Uniqueness under spectral variation in Banach algebras

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Notation

A: complex unital Banach algebra, with unit $\mathbf{1}$

- $\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \text{ is not invertible}\}$
  [Spectrum of a]
- $\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$
  [Spectral radius of a]
- $\delta(a) = \sup\{|\lambda - \mu| : \lambda, \mu \in \sigma(a)\}$
  [Spectral diameter of a]
- $\#\sigma(a)$
  [Number of elements in the spectrum of a]
- $\#\sigma'(a)$
  [Number of nonzero elements in the spectrum of a]

The following theorem of Aupetit is crucial in the proofs of some results and is sometimes referred to as the *Scarcity Principle:*

Theorem (B. Aupetit)

If f is analytic from a domain $D \subseteq \mathbb{C}$ into a Banach algebra A, then either

$$D_F = \{\lambda \in D : \sigma(f(\lambda)) \text{ is finite } \}$$

is a Borel set with zero capacity, or there is $n \in \mathbb{N}$ and a closed, discrete subset $E \subset D$ such that $\#\sigma(f(\lambda)) = n$ for $\lambda \in D \backslash E$ and $\#\sigma(f(\lambda)) < n$ for $\lambda \in E$. In this case the n points of $\sigma(f(\lambda))$ are locally holomorphic on $D \backslash E$.

Objective

On which subsets L of A can the pair of spectrum functions

$$x \mapsto \sigma_A(ax) \quad \text{and } x \mapsto \sigma_A(bx), \;\; x \in L \tag{1}$$

or, alternatively, the pair

$$x \mapsto \sigma_A(a+x) \quad \text{and } x \mapsto \sigma_A(b+x), \;\; x \in L \tag{2}$$

distinguish between a and b?

Theorem

Let A be a semisimple Banach algebra and $a, b \in A$. Then $a = b$ if and only if any one of the following holds:

(i) $\sigma(ax) = \sigma(bx)$ *for all* $x \in A$

(ii) $\sigma(a + x) = \sigma(b + x)$ *for all* $x \in A$

Theorem

Let A be a semisimple Banach algebra.

(i) *If $b \in A$ is invertible and $\#\sigma(ax) \leq \#\sigma(bx)$ for all x in a neighborhood of b^{-1}, then $a = \alpha b$ for some $\alpha \in \mathbb{C}$. In particular if $\sigma(ax) = \sigma(bx)$ for all x in a neighborhood of b^{-1}, then $a = b$.*

(ii) *If $\#\sigma(a + x) \leq \#\sigma(b + x)$ for all x in a neighborhood of $-b$, then $a = b + \alpha\mathbf{1}$ for some $\alpha \in \mathbb{C}$. In particular if $\sigma(a + x) = \sigma(b + x)$ for all x in a neighborhood of $-b$, then $a = b$.*

Theorem

Let A be a semisimple Banach algebra. If $\sigma(ax)$ and $\sigma(bx)$ are finite and equal for all x in some open set N, then $a = b$. In particular the above characterization holds if A is finite dimensional.

Corollary

Let A be a semisimple Banach algebra. If $\sigma(a+x)$ *and* $\sigma(b+x)$ *are finite and equal for all* x *in some open set* N*, then* A *is finite dimensional and* $a=b$*.*

Theorem

Let A be a semisimple Banach algebra, and suppose $\sigma(ax)$ *and* $\sigma(bx)$ *have at most* 0 *as accumulation point for all* $x \in A$*. If* $\sigma(ax) = \sigma(bx)$ *for all* x *in some open set* N*, then* $a = b$*.*

Theorem

Let $(A, \|\cdot\|)$ be a semisimple Banach algebra and let $a, b \in A$. Then $a = b$ if and only if any one of the following conditions holds:

(i) *For each Banach algebra norm $\|\cdot\|_0$ equivalent to $\|\cdot\|$,*

$$\|x - \mathbf{1}\|_0 < 1 \Rightarrow \sigma(ax) = \sigma(bx).$$

(ii) $\sigma(ax) = \sigma(bx)$ *for all x satisfying $\rho(x - \mathbf{1}) < 1$.*

(iii) $\sigma(ax) = \sigma(bx)$ *for all exponentials $x \in A$.*

(iv) $\sigma(a + x) = \sigma(b + x)$ *for all exponentials $x \in A$.*

Example

(A) Let $B_0 = B(0,1)$ *and* $B_2 = B(2, \frac{1}{2})$ *and let* A *be the semisimple algebra of complex functions (under the usual pointwise operations) which are continuous on* $\overline{B_0} \cup \overline{B_2}$ *and holomorphic on* B_0*. Define a norm on* A *by*

$$\|f\| = \rho(f) + \delta(f).$$

Define

$$a(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in \overline{B_0} \\ 0 & \text{if } \lambda \in \overline{B_2} \end{cases}$$

and

$$b(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in \overline{B_0} \\ \frac{1}{4}(\lambda - \frac{3}{2}) & \text{if } \lambda \in \overline{B_2} \end{cases}$$

Then $\sigma(af) = \sigma(bf)$ *for all* f *satisfying* $\|f - \mathbf{1}\| < 1$ *but* $a \neq b$*.*

(B) Let A be the same algebra as in (A), but with the norm on A given by the spectral radius. Fix any $0 < r < 1$. Define

$$a(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in \overline{B_0} \\ 0 & \text{if } \lambda \in \overline{B_2} \end{cases}$$

and

$$b(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in \overline{B_0} \\ \frac{1-r}{4}(\lambda - \frac{3}{2}) & \text{if } \lambda \in \overline{B_2} \end{cases}$$

Then $\sigma(af) = \sigma(bf)$ for all f satisfying $\rho(f - \mathbf{1}) < r$ but $a \neq b$.

(C) Let $0 < r \in \mathbb{R}$ be arbitrary but fixed. Let A be the Banach algebra in (A) with the spectral radius norm. Define

$$a(\lambda) = \begin{cases} 7r\lambda & \text{if } \lambda \in \overline{B_0} \\ \frac{r}{2}(\lambda - \frac{3}{2}) & \text{if } \lambda \in \overline{B_2} \end{cases}$$

$$b(\lambda) = \begin{cases} 7r\lambda & \text{if } \lambda \in \overline{B_0} \\ \frac{r}{2}(\lambda - 2) & \text{if } \lambda \in \overline{B_2} \end{cases}$$

Then $\sigma(a + f) = \sigma(b + f)$ for all f satisfying $\|f\| < r$ but $a \neq b$.

Following a paper by Aupetit and Mouton
[Trace and determinant in Banach algebras, 1996]:
The **rank** of $a \in A$ is defined by

$$\operatorname{rank}(a) = \sup_{x \in A} \#\sigma'(ax) \leq \infty.$$

The set

$$E(a) = \{x \in A : \#\sigma'(ax) = \operatorname{rank}(a)\}$$

is dense and open in A. Moreover, the **socle** of A is given by

$$\{a \in A : \operatorname{rank}(a) < \infty\} = \operatorname{soc}(A).$$

An element $a \in \operatorname{soc}(A)$ is said to be **maximal rank** if $\operatorname{rank}(a) = \#\sigma'(a)$.

The **trace** of $a \in \mathrm{soc}(A)$ is defined by

$$\mathrm{tr}(a) = \sum_{\lambda \in \sigma(a)} \lambda \, m(\lambda, a)$$

where $m(\lambda, a)$ is the **multiplicity of** a **at** λ.

A brief description of the notion of multiplicity: Let $a \in \mathrm{soc}(A)$, $\lambda \in \sigma(a)$ and let V_λ be an open disk centered at λ such that V_λ contains no other points of $\sigma(a)$. It can be shown that there exists an open ball, say $U \subset A$, centered at 1 such that $\# [\sigma(ax) \cap V_\lambda]$ is constant as x runs through $E(a) \cap U$. This constant integer is the multiplicity of a at λ.

- It can be shown that

$$\mathrm{tr}(a+b) = \mathrm{tr}(a) + \mathrm{tr}(b).$$

- The trace of a maximal rank element is simply the sum of its spectral values:

$$\mathrm{tr}(a) = \sum_{\lambda \in \sigma(a)} \lambda.$$

- If f is an analytic function from domain D of $\mathbb{C}$ into $\mathrm{soc}(A)$, then $\mathrm{tr}(f(\lambda))$ is holomorphic on D.
- For a fixed $a \in \mathrm{soc}(A)$, if $\mathrm{tr}(ax) = 0$ for all $x \in \mathrm{soc}(A)$ then $a = 0$.