HI Augmentations of $\mathcal{L}_\infty$ spaces and spaces with very few operators

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The space $\mathcal{L}(X)$

Problem

Find methods of constructing Banach spaces X such that the elements of $\mathcal{L}(X)$ have desirable properties.

- Although there is a deep understanding on methods constructing Banach spaces with desirable properties the corresponding problem for the space $\mathcal{L}(X)$ does not have sufficient general answer.

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- Recent results yield Banach spaces with very few operators (i.e. spaces with the "scalar-plus-compact" property). The goal of the present talk is to discuss these recent developments.

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Hereditarily Indecomposable Banach spaces

- A Banach space X is said to be Hereditarily Indecomposable if no closed infinite dimensional subspace admits non trivial bounded linear projection.
- Every HI space does not contain subspaces with an unconditional basis.
- Also it is not a subspace of a space with an unconditional basis.
- Hence the two classes are completely separated and Gowers' dichotomy explains that every Banach space should contain a member of one of the above classes.

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Strictly Singular Operators

- An operator $T : X \to Y$ is called strictly singular, if its restriction on every infinite dimensional subspace of X is not an isomorphism.
- The class of strictly singular operators is very similar to the class of compact ones.
- For example, in many spaces X the ideal of strictly singular operators $\mathcal{S}(X)$ and the one of compact operators $\mathcal{K}(X)$ coincide. In particular, this is the case in $\ell_p(\mathbb{N})$ spaces.
- In the space $L^p(0,1), p \neq 2$ and $C[0,1]$ the ideals $\mathcal{S}(X)$ and $\mathcal{K}(X)$ are different.

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Operators on HI Banach spaces

Among the seminal properties of Hereditarily Indecomposable Banach spaces is that they admit few operators in the following sense:

- (W. T. Gowers - B. Maurey) For every complex HI space X $dim\mathcal{L}(X)/\mathcal{S}(X) = 1$
- (V. Ferenczi) For every real Banach X one of the following holds

$$\mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{R}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{C}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{H}$$

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The scalar-plus-compact problem

Problem (Scalar-plus-Compact)

Does there exist a Banach space X such that every operator $T : X \to X$ is of the form $\lambda I + K$, with $K : X \to X$ be a compact operator.

- This problem was stated by J. Lindenstrauss in 1969.
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The solution of the "Scalar-plus-Compact" problem

Theorem (*S. A., R. Haydon, Acta Math. 2011*)

There exists a $\mathcal{L}_\infty$ Hereditarily Indecomposable Banach space $\mathfrak{X}_K$ such that $\mathfrak{X}_K^*$ is isomorphic to $\ell_1(\mathbb{N})$ and every $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with K a compact operator.

- A separable space is a $\mathcal{L}_\infty$ space, if it is the closure of the union of an increasing sequence of finite dimensional spaces, each one isomorphic to ℓ_∞ of its dimension, with a uniformly bounded constant.
- Since the space $\mathfrak{X}_K$ has separable dual and it has a Schauder basis, we have that the space $\mathcal{L}(\mathfrak{X}_K)$ is separable.
- This property is not preserved for the space $\mathcal{L}(Y)$, where Y is a closed subspace of $\mathfrak{X}_K$.

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The solution of the "Scalar-plus-Compact" problem

- To build the space $\mathfrak{X}_K$ we combine two fundamental methods for constructing non-classical Banach spaces.
- The first method due to *J. Bourgain* and *F. Delbaen* (Acta Math, 1980) concerns spaces with minimal $\mathcal{L}_\infty$ structure.
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The solution of the "Scalar-plus-Compact" problem

- It is notable that the space $\mathfrak{X}_K$ belongs to the class of HI spaces while its dual $\mathfrak{X}_K^* \cong \ell_1(\mathbb{N})$ is a space with an unconditional basis. This fact describes the extreme structure of $\mathfrak{X}_K$.

More spaces with the “scalar-plus-compact” property

Theorem (*S. A., R. Haydon, Th. Raikoftsalis*)

There exists a $\mathcal{L}_\infty$ space $\mathfrak{X}$ with the “scalar-plus-compact” property and $\ell_1 \hookrightarrow \mathfrak{X}$.

- This is clearly a non HI space with non separable dual and with very few operators.

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Theorem (*S. A., D. Freeman, R. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht, D. Zisimopoulou)*

For every separable space X with X^* separable , there exists a $\mathcal{L}_\infty$ space $\mathfrak{X}$ with separable dual such that $\mathfrak{X}$ contains a subspace $\tilde{X}$ which is isomorphic to X, the quotient space $\mathfrak{X}/_{\tilde{X}}$ is hereditarily indecomposable and satisfies the “scalar-plus-compact” property.

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For every separable space X with X^* separable and X^{**} does not contain c_0, there exists a $\mathcal{L}_\infty$ space $\mathfrak{X}$ with separable dual, the "scalar-plus-compact" property and $X \hookrightarrow \mathfrak{X}$.

The compact operators of $\mathfrak{X}$

Theorem (J. Lindenstrauss)

If $\mathfrak{X}$ is a separable $\mathcal{L}_\infty$ space and X a subspace of $\mathfrak{X}$, then the subalgebra $\mathcal{A}$ of $\mathcal{K}(\mathfrak{X})$ consisting of all operators having X as an invariant subspace, has as quotient the compact operators on X.

General Bourgain-Delbaen construction

- The classical $\mathcal{L}_\infty$ spaces are the spaces $C(K)$ with K a compact set.
- The BD $\mathcal{L}_\infty$ spaces are spaces with minimal $\mathcal{L}_\infty$ structure.
- For example, the spaces which will be presented admit an FDD i.e. $\mathfrak{X} = (\sum_{n\in\mathbb{N}} \oplus M_n)$ and for every $L \subset \mathbb{N}$ such that both L, $\mathbb{N}\setminus L$ are infinite the subspace $Y = \sum_{n\in L} \oplus M_n$ is a reflexive one.

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- BD $\mathcal{L}_\infty$ spaces are exotic Banach spaces.
- The construction involves the choice of a separable subspace of $\ell_\infty(\mathbb{N})$ while the norm is the usual supremum norm.

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By induction we choose a sequence of finite sets $(\Delta_n)_{n=0}^{\infty}$ and we set $\Gamma_n = \cup_{k=0}^{n}\Delta_k$, $\Gamma = \cup_{n=0}^{\infty}\Delta_n$.

- Also by induction we choose linear

$$\phi_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Delta_{n+1})$$

and we define $i_{n,n+1} : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma_{n+1})$ as follows.

$$i_{n,n+1}(f)(\gamma) = \begin{cases} f(\gamma), & \text{if } \gamma \in \Gamma_n, \\ \phi_n(f)(\gamma), & \text{if } \gamma \in \Delta_{n+1} \end{cases}$$

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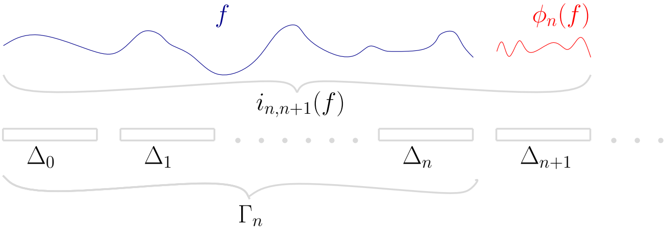

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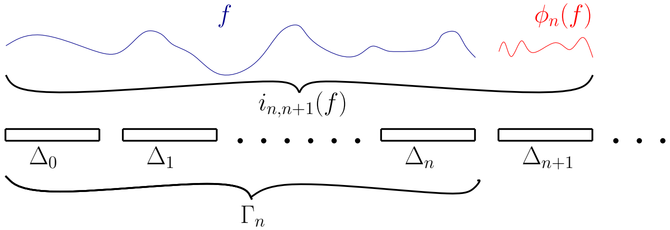

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For every $n < k$ we define $i_{n,k} : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma_k)$ to be the composition described as follows.

$$\ell_\infty(\Gamma_n) \xrightarrow{i_{n,n+1}} \ell_\infty(\Gamma_{n+1}) \longrightarrow \cdots \longrightarrow \ell_\infty(\Gamma_{k-1}) \xrightarrow{i_{k-1,k}} \ell_\infty(\Gamma_k)$$

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General Bourgain-Delbaen construction

A critical required property for the operators $i_{n,k}$ is that there exists a $C \geqslant 1$ such that $\|i_{n,k}\| \leqslant C$.

We set $\Gamma = \cup_{n=0}^{\infty} \Delta_n$ and we define

$$i_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma) \text{ as } i_n(f) = \varinjlim i_{n,k}(f).$$

The operators i_n are well defined since for every $n < p \leqslant k$, if $f \in \ell_\infty(\Gamma_n)$ and $\gamma \in \Delta_p$ we have that

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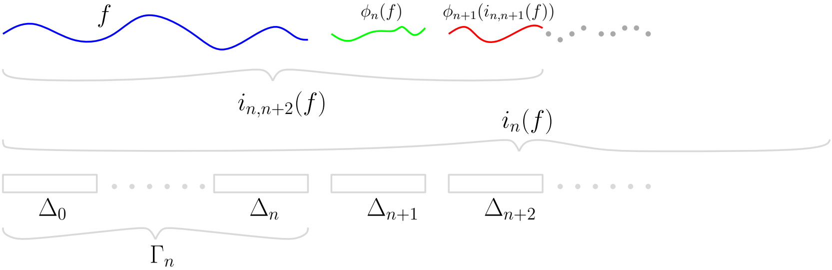

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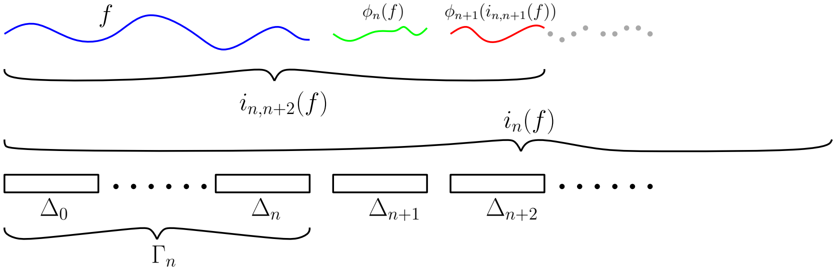

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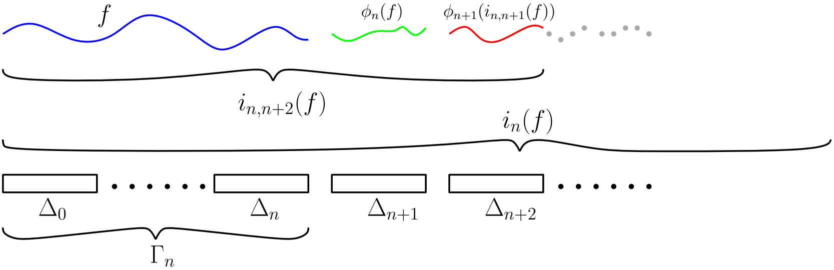

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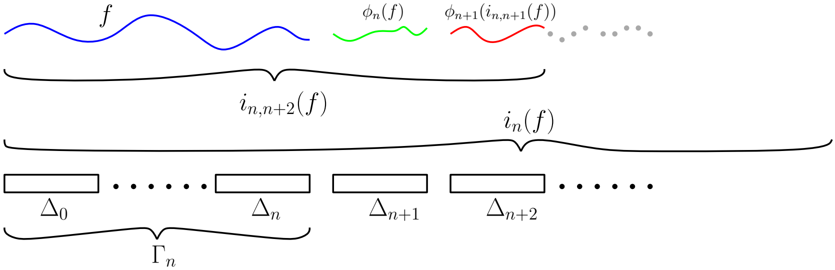

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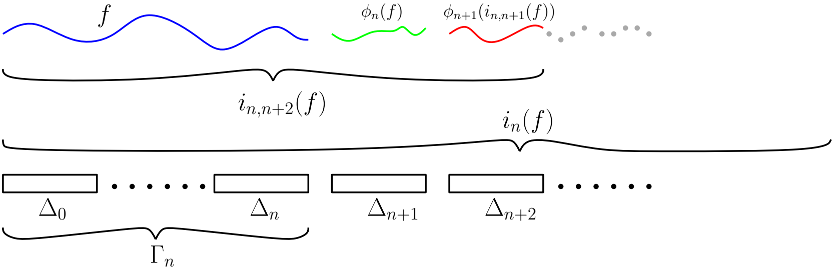

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We set $\mathfrak{X}(\Gamma) = \overline{\cup_{n=1}^{\infty} i_n(\ell_\infty(\Gamma_n))} \hookrightarrow \ell_\infty(\Gamma)$.

- Clearly the space $\mathfrak{X}(\Gamma)$ is $\mathcal{L}_\infty$ space.
- The variety of BD- $\mathcal{L}_\infty$ spaces arises from the different $\phi_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Delta_{n+1})$ one can define.
- In general, each ϕ_n is determined by a finite family $\{c_\gamma^* : \gamma \in \Delta_{n+1}\}$ where each $c_\gamma^* : \ell_\infty(\Gamma_n) \to \mathbb{R}$.

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- The choice of $\{c^*_\gamma : \gamma \in \Gamma\}$ such that the resulting $\mathcal{L}_\infty$ space is HI.
- The space will be depended on two sequences $(\lambda_j)_j$, $(F_j)_j$ of parameters where
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$$id(\gamma) = (n, j(\gamma), F(\gamma)),$$

such that

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We define $c_\gamma^* : \ell_\infty(\Gamma_{n-1}) \to \mathbb{R}$ as

- $c_\gamma^*(f) = f(\eta) + \lambda_j b^*(f - i_{q,n}(f|_{\Gamma_q}))$.

  $\eta \in \Delta_q$, $id(\eta) = (q, j(\eta) = j(\gamma), F(\eta) = F(\gamma) \setminus \{n\})$
- $b^* \in B_{\ell_1(\Gamma_n \setminus \Gamma_q)}$.
- If $j = 2k - 1$, then $b^* = e_\xi^*$ with $weight(\xi)$ uniquely determined by η.
- The definition of c_γ^* imposes the HI structure in the space $\mathfrak{X}_K$. In particular, the c_γ^* with $weight(\gamma)$ odd are the conditional functionals and they are closely related to the definition of the norming set in the Gowers Maurey construction.

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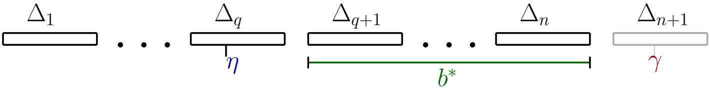

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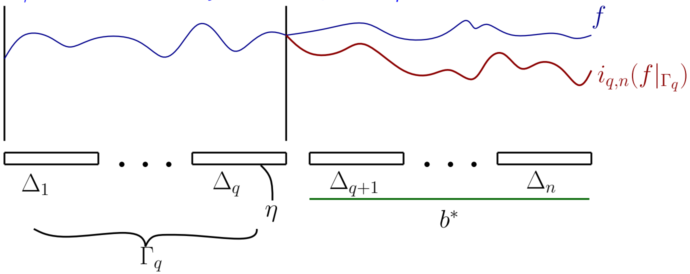

Argyros Haydon $\mathcal{L}_\infty$ space

- The Argyros Haydon hereditarily indecomposable and $\mathcal{L}_\infty$ space $\mathfrak{X}_k$ is a BD construction of the previous form which is based in the following parameters.
- A strictly increasing sequence of natural numbers $(m_j)_j$ with $m_1 \geqslant 4$. (where $\lambda_j = \frac{1}{m_j}$)
- The sequence of compact families $\mathcal{A}_{n_j}$ such that $(n_j)_j$ is strictly increasing sequences of natural numbers growing faster than $(m_j)_j$, where

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The scalar-plus-compact property of $\mathfrak{X}_K$

- The space $\mathfrak{X}_K$ is HI. Therefore every $T \in \mathcal{L}(\mathfrak{X}_K)$ is of the form $\lambda I + S$ with S strictly singular operator.
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- A sequence $(x_k)_{k\in\mathbb{N}}$ is a block sequence if there exist:
- $p_1 < q_1 < \ldots < p_k < q_k < \ldots$ and $y_k \in \ell_\infty(\cup_{k=p_k}^{q_k}\Delta_k)$ such that $x_k = i_{q_k}(y_k)$.
- We define the local support x_k as $supp_{loc}(x_k) = supp(y_k) = \{\gamma \in \Gamma : \ y_k(\gamma) \neq 0\}$ and

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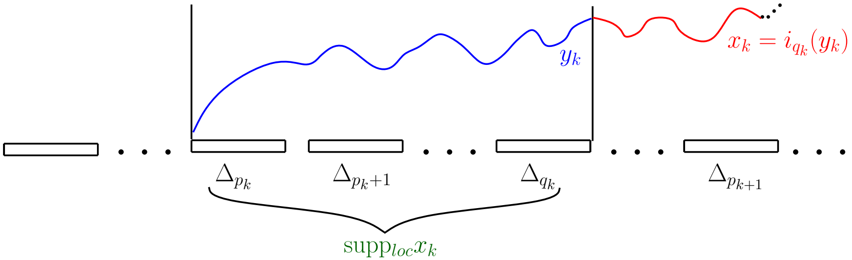

The scalar-plus-compact property of $\mathfrak{X}_K$

Proposition

Let $(x_k)_{k\in\mathbb{N}}$ be a normalized block sequence such that either
(i) the sequence $(G_{x_k})_{k\in\mathbb{N}}$ is uniformly bounded, or
(ii)The sequence $(G_{x_k})_{k\in\mathbb{N}}$ is strictly increasing.
Then for every strictly singular operator $S : \mathfrak{X}_k \to \mathfrak{X}_k$ we have that $\|S(x_k)\| \to 0$.

The scalar-plus-compact property of $\mathfrak{X}_K$

Proposition

Let $T : \mathfrak{X}_k \to \mathfrak{X}_k$ be a bounded linear non compact operator. Then there exists a normalized block sequence $(x_k)_{k\in\mathbb{N}}$ such that $(G_{x_k})_{k\in\mathbb{N}}$ is either uniformly bounded or strictly increasing and $\|T(x_k)\| \nrightarrow 0$.

The scalar-plus-compact property of $\mathfrak{X}_K$

Theorem

Let $T : \mathfrak{X}_k \to \mathfrak{X}_k$ be a bounded linear operator. Then there exists a scalar λ such that the operator $K = T - \lambda I$ is compact, where $I : \mathfrak{X}_k \to \mathfrak{X}_k$ is the identity operator.

HI- Augmentations

- Let $\mathfrak{X}(\Gamma)$, $\Gamma = \cup_{n=0}^{\infty}\Delta_n$ be a BD $\mathcal{L}_\infty$ space with separable dual (i.e. $\mathfrak{X}(\Gamma)^* \simeq \ell_1$).
- Let also X be a subspace of $\mathfrak{X}(\Gamma)$ such that $X^\perp$ is of infinite dimension and $\mathbb{Q}$ well disposed in $\ell_1(\Gamma)$, i.e.

$$\{f \in X^\perp : \quad f \ \text{ finite rational combination of } \ e_\gamma^*\}$$

is dense in $X^\perp$.

- Combining a recent result of D. Freeman, E. Odell, Th. Schlumprecht, (Math.Ann., 2011) with some classical results of W.B. Johnson, H.P. Rosenthal and M. Zippin we conclude that for every Banach space X with separable dual there exists a BD $\mathcal{L}_\infty$ space $\mathfrak{X}(\Gamma)$ such that $X^\perp$ is $\mathbb{Q}$ well disposed in $\ell_1(\Gamma)$.

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Let $\mathfrak{X}(\Gamma)$, X be as before.

A BD $\mathcal{L}_\infty$ space $\mathfrak{X}(\tilde{\Gamma})$ is an HI Augmentation of the pair $(\mathfrak{X}(\Gamma), X)$ if the following are satisfied:

- $\tilde{\Gamma} = \cup_{n=0}^{\infty} \tilde{\Delta}_n$ and $\Delta_n \subset \tilde{\Delta}_n$ (Hence $\Gamma \subset \tilde{\Gamma}$).
- There exists a linear bounded operator $\Phi : \mathfrak{X}(\tilde{\Gamma}) \to \mathfrak{X}(\Gamma)$.
- There exists a subspace $\tilde{X}$ of $\mathfrak{X}(\tilde{\Gamma})$ such that $\Phi(\tilde{X}) = X$ and $\Phi|_{\tilde{X}}$ is an isomorphism.
- The quotient space $\mathfrak{X}(\tilde{\Gamma})/_{\tilde{X}}$ is HI and satisfies the scalar-plus-compact property.
- If ℓ_1 does not embed complementary into X^*, then $\mathfrak{X}(\tilde{\Gamma})$ satisfies the scalar-plus-compact property.

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Theorem (*S. A., D. Freeman, R. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht, D. Zisimopoulou*)

For every separable space X with X^* separable , there exists a $\mathcal{L}_\infty$ space $\mathfrak{X}$ with separable dual such that $\mathfrak{X}$ contains a subspace $\tilde{X}$ which is isomorphic to X, the quotient space $\mathfrak{X}/_{\tilde{X}}$ is hereditarily indecomposable and satisfies the "scalar-plus-compact" property.

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Theorem (*S. A., D. Freeman, R. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht, D. Zisimopoulou*)

For every separable space X with X^* separable and X^{**} does not contain c_0, there exists a $\mathcal{L}_\infty$ space $\mathfrak{X}$ with separable dual, the "scalar-plus-compact" property and $X \hookrightarrow \mathfrak{X}$.

Corollary

Every separable reflexive Banach space embeds to a $\mathcal{L}_\infty$ space with the scalar-plus-compact property

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- The HI Augmentation of the pair $(\mathfrak{X}(\Gamma), X)$ essentially augments the initial set Γ.
- We start with two sequences of parameters $(\lambda_j)_j$, $(F_j)_j$, where $(\lambda_j)_j$ is a decreasing to zero sequence of positive reals and $(F_j)_j$ is a sequence of compact families of $\mathbb{N}$ with complexity greater than the local ℓ_1 complexity of $\mathfrak{X}(\Gamma)$.
- The sets $\tilde{\Delta}_n$ are recursively chosen and for every $\gamma \in \tilde{\Delta}_n$ we define a BD functional $\tilde{c}^*_\gamma : \ell_\infty(\tilde{\Gamma}_{n-1}) \to \mathbb{R}$ such that:

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- If $\gamma \in \tilde{\Delta}_n \backslash \Delta_n$ then $\tilde{c}^*_\gamma(f) = f(\eta) + \lambda_j b^*(f - i_{q,n}(f|_{\tilde{\Gamma}_q}))$, where

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- $b^* \in B_{\ell_1(\tilde{\Gamma}_n \backslash \tilde{\Gamma}_q)} \cap X^\perp$, where we consider X as a subspace of $\mathfrak{X}(\Gamma) \hookrightarrow \ell^\infty(\Gamma)$ and thus naturally embedded in $\ell_\infty(\Gamma \cup_{p=0}^{n-1} \tilde{\Delta}_p)$
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- We consider the following subspace of $\mathfrak{X}(\tilde{\Gamma}) \hookrightarrow \ell_\infty(\tilde{\Gamma})$

$$\tilde{X} = \{y \in \mathfrak{X}(\tilde{\Gamma}) : \quad y|_{\tilde{\Gamma}\setminus\Gamma} = 0, \quad y|_\Gamma \in X\}.$$

- We also define $\Phi : \mathfrak{X}(\tilde{\Gamma}) \to \mathfrak{X}(\Gamma)$ by the rule

$$\Phi(z) = z|_\Gamma$$

- It follows from our construction that $\tilde{X}$, Φ are well defined and

$$\Phi(\tilde{X}) = X.$$

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