Topological radical for Banach modules

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General theory of radicals

There are axiomatic theories and examples of radicals

(1) for rings;

(2) for modules over rings (with generalization to Abelian categories)

(3) for Banach algebras (see P. Dixon topological version of axioms and new results in works of V. Shulman and Yu. Turovskii);

But for Banach modules — nothing!

Jacobson radical has a good extension to modules in Theory of Rings.

Our goal is to generalize Jacobson radical from Banach algebras to Banach modules.

The Jacobson radical of a unital ring

(1) Rad is the *intersection of all maximal left ideals* (from outside)

(2) Rad is the *set of all* r *such that* $1 + ar$ *is invertible for every* a (from inside).

For a unital Banach algebra A:

(1) *every maximal left ideal is closed*

(2) $1 + ar$ is invertible for every $a \in A$ iff ar *is topologically nilpotent (i.e.* $\|(ar)^n\|^{1/n} \to 0$*) for every* $a \in A$*)*.

The radical of a module

A submodule Y in a module X over a unital ring is called *small* ($\equiv$ 'superfluous' $\equiv$ 'co-essential') if for every submodule Z, $Y+Z=X$ implies $Z=X$.

The radical of a unital module X is $\cap$ *of all maximal submodules* and $\cup$ *of all small submodules* (the notation is $\operatorname{rad} X$).

If r is in a unital ring A, Ar is small $\Leftrightarrow$ $1+ar$ is invertible for every a.

The notion is dual to the notion of the socle.

Radical is useful in structural theory. For example, a module X is Artinian and $\operatorname{rad} X = 0$ $\Leftrightarrow$ X is semi-simple and finitely generated.

See also projective covers, perfect and semi-perfect rings, semi-perfect modules etc.

Properties:

(1) If rad is a functor.

(2) $\operatorname{rad}(X/\operatorname{rad} X) = 0$.

(3) If Z is a submodule in X s.t. $\operatorname{rad}(X/Z) = 0$ then $\operatorname{rad} X \subset Z$.

(4) $R \cdot X \subset \operatorname{rad} X$, where $R = \operatorname{Rad} A$.

(5) $A \cdot x_0$ is small $\Leftrightarrow x_0 \in \operatorname{rad} X$.

(6a) X is fin. gen. $\Rightarrow \operatorname{rad} X$ is small in X.

(6b) X is fin. gen. and $X \neq 0 \Rightarrow \operatorname{rad} X \neq X$.

(7a) P is projective $\Rightarrow R \cdot P = \operatorname{rad} P$.

(7b) P is projective and $P \neq 0 \Rightarrow \operatorname{rad} P \neq P$ (not obvious).

But $\operatorname{rad}(\operatorname{rad} X)) \neq \operatorname{rad} X$ in general.

Maximal submodules, small submodules, and the radical itself in a Banach module need to be closed. But it works well in the finitely-generated case at least. (O. Yu. Aristov, *Projective covers of finitely generated Banach modules and the structure of some Banach algebras*// Extracta mathematicae V 21, N. 1, 2006, P. 1–26)

Small morphism of Banach modules

Notation 1 *Let $\varphi : Y \to X$ and $\psi : Z \to X$ be morphisms of Banach modules. Denote by $\varphi \dotplus \psi$ the morphism*

$$Y \oplus Z \to X \colon (y, z) \mapsto \varphi(y) + \psi(z).$$

Def. 2 *We say that a morphism $\psi \colon X_0 \to X$ of Banach modules is* small *if for every morphism $\varphi \colon Y \to X$ such that $\varphi \dotplus \psi$ is surjective φ is surjective also.*

A Banach algebra R is topologically nilpotent $\Leftrightarrow$ for every bounded sequence $(r_n) \subset R$,

$$\lim_{n\to\infty} \|r_1 r_2 \cdots r_n\|^{1/n} = 0.$$

Th. 3 (P. Dixon) *If X is a non-trivial left Banach module over a topologically nilpotent Banach algebra R and*

$$\pi \colon R \widehat{\otimes} X \to X \colon r \otimes x \mapsto r \cdot x,$$

then $\operatorname{Im} \pi \neq X$.

Modifying Dixon's argument we get

Th. 4 *For every left Banach module X over a topologically nilpotent Banach algebra the morphism π is small.*

Th. 5 *Let I be a closed left ideal in a unital Banach algebra A, and let $\iota : I \to A$ be the natural inclusion. The following conditions are equivalent.*

(A) *I is topologically nilpotent.*

(B) *For every unital left Banach A-module X the morphism of Banach A-modules*

$$I \mathbin{\widehat{\otimes}}_A X \to X \colon a \otimes_A x \mapsto a \cdot x$$

is small.

(C) *For every strictly projective unital left Banach A-module P the morphism of Banach A-modules $I \mathbin{\widehat{\otimes}}_A P \to P \colon a \otimes_A x \mapsto a \cdot x$ is small.*

(D) *The morphism of left Banach A-modules $(\iota \otimes 1) \colon I \mathbin{\widehat{\otimes}} \ell^1 \to A \mathbin{\widehat{\otimes}} \ell^1$ is small.*

Maximal contractive monomorphisms

Def. 6 *We say that a contractive monomorphism of left unital Banach A-modules $\alpha\colon Y \to X$ is* maximal *if*

(1) α *is not surjective,*

(2) *for every non-surjective contractive monomorphism β and every contractive morphism $\varkappa$ the equality $\alpha = \beta\varkappa$ implies that $\varkappa$ is an isometric isomorphism.*

Maximal monomorphisms can be described as embeddings of closed maximal submodules.

$\varepsilon : Y \to X$ is a C-epimorphism ($C \geq 1$) if for every $x \in X$ there exist $y \in Y$ such that $x = \varphi(y)$ and $\|y\| \leq C\|x\|$.

Prop. 7 *Set $\tau : A \to X : a \to a \cdot x_0$ where $x_0 \in X$. Suppose that $\varphi\colon Y \to X$ is a morphism such that $x_0 \notin \operatorname{Im}\varphi$ and $\varphi \dot{+} \tau$ is a C-epimorphism for $C \geq 1$. Then* $\operatorname{dist}(x_0, \operatorname{Im}\varphi) \geq 1/C$.

Prop. 8 *Let $C \geq 1$, and let φ be a contr. morphism with range in X. Denote by Γ a family of all contr. mono α with range in X s.t.*

(1) *α is not surjective;*

(2) *$\alpha \dotplus \varphi$ is a C-epimorphism.*

Suppose that $\exists \delta > 0$, $\exists x_0 \in X$ s.t. $\forall \alpha \in \Gamma$ $\mathrm{dist}(x_0, \mathrm{Im}\,\alpha) \geq \delta$ *. Then $\forall \alpha_0 \in \Gamma$ $\exists$ a maximal contr. mono γ such that $\gamma \in \Gamma$ and $\gamma \succeq \alpha_0$.*

Equivalence classes of contractive morphism form a lattice. There is a standard way to define a radical in a lattice using small and maximal elements.

Difficulties: (1) we define small and maximal morphism in different categories (topological and metric); (2) there are no sufficiently many 'compact elements' in our lattice.

Using Proposition 7 and 8 we can find a topological interplay between small and maximal morphisms.

Topological radical of a Banach module

Th. 9 *Let X be a left unital Banach A-module. Set*

$$X_1 := \bigcup \operatorname{Im} \psi,$$

where ψ are small morphisms to X,

$$X_2 := \bigcap \operatorname{Im} \iota,$$

where ι are maximal contractive mono to X.

Then

(1) $X_1 = X_2$*;*

(2) *this submodule is closed.*

Def. 10 *Let X be a left unital Banach A-module. We say that the closed submodule of X from Theorem 9 is a* topological radical *of X and denote it by* t-rad X.

Properties of the topological radical:

(1) If t-rad is a functor.

(2) $\text{t-rad}(X/\,\text{t-rad}\,X) = 0$.

(3) If Z is closed in X and $\text{t-rad}(X/Z) = 0$ then $\text{t-rad}\,X \subset Z$.

(4) $\overline{R \cdot X} \subset \text{t-rad}\,X$, where $R = \text{Rad}\,A$.

(5) $\tau\colon A \to X : a \to a{\cdot}x_0$ is small $\Leftrightarrow x_0 \in \text{t-rad}\,X$.

(6a) X is fin. gen. $\Rightarrow$ $\text{t-rad}\,X \to X$ is small.

(6b) X is fin. gen. and $X \neq 0 \Rightarrow \text{t-rad}\,X \neq X$.

(7a) P is a projective Banach module with the approximation property $\Rightarrow$ $\text{t-rad}\,P = \overline{R \cdot P}$.

$\text{t-rad}(\text{t-rad}\,X)) \neq \text{t-rad}\,X$ in general.

The analogue of (7b) is open.

Questions 11 (1) *Does exist a non-trivial projective Banach module P s.t.* t-rad $P = P$*?*

(2) *Does exist a non-trivial projective Banach module P s.t. $P = \overline{R \cdot P}$?*

Comparison with the algebraic radical:

(1) rad $X \subset$ t-rad X for every X.

(2) X is finitely-generated $\Rightarrow$ t-rad $X =$ rad X. In particular, t-rad $A =$ Rad A for a unital Banach algebra.

(3) Consider a radical Banach algebra R as a left unital Banach module over R_+. Then rad $R = R^2$ and $\overline{R^2} \subset$ t-rad R.

$L^1[0,1]$ and $C[0,1]$ are Banach algebras with respect to the cut-off convolution $*$.

Since $R = (L^1[0,1], *)$ admits a b.a.i.,

$$\operatorname{t-rad} R = \operatorname{rad} R = R^2 = R.$$

If $R = (C[0,1], *)$ then

$$\operatorname{t-rad} R = \overline{R^2} \neq \operatorname{rad} R = R^2.$$

An example of t-rad $X \neq \overline{R \cdot X}$.

If B is a semi-simple Banach algebra it is sufficient to find a unital B_+-module such that t-rad $X \neq 0$.

Let I is a proper ideal in B s.t. B/I is a radical Banach algebra.

Images of B_+-module morphisms to B/I are exactly images of B_+/I-module morphisms $\Rightarrow$ t-rad$(B/I)_{B_+} =$ t-rad$(B/I)_{B_+/I}$.

Since $B_+/I \cong (B/I)_+$,

$$\text{t-rad}(B/I)_{B_+/I} = B/I \neq 0.$$

Reference: O. Aristov arXiv:1203.4760v2