding crossed module is the symmetry of the rotation algebra A_ϑ. This gives a 2-groupoid with a single object, 1-morphisms $\mathbb{T}$ and 2-morphisms $\mathbb{Z} \times \mathbb{T}$.

algebraic 2-groupoid

Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a 2-groupoid, then $\mathcal{G}$ is called 1-principal if the map $(r, d) : \mathcal{G}^1 \to \mathcal{G}^0 \times \mathcal{G}^0$ is one-to-one, 2-principal if the map $(r, d) : \mathcal{G}^2 \to \mathcal{G}^1 \times \mathcal{G}^1$ is one-to-one, and 1+2-principal if both 1-principal and 2-principal. If we replace one-to-one with onto, we get the notions of 1-transitive, 2-transitive, and 1+2-transitive. Note that 2-transitivity here is different from the property of each two nodes being connected by paths od length 2.

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For each $x \in \mathcal{G}^0$ and $u \in \mathcal{G}^1$ $\mathcal{G}_x^x = \{u \in \mathcal{G}^1 : d(u) = r(u) = x\}$,
$\mathcal{G}_u^u = \{a \in \mathcal{G}^2 : d(a) = r(a) = u\}$, and
$\mathcal{G}_{u,x}^{u,x} = \{a \in \mathcal{G}^2 : d(a) = r(a) = u, d^2(a) = r^2(a) = x\}$.
We also have the isotropy groupoid $\mathcal{G}(x) = (\mathcal{G}^2(x), \mathcal{G}^1(x))$ where $\mathcal{G}^2(x) = \{a \in \mathcal{G}^2 : d^2(a) = r^2(a) = x\}$ and $\mathcal{G}^1(x) = \{r(a) : a \in \mathcal{G}^2(x)\}$ with vertical multiplication.

Example

We give three basic examples of 2-groupoids.
(*i*) (Transformation 2-group) Let S be an additive group with identity 0 acting from right on a set U and put $\mathcal{G}^1 = U \times S$ and $\mathcal{G}^0 = U \times \{o\}$. Let T be a multiplicative group with identity 1 acting from left on S and acting trivially from right on U and put $\mathcal{G}^2 = T \times U \times S$ and identify $U \times S$ $\{1\} \times U \times S$. Assume that the left action of T on S is distributive

$$t \cdot (s + s^{'}) = t \cdot s + t \cdot s^{'},$$

for $s, s^{'} \in S$ and $t \in T$. Define $r(u, s) = (u, 0)$ and $d(u, s) = (u \cdot s, 0)$ and partial multiplication by $(u, s).(u \cdot s, s^{'}) = (u, s + s^{'})$ with $(u, s)^{-1} = (u \cdot s, -s)$. Also define $r(t, u, s) = (1, u, s)$ and $d(t, u, s) = (1, t \cdot s)$ and vertical multiplication by $(t, u, t^{'} \cdot s^{'}) \cdot_{\mathrm{v}} (t^{'}, u, s^{'}) = (tt^{'}, u, s^{'})$ with $(t, u, s)^{-\mathrm{v}} = (t^{-1}, u, t \cdot s)$ and horizontal multiplication by $(t, u, s) \cdot_{\mathrm{h}} (t, u \cdot s, s^{'}) = (t, u, s + s^{'})$ with $(t, u, s)^{-\mathrm{h}} = (t, u \cdot s, -s)$.

Example

(*ii*) (Principal 2-groupoid) Let X be a set and put $\mathcal{G}^2 = X^{(5)}$, $\mathcal{G}^1 = X^{(3)}$, $\mathcal{G}^0 = X$. Define $r(x, y, z) = z$ and $d(, y, z) = x$ and $(x, y, z) \cdot (z, u, v) = (x, y, v)$ with $(x, y, z)^{-1} = (z, y, x)$. Define $r(x, y, z, u, v) = (x, u, v)$ and $d(x, y, z, u, v) = (x, y, v)$ and vertical multiplication by $(x, y, z, u, v) \cdot_{\mathrm{v}} (x, u, s, t, v) = (x, y, z, t, v)$ with $(x, y, z, u, v)^{-\mathrm{v}} = (x, u, z, y, v)$ and horizontal multiplication by $(x, y, z, u, v) \cdot_{\mathrm{h}} (v, w, s, t, r) = (x, y, s, u, r)$ with $(x, y, z, u, v)^{-\mathrm{h}} = (v, u, z, y, x)$.
(*iii*) (Groupoid bundle) If $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ satisfies $d(u) = r(u)$ for each $u \in \mathcal{G}^1$ then $\mathcal{G} = \bigsqcup_{x \in \mathcal{G}^0} \mathcal{G}(x)$ is a groupoid bundle.

Similarity

For 2-groupoids $\mathcal{G}$ and $\mathcal{H}$, a vertical morphism $\varphi : \mathcal{G} \to \mathcal{H}$ of 2-groupoids is a pair $\varphi = (\varphi^2, \varphi^1)$ such that $\varphi^2(a \cdot_{\mathrm{v}} b) = \varphi^2(a) \cdot_{\mathrm{v}} \varphi^2(b)$ and $\varphi^1(uv) = \varphi^1(u)\varphi^2(v)$, for $a, b \in \mathcal{G}^2$ and $u, v \in \mathcal{G}^1$, whenever both sides are defined. Two vertical morphisms φ, ψ from $\mathcal{G}$ to $\mathcal{H}$ are called similar if there are maps $\vartheta^2 : \mathcal{G}^1 \to \mathcal{H}^2$ and $\vartheta^1 : \mathcal{G}^0 \to \mathcal{H}^1$ such that

$$d(\vartheta^2(u)) = \vartheta^1(d(u)),\ r(\vartheta^2(u)) = \vartheta^1(r(u))$$

and

$$\vartheta^2 \circ r(a) \cdot_{\mathrm{v}} \varphi^2(a) = \psi^2(a) \cdot_{\mathrm{v}} \vartheta^2 \circ d(a),\ \vartheta^1 \circ r(u)\varphi^1(u) = \psi^1(u)\vartheta^1 \circ r(u)$$

for $u \in \mathcal{G}^1$ and $a \in \mathcal{G}^2$. We write $\varphi \sim_{\mathrm{v}} \psi$. We say that $\mathcal{G}$ and $\mathcal{H}$ are v-similar if there are vertical morphisms $\varphi : \mathcal{G} \to \mathcal{H}$ and $\psi : \mathcal{H} \to \mathcal{G}$ such that $\varphi \circ \psi \sim_{\mathrm{v}} id_{\mathcal{H}}$ and $\psi \circ \varphi \sim_{\mathrm{v}} id_{\mathcal{G}}$. The notions of horizontal morphisms and h-similarity are defined similarly and the latter is denoted by $\sim_{\mathrm{h}}$.

Definition

Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a 2-groupoid and $\mathcal{E} = (\mathcal{E}^1, \mathcal{E}^0)$ with $\mathcal{E}^0 \subseteq \mathcal{G}^0$ and $\mathcal{E}^1 \subseteq \{u \in \mathcal{G}^1 : d(u), r(u) \in \mathcal{E}^0\}$, the 2-groupoid $\mathcal{G}_{\mathcal{E}} = (\mathcal{E}^2, \mathcal{E}^1, \mathcal{E}^0)$, where $\mathcal{E}^2 = \{a \in \mathcal{G}^2 : d(a), r(a) \in \mathcal{E}^1\}$, is called the restriction of $\mathcal{G}$ to $\mathcal{E}$. We say that $\mathcal{E}$ is full if $\mathcal{E}^0$ meets each equivalence class in $\mathcal{G}^0$ and $\mathcal{E}^1$ meets each equivalence class in $\mathcal{G}^1$.

The next lemma is proved by Ramsay for groupoids (Ramsay, 1971).

Lemma

If $\mathcal{E}$ is full then $\mathcal{G}_{\mathcal{E}} \sim_{\mathrm{v}} \mathcal{G}$.

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Lemma

If $\mathcal{E}$ is full then $\mathcal{G}_{\mathcal{E}} \sim_{\mathrm{v}} \mathcal{G}$.

Corollary

Every 2-groupoid is v-similar to a groupoid bundle. A 2-groupoid is v-similar to a groupoid if and only if its objects consists of only one equivalence class.

Identification

We identify $\mathcal{G}^0$ with a subset of $\mathcal{G}^1$ and $\mathcal{G}^1$ with a subset of $\mathcal{G}^2$ by identifying $x \in \mathcal{G}^0$ with 1_x and $u \in \mathcal{G}^1$ with 1_u.

Definition

A topological 2-groupoid is a 2-groupoid $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ and a topology on $\mathcal{G}^2$ such that
(i) The maps $u \mapsto u^{-1}$ and $a \mapsto a^{-\mathrm{v}}$, $a \mapsto a^{-\mathrm{h}}$ are continuous on $\mathcal{G}^1$ and $\mathcal{G}^2$.
(ii) The maps $(u, v) \mapsto uv$ and $(a, b) \mapsto a \cdot_{\mathrm{v}} b$, $(a, b) \mapsto a \cdot_{\mathrm{h}} b$ are continuous on their domains.

Lemma

For any topological 2-groupoid $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$,
(i) The maps $u \mapsto u^{-1}$ and $a \mapsto a^{-\mathrm{v}}$, $a \mapsto a^{-\mathrm{h}}$ are homeomorphisms on $\mathcal{G}^1$ and $\mathcal{G}^2$.
(ii) The source and range maps d, r are continuous on $\mathcal{G}^1$ and $\mathcal{G}^2$.
(iii) If $\mathcal{G}^1$ is Hausdorff, $\mathcal{G}^0 \subseteq \mathcal{G}^1$ is closed, and if $\mathcal{G}^2$ is Hausdorff, $\mathcal{G}^0 \subseteq \mathcal{G}^1$, $\mathcal{G}^1 \subseteq \mathcal{G}^2$ and $\mathcal{G}^0 \subseteq \mathcal{G}^2$ are closed.
(iv) If $\mathcal{G}^0$ is Hausdorff, $\mathcal{G}^{(1)} \subseteq \mathcal{G}^1 \times \mathcal{G}^1$ is closed, and if $\mathcal{G}^1$ is Hausdorff, $\mathcal{G}^{(2\mathrm{v})} \subseteq \mathcal{G}^2 \times \mathcal{G}^2$ and $\mathcal{G}^{(2\mathrm{h})} \subseteq \mathcal{G}^2 \times \mathcal{G}^2$ are closed.
(v) For the range equivalence $a \sim_r b$ defined by $r(a) = r(b)$, the orbit space $\mathcal{G}^2/\sim_r$ is homeomorphic to $\mathcal{G}^1$. Similarly $\mathcal{G}^1/\sim_r$ is homeomorphic to $\mathcal{G}^0$.

Definition

A locally compact 2-groupoid is a topological 2-groupoid $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ such that $\mathcal{G}^0$, $\mathcal{G}^1$ are Hausdorff Borel subsets of $\mathcal{G}^2$ and every point of $\mathcal{G}^2$ has an open, relatively compact, Hausdorff neighborhood.

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For the rest of this talk, $\mathcal{G}$ is a locally compact 2-groupoid. We put

$$C_c(\mathcal{G}) = \{f : \mathcal{G}^2 \to \mathbb{C} : f \text{ is continuous and } supp(f) \subseteq \mathcal{G}^2 \text{ is compact}\},$$

where $supp(f)$ is the complement of the union of open Hausdorff subsets of $\mathcal{G}^2$ on which f vanishes. By assumption $\mathcal{G}^2$ is a union of compact Hausdorff sets K and on the algebraic direct limit $C_c(\mathcal{G})$ is endowed with an inductive limit topology.

Definition

Let $\mathcal{G}$ be a locally compact 2-groupoid. A continuous left 2-Haar system on $\mathcal{G}$ consists of two families of positive Borel measures $\{\lambda_{\mathrm{v}}^{u}\}$and $\{\lambda_{\mathrm{h}}^{x}\}$on $\mathcal{G}^2$, where u ranges over $\mathcal{G}^1$ and x ranges over $\mathcal{G}^0$, such that
(i) $supp(\lambda_{\mathrm{v}}^{u}) = \mathcal{G}^u$ and $supp(\lambda_{\mathrm{h}}^{x}) = \mathcal{G}^x$, for each $u \in \mathcal{G}^1$ and $x \in \mathcal{G}^0$.
(ii) For any $f \in C_c(\mathcal{G})$, the map $u \mapsto \int f d\lambda_{\mathrm{v}}^{u}$ is continuous on $\mathcal{G}^1$ and the map $x \mapsto \int f d\lambda_{\mathrm{h}}^{x}$ is continuous on $\mathcal{G}^0$.
(iii) For any $f \in C_c(\mathcal{G})$,

$$\int f(a \cdot_{\mathrm{v}} b) d\lambda_{\mathrm{v}}^{d(a)}(b) = \int f(b) d\lambda_{\mathrm{v}}^{r(a)}(b)$$

and

$$\int f(a \cdot_{\mathrm{h}} b) d\lambda_{\mathrm{h}}^{d^2(a)}(b) = \int f(b) d\lambda_{\mathrm{h}}^{r^2(a)}(b).$$

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and

$$\int f(a \cdot_{\mathrm{h}} b) d\lambda_{\mathrm{h}}^{d^2(a)}(b) = \int f(b) d\lambda_{\mathrm{h}}^{r^2(a)}(b).$$

$$\int f(uv) d\lambda_{\mathrm{v}}^{d(u)}(v) = \int f(v) d\lambda_{\mathrm{v}}^{r(u)}(v).$$

Proposition

If $\mathcal{G}$ has a continuous 2-Haar system, we have the continuous surjections:

$$\lambda_{\rm v} : C_c(\mathcal{G}^2) \to C_c(\mathcal{G}^1);\ f \mapsto \lambda_{\rm v}(f),\ \lambda_{\rm v}(f)(u) = \int f d\lambda_{\rm v}^u,$$

and

$$\lambda_{\rm h} : C_c(\mathcal{G}^2) \to C_c(\mathcal{G}^0);\ f \mapsto \lambda_{\rm h}(f),\ \lambda_{\rm h}(f)(x) = \int f d\lambda_{\rm h}^x.$$

Moreover the maps $r : \mathcal{G}^2 \to \mathcal{G}^1$, $r : \mathcal{G}^1 \to \mathcal{G}^0$ and $r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ are open and the associated equivalence relations on $\mathcal{G}^1$ and $\mathcal{G}^0$ are open.

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Example

The 2-Haar systems of the above examples are as follows:

(i) (Transformation 2-group) Let S, T be locally compact groups with Haar measures λ_S and λ_T acting continuously on a locally compact Hausdorff space U as in Example 3.1(i) and $\mathcal{G}^2 = T \times U \times S$, then the vertical and horizontal left Haar systems on $\mathcal{G}$ are given by

$$\lambda_{\mathrm{v}}^{(1,u,s)} = \lambda_T \times \delta_u \times \lambda_1, \quad \lambda_{\mathrm{h}}^{(1,u,0)} = \lambda_2 \times \delta_u \times \lambda_S \quad (u \in U, s \in S),$$

where λ_1, λ_2 are arbitrary Borel measures with full support on S, T, respectively.

Example

(*ii*) (Principal 2-groupoid) Let X be a locally compact Hausdorff space and $\mathcal{G}^2 = X^{(5)}$. Consider the homeomorphism

$$d : \mathcal{G}^{(x,u,v)} \longrightarrow X^{(2)}; \ (x, y, z, u, v) \mapsto (y, z),$$

let α be any Borel measure on $X^{(2)}$ with full support such that for each $f \in C_c(\mathcal{G})$, the map

$$(x, u, v) \mapsto \int f(x, y, z, u, v) d\alpha(y, z)$$

is continuous on $X^{(3)}$, then $\int f d\lambda_{\mathrm{v}}^{(x,u,v)} = \int f(x, y, z, u, v) d\alpha(y, z)$ defines a vertical left Haar system. The horizontal case is treated similarly.

Example

(iii) (Groupoid bundle) Let $\mathcal{G} = \bigsqcup_{x\in\mathcal{G}^0} \mathcal{G}(x)$ be a locally compact groupoid bundle. The 2-Haar system is essentially unique (if it exists), that is any two systems $\{\lambda_{\rm v}^u, \lambda_{\rm h}^x\}$ and $\{\sigma_{\rm v}^u, \sigma_{\rm h}^x\}$ are related via $\lambda_{\rm v}^u = h(u)\sigma_{\rm v}^u$ and $\lambda_{\rm h}^x = k(x)\sigma_{\rm h}^x$, where $h \in C(\mathcal{G}^1)_+$ and $k \in C(\mathcal{G}^0)_+$.

Definition

A locally compact 2-groupoid $\mathcal{G}$ is called *r-discrete* if $\mathcal{G}^0 \subseteq \mathcal{G}^1$ and $\mathcal{G}^1 \subseteq \mathcal{G}^2$ are open.

Definition

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Lemma

If $\mathcal{G}$ is r-discrete, then
(i) for each $u \in \mathcal{G}^1$ and $x \in \mathcal{G}^0$, $\mathcal{G}^u$ and $\mathcal{G}^x$ are open in $\mathcal{G}^2$,
(ii) if a continuous 2-Haar system exists, it is essentially the system of counting measures. In this case, $d, r : \mathcal{G}^2 \to \mathcal{G}^1$, $d, r : \mathcal{G}^1 \to \mathcal{G}^0$, and $d^2, r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ are local homeomorphisms.

Definition

Let $\mathcal{G}$ be a locally compact 2-groupoid. A subset s of $\mathcal{G}^2$ is called a $\mathcal{G}^1$-set if the restrictions of d and r to s are one-to-one. This is equivalent to $s \cdot_{\mathrm{v}} s^{-1}$ and $s^{-1} \cdot_{\mathrm{v}} s$ being contained in $\mathcal{G}^1$. A subset s of $\mathcal{G}^2$ is called a $\mathcal{G}^0$-set if the restrictions of d^2 and r^2 to s are one-to-one, or equivalently $s \cdot_{\mathrm{h}} s^{-1}$ and $s^{-1} \cdot_{\mathrm{h}} s$ are contained in $\mathcal{G}^0$.

Definition

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In the above definition the products are considered as products of sets. Note that both $\mathcal{G}^1$-sets and $\mathcal{G}^0$-sets form an inverse semigroup, and for each $a \in \mathcal{G}^2$ and $\mathcal{G}^1$-set s, if $d(a) \in r(s)$ (resp. $r(a) \in d(s)$) then the set $a \cdot_{\mathrm{v}} s$ (resp. $s \cdot_{\mathrm{v}} a$) is a singleton, and so defines an element of $\mathcal{G}^2$ denoted again by $a \cdot_{\mathrm{v}} s$ (resp. $s \cdot_{\mathrm{v}} a$). Also there is a map $r(s) \longrightarrow d(s)$; $u \mapsto u \cdot s := d(u \cdot_{\mathrm{v}} s)$, satisfying $u \cdot (s \cdot_{\mathrm{v}} t) = (u \cdot s) \cdot_{\mathrm{v}} t$, for $\mathcal{G}^1$-sets s, t. Similarly, for $a \in \mathcal{G}^2$ and $\mathcal{G}^0$-set s with $d^2(a) \in r^2(s)$ (resp. $r^2(a) \in d^2(s)$) the element $a \cdot_{\mathrm{h}} s$ (resp. $s \cdot_{\mathrm{v}} a$) of $\mathcal{G}^2$ is defined, and the map $r^2(s) \longrightarrow d^(s)$; $x \mapsto x \cdot s := d^2(x \cdot_{\mathrm{h}} s)$, satisfies $x \cdot (s \cdot_{\mathrm{h}} t) = (x \cdot s) \cdot_{\mathrm{h}} t$, for $\mathcal{G}^0$-sets s, t.

Proposition

For a locally compact 2-groupoid $\mathcal{G}$, the following are equivalent:
(i) $\mathcal{G}$ is r-discrete and has a continuous left 2-Haar system,
(ii) The maps $r : \mathcal{G}^2 \to \mathcal{G}^1$ and $r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ are local homeomorphisms,
(iii) The product maps $\mathcal{G}^{(1)} \to \mathcal{G}^1$, $\mathcal{G}^{(2\mathrm{v})} \to \mathcal{G}^1$ and $G^{(2\mathrm{h})} \to \mathcal{G}^0$ are local homeomorphisms,
(iv) $\mathcal{G}^2$ has an open basis consisting of open G^1-sets and one consisting of open $\mathcal{G}^0$-sets.

Associated measures

Let $\mathcal{G}$ be a locally compact 2-groupoid with continuous left 2-Haar system $\{\lambda_{\rm v}^u\}$and $\{\lambda_{\rm h}^x\}$, let $\{\lambda_{{\rm v}u}\}$ and $\{\lambda_{{\rm h}x}\}$ be the images of this system under the inverse maps $a \mapsto a^{-{\rm v}}$ and $a \mapsto a^{-{\rm h}}$. Then the latter is a continuous right 2-Haar system. Borel measures μ^1 and μ^0 on $\mathcal{G}^1$ and $\mathcal{G}^0$ induce measures

$$\nu_{\rm v} = \int \lambda_{\rm v}^u \, d\mu^1(u), \ \nu_{\rm h} = \int \lambda_{\rm h}^x \, d\mu^0(x)$$

with images

$$\nu_{\rm v}^{-1} = \int \lambda_{{\rm v}u} \, d\mu^1(u), \ \nu_{\rm h}^{-1} = \int \lambda_{{\rm h}x} \, d\mu^0(x)$$

and induced measures

$$\nu_{\rm v}^2 = \int \lambda_{\rm v}^u \times \lambda_{{\rm v}u} \, d\mu^1(u), \ \nu_{\rm h}^2 = \int \lambda_{\rm h}^x \times \lambda_{{\rm h}x} \, d\mu^0(x).$$

Definition

The Borel measure μ^1 on $\mathcal{G}^1$ is called quasi-invariant if $\nu_{\mathrm{v}} \sim \nu_{\mathrm{v}}^{-1}$. The Borel measure μ^0 on $\mathcal{G}^0$ is called quasi-invariant if $\nu_{\mathrm{h}} \sim \nu_{\mathrm{h}}^{-1}$.

Definition

The Borel measure μ^1 on $\mathcal{G}^1$ is called quasi-invariant if $\nu_{\mathrm{v}} \sim \nu_{\mathrm{v}}^{-1}$. The Borel measure μ^0 on $\mathcal{G}^0$ is called quasi-invariant if $\nu_{\mathrm{h}} \sim \nu_{\mathrm{h}}^{-1}$.

from the uniqueness of the Radon-Nikodym derivative we have the following result which defines vertical and horizontal modular functions. We put $\nu_{\mathrm{v}0} = D_{\mathrm{v}}^{\frac{1}{2}} \nu_{\mathrm{v}}$ and $\nu_{\mathrm{h}0} = D_{\mathrm{h}}^{\frac{1}{2}} \nu_{\mathrm{h}}$.

Proposition

For quasi-invariant measure μ^1 on $\mathcal{G}^1$, there is a locally $\nu_{\rm v}$-integrable positive function $D_{\rm v}$ such that $\nu_{\rm v} = D_{\rm v}\nu_{\rm v}^{-1}$ and
(i) $D_{\rm v}(a \cdot_{\rm v} b) = D_{\rm v}(a)D_{\rm v}(b)\,(\nu_{\rm v}^2\text{-}a.e)$, $D_{\rm v}(a^{-{\rm v}}) = D_{\rm v}(a)^{-1}\,(\nu_{\rm v} - a.e)$,
(ii) if $\mu'^1 = g^1\mu^1$ where g^1 is positive and locally μ^1-integrable then $D'_{\rm v} = (g^1 \circ r)D_{\rm v}(g^1 \circ d)^{-1}$ satisfies $\nu'_{\rm v} = D'_{\rm v}\nu'^{-1}_{\rm v}$.
Similarly, for quasi-invariant measure μ^0 on $\mathcal{G}^0$, there is a locally $\nu_{\rm v}$-integrable positive function $D_{\rm h}$ such that $\nu_{\rm h} = D_{\rm h}\nu_{\rm h}^{-1}$ and relations similar to (i) and (ii) above hold.

Non singular units

For locally compact topological spaces X and Y and surjective map $p : X \to Y$, a measure class $\mathcal{C}$ on X and (probability) measure $\mu \in \mathbb{C}$, $p_*\mathbb{C}$ is the measure class of $p_*\mu := \mu \circ p^{-1}$. A pseudo-image of $\mu \in \mathbb{C}$ is a measure in $p_*\mathbb{C}$. If (X, μ) and (Y, ν) are measure spaces and $s : X \to Y; x \mapsto x \cdot s$ is a bi-measurable bijection, then μ lifts to a measure $\mu \cdot s$ on Y defined by

$$\int f(y) d\mu \cdot s(y) = \int f(x \cdot s) d\mu(x) \quad (f \in C_c(Y))$$

and when $\mu \cdot s \ll \nu$ we denote the corresponding Radon-Nikodym derivative by $d\mu \cdot s / d\nu$ and say that s is non singular if it induces an isomorphism of the corresponding measure algebras.

Ergodic measures

For quasi-invariant measures μ^1 and μ^0, subsets $A^1 \subseteq \mathcal{G}^1$ and $A^0 \subseteq \mathcal{G}^0$ are called almost invariant if $r(a) \in A^1$ is equivalent to $d(a) \in A^1$ ($\nu_{\rm v}$-a.e.) and $r^2(a) \in A^0$ is equivalent to $d^2(a) \in A^0$ ($\nu_{\rm h}$-a.e.). The measures μ^1 and μ^0 are called ergodic if every almost invariant set is null or co-null.
For arbitrary Borel measures μ^1 and μ^0, the pseudo-images $[\mu^1]$ and $[\mu^0]$ of $\nu_{\rm v}$ and $\nu_{\rm h}$ under d and d^2 are quasi-invariant and in the same measure class as μ^1 and μ^0 if and only if μ^1 and μ^0 are quasi-invariant .
If $\alpha^u_{\rm v}$ and $\alpha^x_{\rm h}$ are a pseudo-images of $\lambda^u_{\rm v}$ and $\lambda^x_{\rm h}$ then the measure class of $\alpha^u_{\rm v}$ and $\alpha^x_{\rm h}$ depend only on the orbits of u and x in $\mathcal{G}^1$ and $\mathcal{G}^0$ and $\alpha^u_{\rm v}$ and $\alpha^x_{\rm h}$ are ergodic, and every quasi-invariant pair carried by the orbits of u and x are equivalent to $\alpha^u_{\rm v}$ and $\alpha^x_{\rm h}$.

Modular functions

Let μ^1 be a Borel measure on $\mathcal{G}^1$ with induced measure ν_{v} and s be a ν_{v}-measurable $\mathcal{G}^1$-set. The measure ν_{v} is called quasi-invariant under s if the map $a \mapsto a \cdot_{\mathrm{v}} s^{-\mathrm{v}}$ is non singular from $(d^{-1}(d(s)), \nu_{\mathrm{v}})$ to $(d^{-1}(r(s)), \nu_{\mathrm{v}})$. Let $\delta_{\mathrm{v}}(\cdot, s) = d(\nu_{\mathrm{v}} \cdot s^{-\mathrm{v}})/d\nu_{\mathrm{v}}$ be the corresponding Radon-Nikodym derivative. The measure μ^1 is called quasi-invariant under s if the map $u \mapsto u \cdot s^{-\mathrm{v}}$ is non singular from $d(s), \mu^1)$ to $r(s), \mu^1$ and $\Delta_{\mathrm{v}}(\cdot, s) = d(\mu^1 \cdot s^{-\mathrm{v}})/d\mu^1$ is the corresponding Radon-Nikodym derivative. For a Borel measure μ^0 on $\mathcal{G}^0$, The horizontal functions δ_{h} and Δ_{h} are defined similarly.

Lemma

Under the above quasi-invariance properties,
(i) $\delta_{\mathrm{v}}(s(a), s) = \delta_{\mathrm{v}}(a, s) \quad (\nu_{\mathrm{v}}\text{-}a.e.\ a \in d^{-1}(r(s))$,
(ii) $\delta_{\mathrm{v}}(u, s) = D_{\mathrm{v}}(u \cdot s)\Delta_{\mathrm{v}}(u, s) \quad (\mu^1\text{-}a.e.\ u \in r(s))$,
and the same for δ_{h} and Δ_{h}.

Invariant sets

A $\mathcal{G}^1$-set s is said to be Borel (continuous) if the restrictions of d and r to s are Borel isomorphisms (homeomorphisms) onto a Borel (open) subset of G^1. It is called non singular if there is a Borel (continuous) positive function $\delta_{\mathrm{v}}(\cdot, s)$ on $r(s)$, bounded above and below on compact subsets of $\mathcal{G}^1$, such that $\delta_{\mathrm{v}}(d(a), s) = d(\lambda^u_{\mathrm{v}} \cdot s^{-\mathrm{v}})/d\lambda^u_{\mathrm{v}}(a)$ for every $u \in \mathcal{G}^1$ and λ^u_{v}-a.e. $a \in d^{-1}(r(s))$. A non singular Borel $\mathcal{G}^1$-set s is also non singular with respect to the induced measure ν_{v} of any Borel measure μ^1 on $\mathcal{G}^1$ and $\delta_{\mathrm{v}}(d(a), s) = d(\nu_{\mathrm{v}} \cdot s^{-\mathrm{v}})/d\nu_{\mathrm{v}}(a)$ for ν_{v}-a.e. $a \in d^{-1}(r(s))$. The set of non singular Borel G^1-sets also form an inverse semigroup and

$$\delta_{\mathrm{v}}(u, s \cdot_{\mathrm{v}} t) = \delta_{\mathrm{v}}(u, s)\delta_{\mathrm{v}}(u \cdot s, t) \quad (u \in r(s \cdot_{\mathrm{v}} t),$$

$$ddv(u, s^{-\mathrm{v}}) = \delta_{\mathrm{v}}(u \cdot s^{-\mathrm{v}}, s)^{-1} \quad (u \in d(s)).$$

Notation

Let $\mathcal{G}$ be a locally compact 2-groupoid with a fixed continuous left 2-Haar system $\{\lambda_{\mathrm{v}}^u\}$and $\{\lambda_{\mathrm{h}}^x\}$, for $f, g \in C_c(\mathcal{G})$ put

$$f *_{\mathrm{v}} g(a) = \int f(a \cdot_{\mathrm{v}} b) g(b^{-\mathrm{v}}) d\lambda_{\mathrm{v}}^{d(a)}(b), \quad f^{*_{\mathrm{v}}}(a) = \bar{f}(a^{-\mathrm{v}}),$$

and

$$f *_{\mathrm{h}} g(a) = \int f(a \cdot_{\mathrm{h}} b) g(b^{-\mathrm{h}}) d\lambda_{\mathrm{h}}^{d^2(a)}(b), \quad f^{*_{\mathrm{h}}}(a) = \bar{f}(a^{-\mathrm{h}}),$$

for each $a \in \mathcal{G}^2$.

Lemma

$C_c(\mathcal{G})$ is a topological $*$-algebra with respect to both of the vertical and horizontal convolutions and involutions, denoted by $C_{cv}(\mathcal{G})$ and $C_{ch}(\mathcal{G})$, respectively.

Representation

A representation of $C_{cv}(\mathcal{G})$ on a Hilbert space H is a $*$-homomorphism $L : C_{cv}(\mathcal{G}) \to B(H)$ which is continuous in the inductive limit topology on the domain and weak operator topology on the range. We have the same definition for representations of $C_{ch}(\mathcal{G})$. We only work with non-degenerate representations.

Representation

A representation of $C_{cv}(\mathcal{G})$ on a Hilbert space H is a $*$-homomorphism $L : C_{cv}(\mathcal{G}) \to B(H)$ which is continuous in the inductive limit topology on the domain and weak operator topology on the range. We have the same definition for representations of $C_{ch}(\mathcal{G})$. We only work with non-degenerate representations.

Boundedness

For $f \in C_{cv}(\mathcal{G})$ put

$$\|f\|_{\mathrm{v},r} = \sup_{u \in \mathcal{G}^1} \int |f| d\lambda_{\mathrm{v}}^u, \quad \|f\|_{\mathrm{v},d} = \sup_{u \in \mathcal{G}^1} \int |f| d\lambda_{\mathrm{v}u}$$

and $\|f\|_{\mathrm{v}} = max\{\|f\|_{\mathrm{v},r}, \|f\|_{\mathrm{v},d}\}$.

Boundedness

For $f \in C_{cv}(\mathcal{G})$ put

$$\|f\|_{v,r} = \sup_{u \in \mathcal{G}^1} \int |f| d\lambda_v^u, \quad \|f\|_{v,d} = \sup_{u \in \mathcal{G}^1} \int |f| d\lambda_{vu}$$

and $\|f\|_v = max\{\|f\|_{v,r}, \|f\|_{v,d}\}$. This is a norm on $C_{cv}(\mathcal{G})$ defining a topology coarser than the inductive limit topology. We say that a representation L is v-bounded if there is a constant $M > 0$ such that $\|L(f)\| \leq M\|f\|_v$, for each $f \in C_{cv}(\mathcal{G})$. We put $\|f\|^v = \sup_L \|L(f)\|$, where the supremum is taken over all v-bounded non-degenerate representations. This is a C^*-seminorm on $C_{cv}(\mathcal{G})$ and $\|f\|^v \leq \|f\|_v$, for each $f \in C_{cv}(\mathcal{G})$. The norms $\|f\|_h$ and $\|f\|^h$ are defined similarly on $C_{ch}(\mathcal{G})$ using h-bounded representations.

Definition

A vertical representation of $\mathcal{G}$ (abbreviated as v-representation) consists of a quasi-invariant Borel measure μ^1 on $\mathcal{G}^1$, a $\mathcal{G}^1$-Hilbert bundle $\mathcal{H}$ over $(\mathcal{G}^1, \mu^1)$, and a map $\pi : \mathcal{G}^2 \to Iso(\mathcal{H})$ such that
(i) $\pi(a)$ is a map from $\mathcal{H}_{d(a)}$ to $\mathcal{H}_{d(a)}$ and $\pi(u) = id_{\mathcal{H}_u}$, for all $a \in \mathcal{G}^2$ and $u \in \mathcal{G}^1$,
(ii) $\pi(a \cdot_{\mathrm{v}} b) = \pi(a)\pi(b)$ for ν_{v}^2-a.e. (a, b),
(iii) $\pi(a^{-\mathrm{v}}) = \pi(a)^{-1}$ for ν_{v}-a.e. a,
(iv) $a \mapsto \langle \pi(a)\xi \circ d(a), \eta \circ r(a) \rangle$ is measurable on $\mathcal{G}^2$ for all measurable sections ξ, η.
h-representations are defined similarly using Hilbert bundles over $(\mathcal{G}^0, \mu^0)$.

Equivalence

Two v-representations $(\pi_1, \mathcal{H}_1, \mu_1^1)$ and $(\pi_2, \mathcal{H}_2, \mu_2^1)$ are equivalent if $\mu_1^1 \sim \mu_2^1$ and there is an isomorphism ϕ of Hilbert bundles from $\mathcal{H}_1$ onto $\mathcal{H}_2$ which intertwines π_1 and π_2, that is

$$\pi_2(a)\phi \circ d(a) = \phi \circ r(a)\pi_1(a) \text{ for } \nu_{\mathrm{v}}\text{-a.e. } a \in \mathcal{G}^2.$$

Equivalence

Two v-representations $(\pi_1, \mathcal{H}_1, \mu_1^1)$ and $(\pi_2, \mathcal{H}_2, \mu_2^1)$ are equivalent if $\mu_1^1 \sim \mu_2^1$ and there is an isomorphism ϕ of Hilbert bundles from $\mathcal{H}_1$ onto $\mathcal{H}_2$ which intertwines π_1 and π_2, that is

$$\pi_2(a)\phi \circ d(a) = \phi \circ r(a)\pi_1(a) \text{ for } \nu_{\mathrm{v}}\text{-a.e. } a \in \mathcal{G}^2.$$

Let $(\pi, \mathcal{H}, \mu^1)$ be a v-representation and $\Gamma_{\mathrm{v}}(\mathcal{H})$ be the Hilbert space of square integrable sections with respect to μ^1.

Lemma

Let $(\pi, \mathcal{H}, \mu^1)$ be a v-representation of $\mathcal{G}$, $f \in C_{cv}(\mathcal{G})$ and $\xi, \eta \in \Gamma_{\rm v}(\mathcal{H})$, then

$$\langle \tilde{\pi}(f)\xi, \eta \rangle = \int f(a)\langle \pi(a)\xi \circ d(a), \eta \circ r(a)\rangle \, d\nu_{\rm v0}(a)$$

defines a v-bounded representation of $C_{cv}(\mathcal{G})$ on $\Gamma_{\rm v}(\mathcal{H})$, and two equivalent v-representations of $\mathcal{G}$ induce equivalent v-bounded representations of $C_{cv}(\mathcal{G})$ as above.
When $dim(\mathcal{H}_u)$ is constant, namely there is a Hilbert space H with $\mathcal{H}_u \simeq H$, for all $u \in \mathcal{G}^1$,

$$\tilde{\pi}(f)\xi(u) = \int f(a)\pi(a)\xi \circ d(a) D_{\rm v}^{\frac{1}{2}}(a) \, d\lambda_{\rm v}^{u}(a),$$

μ^1-a.e., for $f \in C_{cv}(\mathcal{G})$ and $\xi \in L^2(\mathcal{G}^1, \mu^1, H)$. In general $\tilde{\pi}$ is a direct sum of representations on constant fields over all possible dimensions. Similar statements hold for h-representations $(\pi, \mathcal{H}, \mu^0)$ and Hilbert space $\Gamma_{\rm h}(\mathcal{H})$ of square integrable sections with respect to μ^0.

Regular representation

Consider the measurable field of Hilbert spaces $L^2(\mathcal{G}^2, \lambda_{\mathrm{v}}^u)$ with square integrable sections $L^2(\mathcal{G}^2, \nu_{\mathrm{v}}) = \int^{\oplus} L^2(\mathcal{G}^2, \lambda_{\mathrm{v}}^u) d\mu^1(u)$ where μ^1 is a quasi-invariant Borel measure on $\mathcal{G}^1$. Then $\pi(a) : L^2(\mathcal{G}^2, \lambda_{\mathrm{v}}^{d(a)}) \to L^2(\mathcal{G}^2, \lambda_{\mathrm{v}}^{r(a)})$; $\pi(a)\xi(b) = \xi(a^{-\mathrm{v}} \cdot_{\mathrm{v}} b)$ is a v-representation of $\mathcal{G}$ and

$$a \mapsto \langle \pi(a)\xi \circ d(a), \eta \circ r(a) \rangle = \int \xi(a^{-\mathrm{v}} \cdot_{\mathrm{v}} b)\bar{\eta}(b) d\lambda_{\mathrm{v}}^{r(a)}(b)$$

is continuous for $\xi, \eta \in \mathbb{C}_c(\mathcal{G})$ and measurable for $\xi, \eta \in L^2(\mathcal{G}^2, \nu_{\mathrm{v}})$. This is called the left regular representation of $\mathcal{G}$ with respect to μ^1. Similarly we could define the left regular representation of $\mathcal{G}$ with respect to a quasi-invariant measure μ^0 on $\mathcal{G}^0$.

Lemma

The topological algebra $C_{\mathrm{cv}}(\mathcal{G})$ has a left approximate identity in the inductive limit topology. Same holds for $C_{\mathrm{ch}}(\mathcal{G})$.

Modular function

The above lemma implies that v-left regular representations with respect to all quasi-invariant measures on $\mathcal{G}^1$ induce a faithful family of v-bounded representations of $C_{cv}(\mathcal{G})$. Also, for each quasi-invariant measure μ^1 on $\mathcal{G}^1$, $C_{cv}(\mathcal{G})$ is a generalized Hilbert algebra under the inner product of $L^2(\mathcal{G}^2, \nu_{\rm v}^{-1})$ whose left regular representation is equivalent to the v-left regular representation with respect to μ^1 [7, 2.1.10] and by Tomita-Takesaki theory we have a modular function $J_{\rm v} : L^2(\mathcal{G}^2, \nu_{\rm v}^{-1}) \to L^2(\mathcal{G}^2, \nu_{\rm v}^{-1})$; $J_{\rm v}\xi(a) = D_{\rm v}^{\frac{1}{2}}(a)\bar{\xi}(a^{-{\rm v}})$ and the modular operator

$$\Delta_{\rm v} : L^2(\mathcal{G}^2, \nu_{\rm v}) \cap L^2(\mathcal{G}^2, \nu_{\rm v}^{-1}) \to L^2(\mathcal{G}^2, \nu_{\rm v}) \cap L^2(\mathcal{G}^2, \nu_{\rm v}^{-1});$$

$$\Delta_{\rm v}\xi(a) = D_{\rm v}(a)\xi(a).$$

The same observations hold for $C_{ch}(\mathcal{G})$.

Definition

The full vertical (resp. horizontal) C^*-algebra of $\mathcal{G}$ is the completion of $C_{cv}(\mathcal{G})$ (resp. $C_{ch}(\mathcal{G})$) in $\|.\|^{\mathrm{v}}$ (resp. $\|.\|^{\mathrm{v}}$).

Lemma

Let $\{L, H\}$ be a representation of $C_{cv}(\mathcal{G})$, there is a unique representation $\{L^1, H^1\}$ of $C_c(\mathcal{G}^1)$ such that

$$L(hf) = L^1(h)L(f), \quad L(fh) = L(f)L^1(h) \quad (h \in \mathbb{C}_c(\mathcal{G}^1), f \in C_{cv}(\mathcal{G}))$$

where

$$hf(a) = h \circ r(a)f(a), \quad fh(a) = f(a)h \circ d(a) \quad (a \in \mathcal{G}^2).$$

Moreover for $f, g \in C_{cv}(\mathcal{G}), h \in \mathbb{C}_c(\mathcal{G}^1)$,

$$f *_{\mathrm{v}} hg = fh *_{\mathrm{v}} g, \; hf *_{\mathrm{v}} g = h(f *_{\mathrm{v}} g), \; (hf)^{*_{\mathrm{v}}} = f^{*_{\mathrm{v}}} h^*,$$

where $h^*(u) = \overline{h}(u)$, for $u \in \mathcal{G}^1$. There is a representation $\{L^0, H^0\}$ of $C_c(\mathcal{G}^0)$ with similar relations to the horizontal convolution.

Corollary

$C^*(\mathcal{G}^1)$ and $C^*(\mathcal{G}^0)$ are subalgebras of the multiplier algebras $M(C^*_{\rm v}(\mathcal{G}))$ and $M(C^*_{\rm v}(\mathcal{G}))$, respectively.

Notation

Every representation of $C_c(\mathcal{G})$ extends to a representation of $B(\mathcal{G})$ of bounded Borel functions on $\mathcal{G}^2$ with vertical or horizontal convolution. For a non singular Borel $\mathcal{G}^1$-set s and $f \in B(\mathcal{G})$ we define

$s \cdot_{\mathrm{v}} f(a) = \delta_{\mathrm{v}}^{\frac{1}{2}}(r(a), s)$ for $a \in r^{-1}(r(s))$, and zero otherwise, and

$f \cdot_{\mathrm{v}} s(a) = \delta_{\mathrm{v}}^{\frac{1}{2}}(d(a), s^{-\mathrm{v}})$ for $a \in d^{-1}(d(s))$, and zero otherwise, then

$$(s \cdot_{\mathrm{v}} (t \cdot_{\mathrm{v}} f) = (st) \cdot_{\mathrm{v}} f, \ (f \cdot_{\mathrm{v}} s) *_{\mathrm{v}} g = f *_{\mathrm{v}} (s \cdot_{\mathrm{v}} g), \ (s \cdot_{\mathrm{v}} f) *_{\mathrm{v}} g = s \cdot_{\mathrm{v}} (f *_{\mathrm{v}} g)$$

and $(f \cdot_{\mathrm{v}} s)^* = s^{-\mathrm{v}} \cdot_{\mathrm{v}} f^*$, for non singular $\mathcal{G}^1$-sets s, t and $f, g \in B(\mathcal{G})$.

Notation

We denote $B(\mathcal{G})$ with vertical convolution by $B_{\mathrm{v}}(\mathcal{G})$. Same relations hold for $B_{\mathrm{h}}(\mathcal{G})$, that is $B(\mathcal{G})$ with horizontal convolution. Also we could find a unique representation V^1 of the Borel ample semigroup of non singular $\mathcal{G}^1$-sets such that

$$L(s \cdot_{\mathrm{v}} f) = V^1(s)L(f),\ L(f \cdot_{\mathrm{v}} s) = L(f)V^1(s),\ V^1(s)L^1(h)V^1(s)^* = L^1(h^s),$$

for non singular $\mathcal{G}^1$-set s, $f \in B_{\mathrm{v}}(\mathcal{G})$ and $h \in C_c(\mathcal{G}^1)$, where $h^s(u) = h(us)$ for $u \in r(s)$, and zero otherwise. same holds for representations L, L^0 and a representation V^0 of the Borel ample semigroup of non singular $\mathcal{G}^0$-sets.

Theorem

If $\mathcal{G}$ is a locally compact second countable 2-groupoid with left 2-Haar system $\{\lambda_{\mathrm{v}}^u\}$and $\{\lambda_{\mathrm{h}}^x\}$with sufficiently many non singular $\mathcal{G}^1$-sets (resp. $\mathcal{G}^0$-sets) then every v-bounded (resp. h-bounded) representation of $C_{c\mathrm{v}}(\mathcal{G})$ (resp. $C_{c\mathrm{h}}(\mathcal{G})$) on a separable Hilbert space is the integration of a v-representation (resp. an h-representation) of $\mathcal{G}$.

Corollary

When $\mathcal{G}$ is second countable with sufficiently many non singular $\mathcal{G}^1$-sets (resp. $\mathcal{G}^0$-sets), every representation of $C_{cv}(\mathcal{G})$ (resp. $C_{ch}(\mathcal{G})$) on a separable Hilbert space is v-bounded (resp. h-bounded) and there is a one-to-one correspondence between $\mathcal{G}^1$-Hilbert bundles (resp. $\mathcal{G}^0$-Hilbert bundles) and separable Hermitian $\mathbb{C}^*_{\rm v}(\mathcal{G})$-modules (resp. $\mathbb{C}^*_{\rm v}(\mathcal{G})$-modules) preserving intertwining operators.

Quotients

Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a locally compact 2-groupoid with left 2-Haar system $\{\lambda_{\mathrm{v}}^u\}$and $\{\lambda_{\mathrm{h}}^x\}$and $\mathcal{H} = (\mathcal{H}^2, \mathcal{H}^1, \mathcal{H}^0)$ be a closed 2-subgroupoid, that is a 2-subgroupoid such that $\mathcal{H}^i \subseteq \mathcal{G}^i$ is closed for $i = 0, 1, 2$, with left 2-Haar system $\{\sigma_{\mathrm{v}}^u\}$and $\{\sigma_{\mathrm{h}}^x\}$such that $\mathcal{G}^1 \subseteq \mathcal{H}^2$ and $\mathcal{G}^0 \subseteq \mathcal{H}^1$. For the equivalence relations $a \sim_{\mathrm{v}} b$ iff $d(a) = r(b)$ and $a \cdot_{\mathrm{v}} b \in \mathcal{H}^2$ and $a \sim_{\mathrm{h}} b$ iff $d^2(a) = r^2(b)$ and $a \cdot_{\mathrm{h}} b \in \mathcal{H}^2$, for $a, b \in \mathcal{G}^2$, the quotient space $\mathcal{H}\backslash\mathcal{G}$ is Hausdorff and locally compact and the quotient map: $\mathcal{G} \to \mathcal{H}\backslash\mathcal{G}$ is open. Also there are continuous open surjections from the quotient spaces to $\mathcal{G}^1$ and $\mathcal{G}^0$ induced by d and d^2, respectively.

Lemma

There are Bruhat approximate vertical and horizontal cross-sections for $\mathcal{G}$ over $\mathcal{H}\backslash\mathcal{G}$, that is non negative continuous functions b_{v}, b_{h} on $\mathcal{G}$ whose supports have compact intersections respectively with $\mathcal{H}^2 \cdot_{\mathrm{v}} K$ and $\mathcal{H}^2 \cdot_{\mathrm{h}} K$ for each compact subset K of $\mathcal{G}^2$ such that

$$\int b_{\mathrm{v}}(c^{-\mathrm{v}} \cdot_{\mathrm{v}} a)d\sigma_{\mathrm{v}}^{r(a)}(c) = 1, \quad \int b_{\mathrm{h}}(c^{-\mathrm{h}} \cdot_{\mathrm{h}} a)d\sigma_{\mathrm{h}}^{r^2(a)}(c) = 1,$$

for each $a \in \mathcal{G}^2$.

Quotients

Consider equivalence relations on $\mathcal{G}^{(2\mathrm{v})}$ and $\mathcal{G}^{(2\mathrm{h})}$, $(a_1, b_1) \sim_{\mathrm{v}} (a_2, b_2)$ iff $b_1 = b_2$ and $a_1 \cdot_{\mathrm{v}} a_2^{-\mathrm{v}} \in \mathcal{H}^2$ and $(a_1, b_1) \sim_{\mathrm{h}} (a_2, b_2)$ iff $b_1 = b_2$ and $a_1 \cdot_{\mathrm{h}} a_2^{-\mathrm{h}} \in \mathcal{H}^2$, then the quotient spaces $\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})}$ and $\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{h})}$ are locally compact 2-groupoids with set of 1-morphisms $\mathcal{H}\backslash\mathcal{G}^1$ and $\mathcal{H}\backslash\mathcal{G}^0$ with left 2-Haar systems $\{\delta_{\dot{a}} \times \lambda_{\mathrm{v}}^{d(\dot{a})}\}$ and $\{\delta_{\dot{a}} \times \lambda_{\mathrm{h}}^{d^2(\dot{a})}\}$ with a ranging respectively over $\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})}$ and $\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{h})}$.

Imprimitivity modules

For $\varphi \in C_c(\mathcal{H})$ and $f \in C_c(\mathcal{G})$,

$$\varphi \cdot_{\mathrm{v}} f(a) = \int \varphi(c) f(c^{-\mathrm{v}} \cdot_{\mathrm{v}} a) d\sigma_{\mathrm{v}}^{r(a)}(c),$$

$$f \cdot_{\mathrm{v}} \varphi(a) = \int f(a \cdot_{\mathrm{v}} c) \varphi(c^{-\mathrm{v}}) d\sigma_{\mathrm{v}}^{d(a)}(c),$$

and

$$\varphi \cdot_{\mathrm{h}} f(a) = \int \varphi(c) f(c^{-\mathrm{h}} \cdot_{\mathrm{h}} a) d\sigma_{\mathrm{h}}^{r^2(a)}(c),$$

$$f \cdot_{\mathrm{h}} \varphi(a) = \int f(a \cdot_{\mathrm{h}} c) \varphi(c^{-\mathrm{h}}) d\sigma_{\mathrm{h}}^{d^2(a)}(c),$$

for $a \in \mathcal{G}^2$.

Imprimitivity modules

Also for $\phi \in C_c(\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})})$, $\psi \in C_c(\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})})$ and $f \in C_c(\mathcal{G})$,

$$\phi \cdot_{\mathrm{v}} f(a) = \int \phi(\dot{a}^{-\mathrm{v}}, a \cdot_{\mathrm{v}} b) f(b^{-\mathrm{v}}) d\lambda_{\mathrm{v}}^{d(a)}(b),$$

$$f \cdot_{\mathrm{v}} \phi(a) = \int f(b)\phi(\dot{b}, b^{-\mathrm{v}} \cdot_{\mathrm{v}} a) d\lambda_{\mathrm{v}}^{r(a)}(b),$$

and

$$\psi \cdot_{\mathrm{v}} f(a) = \int \psi(\dot{a}^{-\mathrm{h}}, a \cdot_{\mathrm{h}} b) f(b^{-\mathrm{h}}) d\lambda_{\mathrm{h}}^{d^2(a)}(b),$$

$$f \cdot_{\mathrm{h}} \psi(a) = \int f(b)\psi(\dot{b}, b^{-\mathrm{h}} \cdot_{\mathrm{h}} a) d\lambda_{\mathrm{h}}^{r^2(a)}(b),$$

for $a \in \mathcal{G}^2$.

Imprimitivity modules

Then $X_{\mathrm{v}} := C_{c\mathrm{v}}(\mathcal{G})$ is a bimodule over $B_{\mathrm{v}} := C_{c,\mathrm{v}}(\mathcal{H})$and $E_{\mathrm{v}} := C_{c,\mathrm{v}}(\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})})$ with commuting actions on opposite sides and the action of $C_{c,\mathrm{v}}(\mathcal{H})$ as double centralizers on $C_{c\mathrm{v}}(\mathcal{G})$ extends to an action on $C^*_{\mathrm{v}}(\mathcal{G})$, giving a $*$-homomorphism of $C_{c,\mathrm{v}}(\mathcal{H})$ into the multiplier algebra $M(C^*_{\mathrm{v}}(\mathcal{G}))$, and the same holds for $C_{c\mathrm{h}}(\mathcal{G})$.

Imprimitivity modules

Consider $X_{\rm v}$ as a left $E_{\rm v}$-module and right $B_{\rm v}$-module with the following vector valued inner products
$\langle f, g\rangle_{B_{\rm v}}(c) = \int \bar{f}(a^{-{\rm v}})g(a^{-{\rm v}} \cdot_{\rm v} c)d\lambda_{\rm v}^{r(c)}(a)$ and

$$\langle f, g\rangle_{E_{\rm v}}(\dot{a}, a^{-{\rm v}} \cdot_{\rm v} b) = \int f(a^{-{\rm v}} \cdot_{\rm v} c)\bar{g}(b \cdot_{\rm v} c)d\sigma_{\rm v}^{r(a)}(c),$$

for $c \in \mathcal{H}^2, a, b \in \mathcal{G}^2$. Then

$$\langle f, gh\rangle_{B_{\rm v}} = \langle f, g\rangle_{B_{\rm v}} h, \ \ \langle ef, g\rangle_{B_{\rm v}} = \langle f, e^* g\rangle_{B_{\rm v}},$$

and

$$\langle ef, g\rangle_{E_{\rm v}} = e\langle f, g\rangle_{E_{\rm v}}, \ \ \langle f, gh\rangle_{E_{\rm v}} = \langle fh^*, g\rangle_{E_{\rm v}},$$

for $f, g \in X_{\rm v}$, $h \in B_{\rm v}$ and $e \in E_{\rm v}$, and $f_1\langle g, f_2\rangle_{B_{\rm v}} = \langle f_1, g\rangle_{E_{\rm v}} f_2$, for $f_1, f_2, g \in X_{\rm v}$. The same holds for the horizontal spaces and modules.

Lemma

The linear span of $\{\langle f, g\rangle_{E_{\mathrm{v}}} : f, g \in X_{\mathrm{v}}\}$ contains a left approximate identity for E_{v} in the inductive limit topology and is dense in E_{v} and $C^*_{\mathrm{v}}(\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})})$. Similarly the linear span of $\{\langle f, g\rangle_{B_{\mathrm{v}}} : f, g \in X_{\mathrm{v}}\}$ is dense in B_{v} and $C^*_{\mathrm{v}}(\mathcal{H})$. Same holds for E_{h} B_{h}.

Corollary

The C^*-algebras $C^*_{\mathrm{v}}(\mathcal{G}^{(2\mathrm{v})})$ and $C^*_{\mathrm{v}}(\mathcal{G}^1)$ are strongly Morita equivalent. Similarly, $C^*_{\mathrm{h}}(\mathcal{G}^{(2\mathrm{h})})$ and $C^*_{\mathrm{v}}(\mathcal{G}^0)$ are strongly Morita equivalent.

Conditional expectation

Now by Rieffel construction, each v-representation of $C^*_{\mathrm{v}}(\mathcal{G}^1)$ induces a v-representation of $C^*_{\mathrm{v}}(\mathcal{G}^{(2\mathrm{v})})$ and then restricts to a v-representation of $C^*_{\mathrm{v}}(\mathcal{G})$ which acts on $C^*_{\mathrm{v}}(\mathcal{G}^{(2\mathrm{v})})$ as double centralizers, in other words, the restriction map $P_{\mathrm{v}} : C_{c,\mathrm{v}}(\mathcal{G}) \to C_{c,\mathrm{v}}(\mathcal{G}^1)$ is a generalized conditional expectation in the sense of (Rieffel, 1974). Similarly we have a generalized conditional expectation $P_{\mathrm{h}} : C_{c,\mathrm{h}}(\mathcal{G}) \to C_{c,\mathrm{h}}(\mathcal{G}^0)$.

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More generally, if $\mathcal{G}$ is second countable and $\mathcal{H}$ is a closed 2-subgroupoid such that both $\mathcal{G}$ and $\mathcal{H}$ have sufficiently many non singular Borel sets, the restriction map from $C_{c\mathrm{v}}(\mathcal{G})$ to $C_{c\mathrm{v}}(\mathcal{H})$ is a generalized conditional expectation, and the same for $C_{c\mathrm{h}}(\mathcal{G})$.

Induced representation

For the representation of $C_{\mathrm{v}}^*(\mathcal{G}^1)$ given by multiplication on $L^2(\mathcal{G}^1, \mu^1)$ the induced representation $Ind\mu^1$ acts on $L^2(\mathcal{G}^1, {\nu_{\mathrm{v}}}^{-1})$ by convolution on the left, namely

$$\langle Ind\mu^1(f)\xi, \eta\rangle = \int\int\int f(a \cdot_{\mathrm{v}} b)\xi(b^{-\mathrm{v}})\bar{\eta}(a)d\lambda_{\mathrm{v}}^u(b)\lambda_{\mathrm{v},u}(a)d\mu^1(u),$$

for $f \in C_{c\mathrm{v}}(\mathcal{G})$ and $\xi, \eta \in L^2(\mathcal{G}^1, {\nu_{\mathrm{v}}}^{-1})$. When μ^1 is quasi-invariant , $Ind\mu^1$ is just the left regular representation on μ^1. In this case, $ker(Ind\mu^1)$ consists of those $f \in C_{c\mathrm{v}}(\mathcal{G})$ that $f = 0$ on $supp({\nu_{\mathrm{v}}}^{-1})$. Since $\mathcal{G}^1$ has a faithful family of quasi-invariant measures, $C_{c\mathrm{v}}(\mathcal{G})$ has a faithful family of v-bounded representations (consisting of induced representations of such quasi-invariant measures).

Reduced C^*-algebras

In particular, $\|f\|^{\mathrm{v}}_{red} := \sup_{\mu^1} \|Ind\mu^1(f)\|$ is a C^*-norm, where μ^1 ranges over all quasi-invariant Borel measures on $\mathcal{G}^1$, and $\|f\|^{\mathrm{v}}_{red} \leq \|f\|^{\mathrm{v}}$, for each $f \in C_{c\mathrm{v}}(\mathcal{G})$. Similarly $\|f\|^{\mathrm{h}}_{red} := \sup_{\mu^0} \|Ind\mu^0(f)\| \leq \|f\|^{\mathrm{h}}$ is a C^*-norm, where μ^0 ranges over all quasi-invariant Borel measures on $\mathcal{G}^0$. The completions $C^*_{\mathrm{v},red}(\mathcal{G})$ and $C^*_{\mathrm{h},red}(\mathcal{G})$ of $C_{c\mathrm{v}}(\mathcal{G})$ and $C_{c\mathrm{h}}(\mathcal{G})$ with respect to these C^*-norms are called the vertical and horizontal reduced C^*-algebras of $\mathcal{G}$, which are quotients of the vertical and horizontal full C^*-algebras $C^*_{\mathrm{v}}(\mathcal{G})$ and $C^*_{\mathrm{h}}(\mathcal{G})$ of $\mathcal{G}$.

Proposition

If a second countable locally compact groupoid $\mathcal{G}$ has two 2-Haar systems $\{\lambda_{\mathrm{v}}^u\}$, $\{\lambda_{\mathrm{h}}^x\}$and $\{\sigma_{\mathrm{v}}^u\}$, $\{\sigma_{\mathrm{h}}^x\}$and it has sufficiently many non singular Borel $\mathcal{G}^1$-sets (resp. $\mathcal{G}^0$-sets) with respect to both systems, then the corresponding C^*-algebras $C_{\mathrm{v}}^*(\mathcal{G}, \lambda)$ and $C_{\mathrm{v}}^*(\mathcal{G}, \sigma)$ (resp. $C_{\mathrm{h}}^*(\mathcal{G}, \lambda)$ and $C_{\mathrm{h}}^*(\mathcal{G}, \sigma)$) are strongly Morita equivalent.

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Lemma

Let $\mathcal{G}$ be an r-discrete 2-groupoids with 2-Haar system and $a \in \mathcal{G}^2$. Let $L = Ind\mu^1$ (resp. $L = Ind\mu^0$) be the representation of $C_{cv}(\mathcal{G})$ (resp. $C_{ch}(\mathcal{G})$) induced by the point mass $\mu^1 = \delta_{d(a)}$ (resp. $\mu^0 = \delta_{d^2(a)}$), then for every $f \in C_{cv}(\mathcal{G})$ (resp. $f \in C_{ch}(\mathcal{G})$),

$$f(a) = \langle L(f)\delta_u, \delta_a \rangle = L(f)\delta_u(a),$$

where $u = d(a)$ (resp. $u = x := d^2(a)$) and δ_u, δ_a are regarded as unit vectors in $L^2(\mathcal{G}, \lambda_{vu})$ (resp. in $L^2(\mathcal{G}, \lambda_{hx})$). In particular, $max\{\|f\|_\infty, \|f\|_2\} \le \|f\|^{v}_{red}$ (resp. the same for $\|f\|^{h}_{red}$) where $\|.\|_2$ is the norm in $L^2(\mathcal{G}, \lambda_{vu})$ (resp. in $L^2(\mathcal{G}, \lambda_{hx})$).

GNS-representation

Now the inclusion map $j_v : C_{cv}(\mathcal{G}) \to C_0(\mathcal{G})$ extends to a norm decreasing linear map $j_v : C^*_{\mathrm{v},red}(\mathcal{G}) \to C_0(\mathcal{G})$. Let us observe that the latter map is still injective: consider the surjection $p : C_{cv}(\mathcal{G}) \to C_c(\mathcal{G}^1)$, for a quasi-invariant probability measure μ^1 on $\mathcal{G}^1$, the induced representation $Ind\mu^1$ is the GNS-representation of $\mu^1 \circ p$, namely $\int p(f) d\mu^1 = \langle Ind\mu^1(f)\xi_0, \xi_0\rangle$ and $Ind\mu^1(f)\xi_0 = f *_{\mathrm{v}} \xi_0 = j_v(f)$ where $\xi_0 \in L^2(\mathcal{G}, {\nu_{\mathrm{v}}}^{-1})$ is the characteristic function of $\mathcal{G}^1$ and j_v is now considered as the inclusion from $C_{cv}(\mathcal{G})$ into $L^2(\mathcal{G}, {\nu_{\mathrm{v}}}^{-1})$, now the above lemma shows that $Ind\mu^1(g)\xi_0 = j_v(g)$ remains valid for $g \in C^*_{\mathrm{v},red}(\mathcal{G})$ and if $j_v(g) = 0$ then $Ind\mu^1(g) = 0$ as ξ_0 is a cyclic vector, and this, being true for all quasi-invariant probability measures μ^1 on $\mathcal{G}^1$, implies that $g = 0$. Also $\|g\|_\infty \le \|g\|^{\mathrm{v}}_{red}$, where on the left hand side g is regarded as a continuous function on $\mathcal{G}$. The same observations hold for $C^*_{\mathrm{h},red}(\mathcal{G})$.

Principal 2-groupoid

A 2-groupoid $\mathcal{G}$ is called essentially v-principal (resp. h-principal), if for every invariant closed subset F of $\mathcal{G}^1$ (resp. $\mathcal{G}^0$) the set of $u \in F$ (resp. $x \in F$) whose isotropy group $\mathcal{G}_u^u$ (resp. $\mathcal{G}_x^x$) is a singleton, is dense in F. It is called essentially principal, if for every invariant closed subset F of $\mathcal{G}^0$ the set of $x \in F$ whose isotropy groupoid $\mathcal{G}(x)$ is a singleton, is dense in F.

Lemma

Let $\mathcal{G}$ be an r-discrete essentially v-principal (resp. h-principal) 2-groupoids with 2-Haar system and $a \in \mathcal{G}^2$. For any quasi-invariant measure μ^1 on $\mathcal{G}^1$ (resp. μ^0 on $\mathcal{G}^0$) with support F, any v-representation (resp. h-representation) π on μ^1 (resp. μ^0), and any $f \in C_{cv}(\mathcal{G})$ (resp. $f \in C_{ch}(\mathcal{G})$) we have $\sup_F f \leq \|\tilde{\pi}(f)\|$.

Correspondence

Let $\mathcal{G}$ be a locally compact groupoid with 2-Haar system. For an invariant open subset U of $\mathcal{G}^1$ (resp. G^0) let $I_{\text{cv}}(U) = \{f \in C_{\text{cv}}(\mathcal{G}) : f(u) = 0 \ (u \notin \mathcal{G}_U)\}$ (resp. $I_{\text{ch}}(U) = \{f \in C_{\text{ch}}(\mathcal{G}) : f(x) = 0 \ (x \notin \mathcal{G}_U)\}$) and $I_{\text{v}}(U)$ (resp. I_{h}) be its closure. Let F be the complement of U in $\mathcal{G}^1$ (resp. G^0) then it follows from [7, 2.4.5] that $I_{\text{v}}(U)$ (resp. I_{h}) is isomorphic to $C^*_{\text{v},red}(\mathcal{G}_U)$ (resp. $C^*_{\text{h},red}(\mathcal{G}_U)$), and it is a closed ideal of $C^*_{\text{v},red}(\mathcal{G})$ (resp. $C^*_{\text{h},red}(\mathcal{G})$) whose quotient is isomorphic to $C^*_{\text{v},red}(\mathcal{G}_F)$ (resp. $C^*_{\text{h},red}(\mathcal{G}_F)$). If μ^1 (resp. μ^0) is a quasi-invariant measure on $\mathcal{G}^1$ (resp. on $\mathcal{G}^0$) with support F, U is the complement of F, then $I_{\text{v}}(U) = ker(Ind\mu^1)$ (resp. $I_{\text{h}} = ker(Ind\mu^0)$). This provides a one-to-one correspondence between invariant open subsets of $\mathcal{G}^1$ (resp. G^0) and a family of closed ideals of $C^*_{\text{v},red}(\mathcal{G})$ (resp. $C^*_{\text{h},red}(\mathcal{G})$). Both sets are a lattice with respect to inclusion.

GNS-representation

When $\mathcal{G}$ is r-discrete and essentially v-principal (resp. h-principal), the above correspondence is an order preserving bijection, namely all closed ideals of $C^*_{\mathrm{v},red}(\mathcal{G})$ (resp. $C^*_{\mathrm{h},red}(\mathcal{G})$) are of the form $I_{\mathrm{v}}(U)$ (resp. I_{h}) for some invariant open subset U of $\mathcal{G}^1$ (resp. G^0) and the correspondence $U \mapsto I_{\mathrm{v}}(U)$ (resp. I_{h}) preserves inclusion. Indeed, in this case, the surjection p defined above is a conditional expectation and $Ind\mu^1$ (resp. $Ind\mu^0$) is the GNS-representation of $\mu^1 \circ p$ (resp. $\mu^0 \circ p$) and so $\|Ind\mu^1(f)\| \leq \|\tilde{\pi}(f)\|$ for $f \in C_{c\mathrm{v}}(\mathcal{G})$ (resp. $\|Ind\mu^0(f)\| \leq \|\tilde{\pi}(f)\|$ for $f \in C_{c\mathrm{h}}(\mathcal{G})$) hence $ker(\tilde{\pi})$ is equal to $I_{\mathrm{v}}(U)$ (resp. I_{h}) where U is the complement of the support of μ^1 (resp. μ^0).

Lemma

Let $\mathcal{G}$ be an r-discrete with 2-Haar system. An element g of $C^*_{\mathrm{v},red}(\mathcal{G})$ (resp. $C^*_{\mathrm{h},red}(\mathcal{G})$) commutes with each element of $C^*_{\mathrm{v}}(\mathcal{G}^1)$ (resp. $C^*_{\mathrm{h}}(\mathcal{G}^0)$) iff it vanishes off the isotropy group bundle $\bigsqcup_{u\in\mathcal{G}^1}\mathcal{G}^u_u$ (resp. $\bigsqcup_{x\in\mathcal{G}^0}\mathcal{G}^x_x$).

Corollary

If $\mathcal{G}$ is an r-discrete with 2-Haar system, $C^*_{\mathrm{v}}(\mathcal{G}^1)$ (resp. $C^*_{\mathrm{h}}(\mathcal{G}^0)$) is a masa in $C^*_{\mathrm{v},red}(\mathcal{G})$ (resp. $C^*_{\mathrm{h},red}(\mathcal{G})$) iff $\mathcal{G}^1$ (resp. $\mathcal{G}^0$) is the interior of the isotropy group bundle $\bigsqcup_{u\in\mathcal{G}^1}\mathcal{G}^u_u$ (resp. $\bigsqcup_{x\in\mathcal{G}^0}\mathcal{G}^x_x$).

Ample semigroup

In the above corollary, if moreover $\mathcal{G}$ is essentially v-principal (resp. h-principal), the restriction map $p : C^*_{\mathrm{v},red}(\mathcal{G}) \to C^*_{\mathrm{v}}(\mathcal{G}^1)$ (resp. $p : C^*_{\mathrm{h},red}(\mathcal{G}) \to C^*_{\mathrm{h}}(\mathcal{G}^0)$) is a faithful surjective conditional expectation and there is a one-to-one correspondence between the ample semigroup of compact open $\mathcal{G}^1$-sets (resp. $\mathcal{G}^1$-sets) and the inverse semigroup of partial homeomorphisms of $C^*_{\mathrm{v}}(\mathcal{G}^1)$ (resp. $C^*_{\mathrm{h}}(\mathcal{G}^0)$) defined by conjugation with respect to the elements in the normalizer of $C^*_{\mathrm{v}}(\mathcal{G}^1)$ (resp. $C^*_{\mathrm{h}}(\mathcal{G}^0)$) in $C^*_{\mathrm{v},red}(\mathcal{G})$ (resp. $C^*_{\mathrm{h},red}(\mathcal{G})$) (c.f. [7, 2.4.8]).

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